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COMPUTATION OF DERIVATIVES IN THE FINITE ELEMENT METHOD x)

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1. Introduction. The finite element method is a generalized Ritz-Galerkin method using special trial functions. Recently many papers have been written on different modifications of this method, e.g. [1] - [17]. The error bounds are mostly in the energy norm. In particular, for differential equations of second order, the energy norm is equivalent to the norm of the Sobolev space W_2^1. It is possible to get error bounds in spaces of lower derivatives as the energy norm, e.g. in the space L_2. See e.g. [2], [6], [10], [11], [12]. It is also possible to get error bound in the spaces with higher derivatives for special domains.

In [10] the authors get the error bounds in the higher derivatives spaces by smoothing the approximative solution. This paper deals with a general procedure for approximation of the derivatives of the solution in the general case. The procedure will be shown on a model problem. The approach is very easy to use in the general case.

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2. Some notions and lemmas.

Let R_m be m -dimensional Euclidean space $\underline{x} \equiv (x_1, \dots, x_m)$, $\|\underline{x}\|^2 = \sum_{i=1}^{m} x_i^2$. Let $S(R_m)$ be the totality of all rapidly decreasing functions (at ∞) with the common topology (see e.g.[18],p.146). The space of generalized functions over $S(R_m)$ will be denoted $S'(R_m)$ [1]. Let $f \in S'$. Then the Fourier transform of f will be denoted by $F(f)$. More about Fourier transforms of generalized functions is in [18] and [19].

Definition 2.1. The space $W_2^{\alpha}(R_m)$, $\alpha \geqslant 0$ will be the space of all functions $f \in S'$ that

$$(2.1) \qquad |F(f)|^2 (1 + \|\underline{x}\|^{2\alpha}) \in L_1(R_m)$$

and

$$(2.2) \qquad (2\pi)^m \|f\|^2_{W_2^{\alpha}(R_m)} = \| |F(f)|^2 (1 + \|\underline{x}\|^{2\alpha}) \|_{L_1(R_m)} .$$

The space $W_2^{\alpha}(R_m)$ is the Sobolev space (in general fractional). Let $\omega(x) \in S'$ be a function with compact support. Then $\omega(x)$ is a convolutor (see e.g.[19] vol. 2,ch.III,§ 3,4) and we may define the operator $A(\omega, h)$

$$(2.3) \qquad A(\omega, h)(\underline{x}) = \frac{1}{h^m} \left(f(\underline{x}) * \omega\left(\frac{\underline{x}}{h}\right) \right) .$$

The operator $A(\omega, h)$ is a linear operator on S' into S'. [See e.g.[19],vol.2,ch.III,§ 3,4 .]

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1) We shall often write S instead of $S(R_m)$ in this and analogous cases.

Lemma 2.1. Let $\omega \in S'$ have compact support. Let $\lambda = F(\omega)$. Further let us assume that

(2.4) $$|\lambda(\underline{x}) - 1| \leq C \|\underline{x}\|^t \quad \text{for } t \geq 1 \text{ and}$$
$\|x\| \leq 1$.

(2.5) $$|\lambda(\underline{x})(1 + \|\underline{x}\|^q)| \leq C \quad \text{for } q \geq 0 .$$

Then

1) for $\beta \leq \alpha + q$, $f \in W_2^\alpha(R_m)$ we have

(2.6) $$\|A(\omega, h)f\|_{W_2^\beta(R_m)} \leq C h^{-\chi} \|f\|_{W_2^\alpha(R_m)}$$

with

(2.7) $$\chi = \max(\beta - \alpha, 0) ,$$

2) for $0 \leq \beta \leq \alpha$, $\beta \geq t$,

(2.8) $$\|A(\omega, h)f - f\|_{W_2^\beta(R_m)} \leq C h^\mu \|f\|_{W_2^\alpha(R_m)}$$

where

(2.9) $$\mu = \min(t - \beta, \alpha - \beta)$$

Proof. 1. Putting $g = A(\omega, h)f$ we have

(2.10) $$(Fg)(x) = \lambda(xh)(Ff)(x)$$

and therefore

(2.11) $$\begin{aligned} \int_{R_m} |Fg|^2 (1 + \|x\|^{2\beta})\, dx &\leq \\ &\leq C \int_{R_m} |Ff|^2 \frac{1 + \|x\|^{2\beta}}{1 + \|xh\|^{2q}}\, dx \\ &\leq C h^{-2\chi} \|f\|^2_{W_2^\alpha(R_m)} \end{aligned}$$

because $\beta - q \leq \alpha$.

1) C will be a generic constant with different values on different places.

2. Putting

$$(2.12) \qquad z = g - f$$

we have

$$(2.13) \qquad (Fz)(x) = (\lambda(xh) - 1)(Ff)(x)$$

and therefore using (2.4) and (2.5) for $q = 0$ we get

$$(2.14) \qquad \begin{aligned} \|z\|^2_{W_2^\beta(R_m)} &\le C\int_{R_m} |\lambda(xh)-1|^2 |(Ff)(x)|^2 (1+\|x\|^{2\beta})\,dx \\ &\le C\Big[\int_{\|x\|<\frac{1}{h}} \ldots + \int_{\|x\|\ge\frac{1}{h}} \ldots\Big] \le \\ &\le Ch^{2\mu}\int_{\|x\|<\frac{1}{h}} |F(f)|^2 (1+\|x\|^{2\beta})\|x\|^{2\mu}\,dx \\ &+ C\int_{\|x\|\ge\frac{1}{h}} |F(f)(x)|^2 (1+\|x\|^{2\beta}) \frac{(1+\|x\|^{2\mu})}{(1+\|x\|^{2\mu})}\,dx \le \\ &\le Ch^{2\mu}\int_{R_m} |F(f)(x)|^2 (1+\|x\|^{2\alpha})\,dx \ . \end{aligned}$$

Our lemma is proved.

__Lemma 2.2__. Let $f \in W_2^\alpha(R_m)$, $g_h \in W_2^\beta(R_m)$, $\beta \le \alpha$, $0 < h < 1$. Further let

$$(2.15) \qquad \|f - g_h\|_{W_2^\beta(R_m)} \le Ch^\sigma \|f\|_{W_2^\alpha(R_m)} \ .$$

Suppose $\omega(x) \in S'$ is the function fulfilling the assumption of Lemma 2.1 and $\gamma \le q + \beta$, $t \ge \gamma$, $\alpha \ge \gamma$. Then

$$(2.16) \qquad \|f - A(\omega, h) g_h\|_{W_2^\gamma(R_m)} \le Ch^\mu \|f\|_{W_2^\alpha(R_m)}$$

where

$$(2.17) \qquad \mu = \min(t-\gamma, \alpha-\gamma, \sigma - \max(\gamma-\beta, 0)) \ .$$

__Proof__. 1. Let

$$(2.18) \qquad z_h = A(\omega, h)(f - g_h) \ ,$$

then we have by Lemma 2.1

(2.19) $$\| z_h \|_{W_2^{\gamma}(R_m)} \le C h^{s-\chi} \| f \|_{W_2^{\alpha}(R_m)}$$

with

$$\chi = \max(\gamma - \beta, 0) .$$

2. We also have by Lemma 2.1

(2.20) $$\| A(\omega, h) f - f \|_{W_2^{\gamma}(R_m)} \le C h^{\mu_1} \| f \|_{W_2^{\alpha}(R_m)}$$

with

$$\mu_1 = \min(t - \gamma, \alpha - \gamma) .$$

3. We may write

(2.21) $$\| f - A(\omega, h) \varphi_h \|_{W_2^{\gamma}(R_m)} \le$$

$\le \| f - A(\omega, h) f \|_{W_2^{\gamma}(R_m)} + \| A(\omega, h)(f - \varphi_h) \|_{W_2^{\gamma}(R_m)}$ and using (2.19) and (2.20) together with (2.21) we get our result.

3. <u>Finite element method</u>.

Let us introduce functions $\varphi_k(x)$, $x \in R_1$, in the following way. Let

(3.1) $$\varphi_{k+1}(x) = \varphi_1(x) * \varphi_k(x)$$

with

$$\begin{aligned} \varphi_1(x) &= 1 \quad \text{for } |x| \le \tfrac{1}{2} , \\ &= 0 \quad \text{for } |x| > \tfrac{1}{2} . \end{aligned}$$

Further for $\underline{x} \in R_m$, $\underline{x} = (x_1, \dots, x_m)$, put

$$(3.2) \qquad \varphi_k(\underline{x}) = \prod_{j=1}^{m} \varphi_k(x_j) \quad .$$

We have

$$(3.3) \qquad (F\varphi_1)(x) = \frac{\sin \frac{x}{2}}{\frac{x}{2}} = \lambda_1(x) \quad ,$$

$$(3.4) \qquad (F\varphi_k)(x) = \lambda_1^k(x)$$

and

$$(3.5) \qquad (F\varphi_k)(x) = \prod_{j=1}^{m} \lambda_1^k(x_j) \quad .$$

For a numerical procedure for $\varphi_k(x)$ see [20].

Let $\Omega \subset R_m$ be a bounded domain with the boundary $\Omega^\circ \subset C^\infty$. The space $W_2^\alpha(\Omega)$ for $\alpha \geq 0$ will be the usual Sobolev fractional space. For α an integer we have

$$(3.6) \qquad \| u \|^2_{W_2^\alpha(\Omega)} = \sum_{|k| \leq \alpha} \| D^k u \|^2_{L_2(\Omega)}$$

where the sum is over all derivatives of the order $0 \leq$ $\leq |k| \leq \alpha$. For $\alpha = [\alpha] + \sigma$, $0 < \sigma < 1$ we introduce the fractional spaces due to Aronszajn [21] and Slobodetskij [22]

$$(3.7) \qquad \| u \|^2_{W_2^\alpha(\Omega)} = \| u \|^2_{W_2^{[\alpha]}(\Omega)} + \sum_{|k| = [\alpha]} \| D^k u \|^2_{W_2^\sigma(\Omega)}$$

where

$$\| u \|^2_{W_2^\sigma(\Omega)} = \int_\Omega \int_\Omega \frac{|u(t) - u(\tau)|^2}{\| t - \tau \|^{m+2\sigma}} \, dt \, d\tau \quad .$$

Putting $\Omega = R_m$ we get the space introduced in Definition 2.1 (with an equivalent norm).

As a model problem let us seek the weak solution of the Neumann problem

$$(3.8) \qquad -\Delta u + u = f \qquad \text{on } \Omega ,$$

$$(3.9) \qquad \frac{\partial u}{\partial n} = 0 \qquad \text{on } \Omega^{\bullet} .$$

The following theorem holds

<u>Theorem 3.1</u>. Let $f \in W_2^{\alpha}(\Omega)$, for $\alpha \geq 0$. Then there exists exactly one solution u of the Neumann problem (3.8), (3.9) and

$$(3.10) \qquad \| u \|_{W_2^{\alpha+2}(\Omega)} \leq C \| f \|_{W_2^{\alpha}(\Omega)} .$$

For the proof see [23],ch.2,§ 7.3.

By the finite element method (special version) we shall understand the following method for computing the approximate solution $u_h(\underline{x})$. We put

$$(3.11) \qquad u_h(x) = \sum_{\underline{j}} c_h(j)\varphi_k\left(\frac{\underline{x} - \underline{j} h}{h}\right) \qquad k \geq 2 ,$$

where $\underline{j}$ is a multiinteger, i.e. $j = (j_1, \dots, j_m)$, j_i, $i = 1, 2, \dots, m$ integers and with the sum taken over all $\underline{j}$ such that

$$(3.12) \qquad \operatorname{supp} \varphi_k\left(\frac{\underline{x} - \underline{j} h}{h}\right) \cap \Omega \neq \emptyset \qquad 1)$$

---

1) $\operatorname{supp} \varphi_k(x)$ means the support of the function $\varphi_k(x)$.

The coefficients $C_h(j)$ in (3.11) are taken so that they minimize the quadratic functional

$$\int_\Omega \left(\sum_{i=1}^m \left(\frac{\partial u_h}{\partial x_i}\right)^2 + u_h^2\right) d\underline{x} - 2\int_\Omega f u_h \, dx$$

over all possible choices of the form (3.11).
The following theorem holds

<u>Theorem 3.2</u>. We have

$$\|u_h - u\|_{W_2^1(\Omega)} \le C h^\mu \|f\|_{W_2^\alpha(\Omega)} \tag{3.13}$$

where

$$\mu = \min(\alpha + 1, k - 1) . \tag{3.14}$$

The theorem follows from the approximation properties of $\varphi_h(\underline{x})$ and the basic properties of the Ritz method, e.g. see [16],[17]. It is also possible to prove

<u>Theorem 3.3</u>. Let $0 \le \gamma \le 1$, $\alpha \ge 0$, then

$$\|u_h - u\|_{W_2^\gamma} \le C h^\mu \|f\|_{W_2^\alpha(\Omega)} \tag{3.15}$$

where $\qquad \mu = \min(\alpha + 2 - \gamma, k - \gamma)$.

The theorem follows by the arguments used in [2],[11], [12] and [17].
A logical question is whether (3.13) holds for $\gamma > 1$.

Let us now show that a small transformation (which is computationely very cheap) of u_h in (3.11) will guarantee convergence in the space $W_2^\gamma(\Omega)$ when $\gamma \le \min(\alpha + 2, k)$.

Define $\gamma_0 = \min(\alpha + 2, k)$ and let $q_0 \geq \gamma_0 - 1$ with q_0 an integer. Let us construct a finite sum

(3.16) $$\omega(x) = \sum a(\underline{k})\varphi_{q_0}(x - k)$$

so that

(3.17) $$|F(\omega)(x) - 1| \leq C\|x\|^{\gamma_0} .$$

Obviously there exist many such functions. In particular we may find one so that the support of (3.14) will lie in apriori prescribed cone with the vertex at the origin.

Let u_h be the approximate solution given by (3.11). By the extension theorem (see e.g.[24]) we may construct the function U_h (resp. U) so that

1. $U_h \in W_2^1(R_m)$ (resp. $U \in W_2^{\alpha+2}(R_m)$),
2. $U_h = u_h$ (resp. $U = u$) on Ω ,
3. $\|U_h\|_{W_2^1(R_m)} \leq C\|u_h\|_{W_2^1(\Omega)}$

   (resp. $\|U\|_{W_2^{\alpha+2}(R_m)} \leq C\|u\|_{W_2^{\alpha+2}(\Omega)}$),
4. $\|U_h - U\|_{W_2^1(R_m)} \leq C\|u_h - u\|_{W_2^1(\Omega)}$.

Applying Lemma 2.2 we get

(3.18) $$\|u - A(\omega, h)U_h\|_{W_2^{\gamma}(\Omega)} \leq \leq C h^{\mu}\|f\|_{W_2^{\alpha}(\Omega)}$$

with

$$\mu = \gamma_0 - \gamma, \quad \gamma \leq \gamma_0 .$$

So if we take $\bar{u}_h^{\omega}(\underline{x}) = (A(\omega, h) U_h)(x)$ then we would get the desired result. The problem now is how to determine $\bar{u}_h^{\omega}(\underline{x})$ because U_h is in practice very hard to construct.

For some $\underline{x}$ the construction is very easy. In fact for $\underline{x}$ far enough from the boundary we have

$$(3.19) \qquad \bar{u}_h^{\omega}(\underline{x}) = \sum_{j} \bar{C}_h(\underline{j})\, \varphi_{k+q_0}(\underline{x})$$

where.

$$(3.20) \qquad \bar{C}_h(\underline{j}) = \sum C_h(\underline{k})\, a(\underline{j} - \underline{k})$$

and the sum is finite (with only a few terms).
Let us now define $Q(\omega, h)$ as the set of all $\underline{x} \in R_m$ that $\bar{u}_h^{\omega}(\underline{x})$ is given in the form (3.19) and (3.20).
If ω has its support in a cone with a proper angle then for h sufficiently small

$$Q(\omega, h) \cap \Omega^{\circ} \neq \emptyset \, .$$

We may so construct a finite set ω_j, $j = 1, 2, \dots, \ell$ with the desired properties so that ($\mathcal{I}nt\, Q(\omega_j, h)$ means interior of $_k Q(\omega_j, h)$)

$$(3.21) \qquad \bigcup_{j=1} \mathcal{I}nt\, Q(\omega_j, h) \supset \bar{\Omega} \quad .$$

Obviously it is very easy to construct functions $\psi_j(\underline{x})$, $j = 1, \dots, \ell$ so that $\psi_j(\underline{x}) \in C^{\infty}$, $\psi_j(\underline{x}) = 0$ outside of $Q(\omega_j, h)$ for all $h < H$ and $\sum_{j=1}^{\ell} \psi(x) = 1$ for $x \in \Omega$. Then we may define

(3.22) $\bar{u}_h(x) = \sum_{j=1}^{\ell} \psi_j(\underline{x})\, \bar{u}_h^{\omega_j}(\underline{x})$

and for $\bar{u}_h(x)$ (3.18) is valid.

The procedure is very simple from a practical standpoint. Let us remark that using interpolation technique (see [25]) we get the error bound in C norm provided that $\gamma_0 \geq m$.

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