Commentationes Mathematicae Universitatis Carolinae

11, 4 (1970)

ON THE FIRST DERIVATIVE OF REAL FUNCTIONS

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(Preliminary communication)

Z. Zahorski in [1] defined the well-known classes M_1 - M_5 of sets of real numbers. (We denote R the set of all real numbers and $(a, b) = (b, a)$ for $a, b \in R$, $a > b$. For $E \subset R$ we denote $|E|$ the outer measure of E.)

The following theorems are proved in [1].

Theorem A: Let f be a continuous function defined on (a, b). Let f possess the derivative (respectively the finite derivative; respectively the bounded derivative) f' on (a, b). Then for each $\alpha \in R$ the sets $\{x \in (a, b), f'(x) > \alpha\}$ and $\{x \in (a, b), f'(x) < \alpha\}$ are elements of M_2 (respectively M_3; respectively M_4).

Theorem B: Let $E \in M_4$. Then there exists a nondecreasing function f which possesses the bounded derivative on R such that $E = \{x \in R, f'(x) > 0\}$.

Zahorski formulated the following problem.

Is the analogy of theorem B valid for classes M_2 and M_3 ?

J.S. Lipiński [2] proved that the answer is negative.

At first, we shall solve the following problem.

Let S, G, E be subsets of R. The problem is to construct such a function f defined on R that f possesses the derivative f' on R, $E = \{x \in R, f'(x) > 0\}$, $G = \{x \in R, f'(x) = +\infty\}$ and S is the set of all $x \in R$ such that f is not continuous at x and $f'(x) > 0$.

Theorem 1 gives some necessary conditions on the sets S, G, E and Theorem 2 says that these conditions are also sufficient.

Theorem 1: Let f be a function defined on (a, b) which possesses the derivative f' on (a, b). Let $\alpha \in R$, $E = \{x \in (a, b), f'(x) > \alpha\}$, $G = \{x \in (a, b), f'(x) = +\infty\}$.

Let S be the set of all $x \in E$ at which f is discontinuous. Then the following conditions are valid:

(i) S is a countable set, G is a G_δ set of measure zero, E is a F_σ set and $S \subset G \subset E$.

(ii) For each $x \in G - S$ and $h \neq 0$ either $|(x, x+h) \cap E| > 0$ or $(x, x+h) \cap S \neq \emptyset$.

(iii) For each $x \in E - G$ and $c > 0$ there exists $\varepsilon > 0$ with the following property:
For every $h, h_1 \in R$ such that $0 < \frac{h}{h_1} < c$, $|h + h_1| < \varepsilon$ either $|(x+h, x+h+h_1) \cap E| > 0$ or $(x+h, x+h+h_1) \cap$ $\cap S \neq \emptyset$.

(iv) For every perfect set $P \subset R - G$ there exists such a portion P_0 of P (i.e. $P_0 = I \cap P \neq 0$ where I is an open interval) that there exist $\eta_n > 0$, F_n closed, $E \cap P_0 = \bigcup F_n$ such that for each $x \in F_n$ and $c > 0$ there exists $\varepsilon > 0$ with the following property (P):

For each $h, h_1 \in R$ with $0 < \frac{h}{h_1} < c$, $|h + h_1| < \varepsilon$, $x + h \in P_o$, $x+h+h_1 \in P_o$ and for each open set $H \subset R - (P_o \cup E)$ such that for every open interval $I \subset R - P_o$ the set $I \cap H$ is connected the inequality

$$(|P_o \cap E \cap (x+h, x+h+h_1)| + |(x+h, x+h+h_1) - (P_o \cup H)|) > \eta_n |h_1|$$

holds.

The proof of the conditions (i) - (iii) is similar to the proof of Zahorski's theorems A,B. The proof of the condition (iv) is based on the fact that if f' is finite on P then there exists a portion P_o of P such that for each $y, z \in P$, $y < z$ we can write the difference $f(z) - f(y)$ as the sum of $\int_{P_o \cap \langle y,z \rangle} f'$ and $\sum_n (f(b_n) - f(a_n))$ where (a_n, b_n) is the sequence of all bounded intervals contiguous to $P_o \cap \langle y, z \rangle$.

<u>Theorem 2:</u> Let S, G, E be sets of real numbers which fulfill the conditions (i) - (iv). Then there exists a function f which possesses the derivative f' on R such that

f is continuous at $x \in R$ if and only if $x \in S$; at each $x \notin S$ the function f is discontinuous from the right as well as from the left

$G = \{x \in R, f'(x) = +\infty\}$,

$E = \{x \in R, f'(x) > 0\}$,

$R - E = \{x \in R, f'(x) = 0\}$,

$f = g + v$, where g is an absolutely continuous nondecreasing function and $v(x) = \sum_{s_n < x} a_n + \sum_{s_n \leq x} a_n$, $a_n > 0$ ($\{s_n\}$ is an enumeration of all elements of S).

We omit the proof of this theorem in this paper; the detailed proofs of all theorems contained in this paper will be published later on.

On the base of Theorem 2 we can easily prove the characterisations of the sets $\{ f'(x) > \alpha \}$. We define the following classes of subsets of R .

$E \in M^*$ if $E \subset R$ is a F_σ set and for each perfect set $P \subset R$ there exists a portion P_0 of P such that a) either $P_0 \subset E$ or $E \cap P_0 = \bigcup F_n$, F_n closed and

b) there exist $\eta_n > 0$ such that for every $x \in F_n$ and $c > 0$ there exists $\epsilon > 0$ with the property (P).

$M_2^* = M_2 \cap M^*$,

$M_3^* = M_3 \cap M^*$.

Theorem 3:1.Let f be a function defined on (a, b) which possesses the derivative on (a, b) . Then for each $\alpha \in R$

$\{x \in (a, b), f'(x) > \alpha\} \in M^*$; $\{x \in (a, b), f'(x) < \alpha\} \in M^*$.

2. Let $E \in M^*$. Then there exists a nondecreasing function f defined on R which possesses the derivative on R such that $E = \{x \in R, f'(x) > 0\}$

3. Let $E_1, E_2 \in M^*$, $E_1 \cap E_2 = \emptyset$. Then there exists a function f which possesses the derivative on R such that

$E_1 = \{x \in R , f'(x) > 0\}$, $E_2 = \{x \in R , f'(x) < 0\}$.

Theorem 4: 1.Let f be a continuous function defined on (a, b) which possesses the derivative f' on (a, b) . Then for each $\alpha \in R$

$\{x \in (a,b),\ f'(x) > \alpha\} \in M_2^*,\ \{x \in (a,b),\ f'(x) < \alpha\} \in M_2^*$.

2. Let $E \in M_2^*$. Then there exists a nondecreasing absolutely continuous function f defined on R which possesses the derivative on R such that $E = \{x \in R,\ f'(x) > 0\}$.

3. Let $E_1, E_2 \in M_2^*$, $E_1 \cap E_2 = \emptyset$. Then there exists an absolutely continuous function f which possesses the derivative on R such that

$E_1 = \{x \in R,\ f'(x) > 0\},\ E_2 = \{x \in R,\ f'(x) < 0\}$.

<u>Theorem 5</u>: 1.Let f be a function defined on (a,b) which possesses the dinite derivative on (a,b) . Then for each $\alpha \in R$

$\{x \in (a,b),\ f'(x) > \alpha\} \in M_3^*,\ \{x \in (a,b),\ f'(x) < \alpha\} \in M_3^*$.

2. Let $E \in M_3^*$. Then there exists a function f defined on R which possesses the finite derivative on R such that f is an absolutely continuous nondecreasing function and $E = \{x \in R,\ f'(x) > 0\}$.

3. Let $E_1, E_2 \in M_3^*$, $E_1 \cap E_2 = \emptyset$. Then there exists an absolutely continuous function f which possesses the finite derivative on R such that

$E_1 = \{x \in R,\ f'(x) > 0\}.\ E_2 = \{x \in R,\ f'(x) < 0\}$.

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[1] Z. ZAHORSKI: Sur la première derivée, Trans.Amer.Math. Soc.69(1950),1-54.

[2] J.S. LIPIŃSKI: Une propriété des ensembles $\{f'(x) > a\}$,

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(Oblatum 10.9.1970)