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A REMARK ON THE THEORY OF DIOPHANTINE APPROXIMATIONS

(Preliminary communication)

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Let β be an irrational number and let $\beta = (b_0; b_1, b_2, \dots)$ express his expansion into a (regular) continued fraction. For t real, $t \geq 1$, put

$$\psi_\beta(t) = \min_{\substack{p,\, q \text{ integer} \\ 0 < q \leq t}} |q\beta - p|$$

and denote

$$\lambda(\beta) = \liminf_{t \to +\infty} t\,\psi_\beta(t), \quad \mu(\beta) = \limsup_{t \to +\infty} t\,\psi_\beta(t) .$$

The numbers $\lambda(\beta)$ which form the so called Lagrange's spectrum, were largely studied in the literature (cf.,e.g.,[1],[3],[4]). Comparably less studied were the values $\mu(\beta)$ (cf.[4], p.37). In 1968, V. Jarník pointed out this fact in his lectures on the theory of Diophantine approximations. Discussions between the present authors have resulted in theorems brought further, altogether proved in the first half of the year 1968.

We can easily see that

$$\mu(\beta) = \frac{1}{1 + \frac{1}{R_\beta}} ,$$

where

$$R_\beta = \limsup_{k \to +\infty} (b_k'; b_{k-1}, \dots, b_1) \cdot (b_{k+1}; b_{k+2}, \dots)$$

($\frac{1}{R_\beta} = 0$ for $R_\beta = +\infty$). Hence it is $\frac{1}{2} \leqq \mu(\beta) \leqq 1$ and $\mu(\beta) = 1$ if and only if the sequence $b_1, b_2, \dots$ is not bounded. For a natural number N, let $\mathcal{L}(N)$ denote the set of all numbers β, for which $\limsup_{k \to +\infty} b_k = N$; designate by $\mathcal{M}(N)$ the set of all R_β with $\beta \in \mathcal{L}(N)$, $\mathcal{M} = \bigcup_{N=1}^{\infty} \mathcal{M}(N)$. So we have $\mu(\beta) < 1$ if and only if $\beta \in \bigcup_{N=1}^{\infty} \mathcal{L}(N)$. A number $\alpha = (a_0, a_1, a_2, \dots)$ is said to be equivalent to a number β, if for some integer n the equality $a_{k+n} = b_k$ holds for all sufficiently large natural numbers k (we write $\alpha \sim \beta$ if α and β are equivalent, and $\alpha \nsim \beta$ otherwise). If $\alpha \sim \beta$, then clearly $\lambda(\alpha) = \lambda(\beta)$ and $\mu(\alpha) = \mu(\beta)$. We designate a period in a continued fraction as usual, e.g., $(\overline{1; 2}) = (1; 2, 1, 2 \dots) = \frac{1}{2}(1 + \sqrt{3})$.

Theorem 1 (Diviš) [1]. Let $c_i = 1$, $i = 0, 1, 2, \dots$, $\alpha_0 = (c_0; c_1, c_2, \dots)$, $\alpha_n = (\overline{2; c_1, c_2, \dots, c_{2n-1}})$, $n = 1, 2, \dots$.

Then it holds

a) $R_{\alpha_0} = \frac{3+\sqrt{5}}{2}$, b) $R_{\alpha_i} < R_{\alpha_{i+1}}, i = 0, 1, 2, \dots$, c) $\lim_{i \to +\infty} R_{\alpha_i} = 2 + \sqrt{5}$.

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1) This theorem has been proved independently by J. Lesca in his paper [5], unattainable to the present authors till March 1970. Thus, the priority belongs to J. Lesca.

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d) If $R_\beta < 2 + \sqrt{5}$ then there exists a non negative integer n such that $\beta \sim \alpha_n$.

A certain information about the structure of sets $\mathfrak{M}(N)$ is given by the following two theorems (analogous theorems for values $\lambda(\beta)$ can be found in [3]):

<u>Theorem 2</u> (Novák). Let $\alpha = (\overline{1; N})$. If $\beta \in \mathcal{L}(N)$, then $R_\beta \geq R_\alpha = \alpha N + 1 = \frac{1}{2}(N + 2 + \sqrt{N^2 + 4N})$. Moreover, there exists a positive constant c depending only on N, that for $\beta \in \mathcal{L}(N)$, $\beta \nsim \alpha$ we have $R_\beta \geq$ $\geq R_\alpha + c$.

<u>Theorem 3</u> (Novák). Let $\alpha = (\overline{1; N})$. If $\beta \in \mathcal{L}(N)$ then $R_\beta \leq N R_\alpha = \frac{N}{2}(N + 2 + \sqrt{N^2 + 4N})$. For $N > 1$ and $\varepsilon > 0$ there exist uncountable sets $\mathfrak{N}$, $\mathfrak{N}_\varepsilon \subset \mathcal{L}(N)$ such that it holds

$$\text{if } \beta, \gamma \in \mathfrak{N}, \beta \neq \gamma \text{ then } \beta \nsim \gamma, R_\beta = N R_\alpha ,$$

$$\text{if } \beta, \gamma \in \mathfrak{N}_\varepsilon, \beta \neq \gamma \text{ then } \beta \nsim \gamma, R_\beta \neq R_\gamma ,$$
$$N R_\alpha - \varepsilon < R_\beta < N R_\alpha .$$

From these two theorems it follows that, for $N >$ > 1, the minimums of $\mathfrak{M}(N)$ is an isolated point of this set, and the maximum of $\mathfrak{M}(N)$ is its point of condensation.

One can show, using the results from [2] (cf.[1], p.44), that there exists λ_0 such that the values $\lambda(\beta)$ run over the whole interval $[0, \lambda_0]$. An analogous assertion of this kind is

Theorem 4 (Diviš). There exists a number R_o (e.g. $R_o = 23,4$) such that $[R_o, +\infty] \subset \mathfrak{M}$.

Detailed proofs together with further results will be published in the shortest time.

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