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GRAPHIC ALGEBRAS

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In this article, certain abstract algebras defining undirected graphs are considered. It is shown how the relationship between those algebras and graphs is changed when further conditions are imposed on those algebras.

1. We shall say that an ordered pair (V, Q) is a graphic algebra if V is a non-empty set and Q is a ternary operation on V such that for every $a, b, c \in V$

(1) $\quad Q(a, a, b) = a$

and

(2) $\quad Q(b, a, c) = Q(a, b, c) = Q(a, c, b)$.

Let $A = (V, Q)$ be a graphic algebra and $G = (V_0, E)$ an undirected graph ([2]). We shall say that G is the graph of A if $V_0 = V$ and for every $a, b \in V$ it holds that

(3) $\quad Q(a, b, c) \in \{a, b\}$ for every $c \in V$ if and only if $(a, b) \in E$.

<u>Proposition 1.</u> The graph of a graphic algebra contains no circuit of length 3.

Let $A = (V, Q)$ be a graphic algebra; we shall say that a mapping $L : V \times V \rightarrow \exp V$ is a localizator of A if for every $a, b, c \in V$ it holds that

(4) $L(a, b) \cap L(b, c) \cap L(a, c) = \{Q(a, b, c)\}$.

<u>Proposition 2.</u> Let $A_1 = (V, Q_1)$, $A_2 = (V, Q_2)$ be graphic algebras having a common localizator L . Then $A_1 = A_2$.

<u>Proposition 3</u>. Let (V, E) be the graph of a graphic algebra A having a localizator L . For every $a, b \in V$ it holds that $\{a, b\} \subset L(a, b)$ .Moreover, $\{a, b\} = L(a, b)$ if and only if $(a, b) \in E$.

<u>Lemma 1.</u> Every graphic algebra has at most one localizator.

<u>Proof</u>. Let (V, Q) be a graphic algebra with localizators L_1 and $L_2 \neq L_1$. Without loss of generality let us assume that there exist $a, b, c \in V$ such that $c \in L_1(a, b) \dot{-} L_2(a, b)$. Thus $L_1(a, b) \cap L_1(b, c) \cap L_1(a, c) = \{c\} \neq L_2(a, b) \cap L_2(b, c) \cap L_2(a, c)$, which is a contradiction and thus the lemma is proved.

2. We shall say that a graphic algebra $A = (V, Q)$ is normal if for every $a, b, c, d \in V$ it holds that

(5) $Q(Q(a, b, c), b, d) = Q(a, b, Q(c, b, d))$.

<u>Proposition 4</u>. In a normal graphic algebra (V, Q) for every $a, b, c \in V$

(6) $\quad Q(Q(a,b,c),b,c) = Q(a,b,c)$

holds.

Theorem 1. The graph of a normal graphic algebra contains no circuit of length 5.

Proof. Let $(a_1,a_2),(a_2,a_3),(a_3,a_4),(a_4,a_5),(a_5,a_1)$ be a circuit of length 5 in the graph of a normal graphic algebra (V,Q). Further, the operations " + " and " - " on $\{1,\dots,5\}$ mean the operations modulo 5. From (3) it follows that $a_i = Q(a_{i-1}, a_i, a_{i+1})$, for $1 \leq i \leq 5$.

Let $1 \leq j \leq 5$. If $a_j = Q(a_j, a_{j+1}, a_{j+3})$ and $a_{j+2} = Q(a_{j+3}, a_{j+2}, a_j)$ then by (2) and (5) we get

$$a_j = Q(a_j,a_{j+3},Q(a_j,a_{j+1},a_{j+2})) = Q(a_j,a_{j+1},Q(a_j,a_{j+2},a_{j+3})) =$$

$= Q(a_j,a_{j+1},a_{j+2}) = a_{j+1}$, which is a contradiction. Analogically, if $a_{j+3} = Q(a_j,a_{j+2},a_{j+3})$, then $Q(a_j,a_{j+1},a_{j+3}) \neq a_{j+1}$. Thus, we shall next assume that $Q(a_j,a_{j+1},a_{j+3}) = a_j$ if and only if $Q(a_{j+3},a_{j+2},a_j) = a_{j+3}$.

Without loss of generality let us discuss the case $Q(a_1,a_2,a_4) = a_1$; then $Q(a_4,a_3,a_1) = a_4$. Let $Q(a_2,a_3,a_5) = a_2$; if $Q(a_5,a_1,a_3) = a_5$, then $Q(a_3,a_2,a_5) = a_3$, which is a contradiction; if $Q(a_5,a_1,a_3) = a_1$, then $Q(a_4,a_3,a_1) = a_3$, which is also a contradiction. Let $Q(a_2, a_3, a_5) = a_3$; if $Q(a_4, a_5, a_2) = a_4$, then $Q(a_1, a_2, a_4) = a_2$, which is a contradiction; if $Q(a_4, a_5, a_2) = a_5$, then $Q(a_2, a_3, a_5) = a_2$, which is a contradiction, too.

<u>Lemma 2</u>. Let (V, Q) be a normal graphic algebra, $a, b, c, d \in V$.

(7) If $Q(a,c,d) = Q(b,c,d)$, then $Q(a,b,c) = Q(a,b,d)$.

<u>Proof</u>. A) Let $p, q, r, s \in V$ and $Q(p,r,s) = Q(q,r,s)$. Then

$$\begin{aligned}
Q(p,q,r) &= Q(s, Q(p,q,r), Q(p,q,r)) = \\
&= Q(s, Q(p,r,q), Q(r, Q(r,p,q), p)) = \\
&= Q(Q(s, Q(p,r,q), r), Q(r,p,q), p) = \\
&= Q(Q(Q(s,p,r), q, r), Q(r,p,q), p) = \\
&= Q(Q(q,r,s), p, Q(r,p,q)) = \\
&= Q(Q(Q(q,r,s), p, r), p, q) = \\
(+) \qquad &= Q(Q(p,r,s), p, q) = \\
(++) \qquad &= Q(Q(p,q,r), p, s) \\
&= Q(Q(q,r,s), p, q) = \\
(+++) \qquad &= Q(Q(p,q,r), q, s) .
\end{aligned}$$

B) Let $t, u, v, w \in V$ and $Q(t,v,w) = Q(u,v,w) = w$. Then by (+) $Q(t,u,v) = Q(Q(t,v,w), t, u) = Q(t,u,w)$.

C) Let $Q(a,c,d) = Q(b,c,d)$. Then by (++)

and (+++) $Q(a,d,Q(a,b,c)) = Q(b,d,Q(a,b,c)) = Q(a,b,c)$.
From B) it follows that $Q(a,b,d) = Q(a,b,Q(a,b,c)) = Q(a,b,c)$ and the proof is complete.

Theorem 2. Let (V,Q) be a normal algebra, $a,b,c,d,e \in V$. When $Q(a,d,e) = Q(b,d,e) = e$, then $Q(Q(a,b,c),d,e) = e$.

Proof. According to (7) $Q(a,b,d) = Q(a,b,e)$. From (1),(2) and (5) it follows that $Q(Q(a,b,c),d,e) = Q(Q(a,b,c),d,Q(a,d,e)) = Q(Q(Q(a,b,c),d,a),d,e) = Q(Q(Q(a,b,d),c,a),d,e) = Q(Q(Q(a,b,e),c,a),d,e) = Q(Q(Q(a,b,c),e,a),d,e) = Q(Q(a,b,c),e,Q(a,d,e)) = Q(Q(a,b,c),e,e) = e$.

Theorem 3. Every normal graphic algebra has a localizator.

Proof. Let $A = (V,Q)$ be a normal graphic algebra; as J we shall designate the mapping $V \times V \to \exp V$ such that

$J(p,q) = \{r \in V \mid r = Q(p,q,r)\}$ for every $p,q \in V$.

Let $a,b,c,d \in V$, $d = Q(a,b,c)$. By (6) $Q(a,b,d) = Q(a,c,d) = Q(b,c,d) = d$. Thus $Q(a,b,c) = d \in J(a,b) \cap J(b,c) \cap J(a,c)$. Now, let $e \in J(a,b) \cap J(b,c) \cap J(a,c)$. Then $e = Q(a,b,e)$, $Q(b,c,e) = Q(a,c,e) = e$; thus, by (6) $d = Q(a,b,c) = Q(a,b,e) = e$. This means that J fulfils (4) and the proof is complete.

Lemma 3. Let (V, Q) be a normal graphic algebra, $n \geq 1$, $a, b_0, \ldots, b_n \in V$ and let it hold that

(8) if $0 < m \leq n$, then $b_m = Q(b_{m-1}, b_m, a) \neq b_{m-1}$.

Then the elements $b_0, \ldots, b_n$ are different from each other.

Proof. The case when $n = 1$ is obvious. Let $n > 1$ and let for $b_0, \ldots, b_{n-1}$ be proved that they are different from each other. Let us assume that there exists k, $0 \leq k < n-1$, such that $b_n = b_k$. Then

$$\begin{aligned} b_n &= Q(b_{n-1}, b_n, a) = Q(Q(b_{n-2}, b_{n-1}, a), b_n, a) = \\ &= Q(Q(\ldots Q(b_k, b_{k+1}, a), \ldots, b_{n-1}, a), b_n, a) = \\ &= Q(Q(\ldots Q(b_n, b_{k+1}, a), \ldots, b_{n-1}, a), b_n, a) = \\ &= Q(\ldots Q(b_k, b_{k+1}, a), \ldots, b_{n-1}, a), b_{n-1} \quad , \end{aligned}$$

which is a contradiction and thus the lemma is proved.

Theorem 4. The graph of a finite normal graphic algebra is connected.

Proof. Let (V, Q) be a normal graphic algebra, (V, E) its proper graph and V finite.

A) Let us assume that there exist $a, b_0 \in V$, $a \neq b_0$ such that for every $c \in V$, $c = Q(a, b_0, c)$, it holds that $(a, c) \notin E$. Then there exists an infinite sequence $b_0, b_1, \ldots$ of elements of V such that $b_m = Q(b_{m-1}, b_m, a) \neq b_{m-1}$ for every $m \geq 1$. Because V is finite, then from Lemma 3 it follows that is a contradiction. Thus, for every $a, b \in V$, $a \neq b$, there exists c, $c = Q(a, b, c)$ such that $(a, c) \in E$.

B) Let us assume that there exist $a, b_0 \in V$ such that there exists no arc connecting a and b_0. Then $a \neq b_0$ and there exists an infinite sequence $b_0, b_1, b_2,$ of elements of V such that $(b_{m-1}, b_m) \in E$ and $Q(b_{m-1}, b_m, a) = b_m$ for every $m \geq 1$. From A), Lemma 2 and the finiteness of V it follows that it is a contradiction. Thus, for every $a, b \in V$ there exists an arc connecting a and b.

3) We shall say that a graphic algebra (V, Q) is simple if for every $a, b, c, d, e \in V$ it holds that

(9) $\quad Q(Q(a, b, c), d, e) = Q(Q(a, d, e), b, Q(c, d, e))$.

<u>Proposition 5</u>. Every simple graphic algebra is normal.

<u>Remark.</u> Algebras with operation of median (see [1], pp.137-138) and tree algebras (see [3], pp.19-22 and also [4]) are simple graphic algebras. In [3], Lemma 2 and Theorem 2 are proved (but in a more special context and by a rather different way), Lemmas 2 and 3 in [4].

Let $G = (V, E)$ be an undirected graph. As $d_G(a, b)$ we shall designate the distance between a and b. For every $a, b \in V$, such that $d_G(a, b) < +\infty$, we shall denote

(10) $D_G(a, b) = \{c \in V \mid d_G(a, c) + d_G(b, c) = d_G(a, b)\}$.

Lemma 4. Let (V, G) be a simple graphic algebra, $G = (V, E)$ its graph, $a, b, c \in V$, $d_G(a, b) < +\infty$, and $c \in D_G(a, b)$. Then $c = Q(a, b, c)$.

Proof. Let us denote $d_G(a, b) = m$. The case when $m \leq 1$ is obvious. Let $m > 1$ and let for every $\bar{a}, \bar{b} \in V$, such that $d_G(\bar{a}, \bar{b}) < m$, the proof has been completed. Evidently, there exists an arc

$(c_0, c_1), \dots, (c_{n-1}, c_n)$, such that $c_0 = a$, $c_n = b$ and $c_k = c$ for some k, $0 \leq k \leq m$. The case when $k = 0$ or $k = m$ is obvious.

Let us assume that $0 < k < m$ and that $Q(a, b, c_k) \neq c_k$. For every i, $0 < i < m$, we shall designate $Q(a, b, c_i)$ as e_i. If $e_1 = c_1$, then

$$e_k = Q(a, b, c_k) = Q(a, b, Q(c_1, c_k, b)) = Q(c_k, b, Q(c_1, a, b)) =$$

$= Q(c_k, b, c_1) = c_k$, which is a contradiction; thus we shall assume that $e_1 = a$ and analogically that $e_{n-1} = b$. If $0 < i < j < m$ and $e_i = e_j$, then by (6) $c_i = c_j$, which is a contradiction; thus $e_1, e_2, \dots, e_{m-2}$, e_{m-1} are different from each other. Let $0 < j < m-1$; $x \in V$; then by (9) $Q(e_j, e_{j+1}, x) = Q(Q(a, b, c_j), Q(a, b, c_{j+1}), x) =$

$$= Q(Q(c_j, c_{j+1}, x), a, b) = \begin{cases} e_j, \text{ if } Q(c_j, c_{j+1}, x) = c_j \\ e_{j+1}, \text{ if } Q(c_j, c_{j+1}, x) = c_{j+1} \end{cases}.$$

This means that $(e_1, e_2), \dots, (e_{m-2}, e_{m-1})$ is an arc connecting a and b and thus $d_G(a, b) \leq m - 2$, which is a contradiction. Thus the lemma is proved.

<u>Lemma 5</u>. Let (V, Q) be a simple graphic algebra, $G = (V, E)$ its graph, $a, b, c \in V$, $d_G(a, b) < +\infty$, $Q(a, b, c) = c$. Then $c \in D_G(a, b)$.

<u>Proof</u>. The case when $d_G(a, b) \leq 1$ is obvious. Let us assume that $d_G(a, b) > 1$ and that for every $\bar{a}, \bar{b} \in V$ such that $d_G(\bar{a}, \bar{b}) < d_G(a, b)$, $c = Q(\bar{a}, \bar{b}, c)$, it has been proved that $c \in D_G(\bar{a}, \bar{b})$. Let us denote

$$X = \{x \in D_G(a, b) \mid (x, a) \in E\},$$

$$Y = \{y \in D_G(a, b) \mid (y, b) \in E\}.$$

From Lemma 4 it follows that $Q(a, x, b) = x$, $Q(a, y, b) = y$, for every $x \in X$ and $y \in Y$.

Let us assume that $c \notin D_G(a, b)$. If there exists $x_0 \in X$ such that $Q(a, x_0, c) = x_0$, then $c = Q(a, b, c) = Q(x_0, b, c)$ and thus $c \in D_G(x_0, b) \subset D_G(a, b)$, which is a contradiction. The case when there exists $y_0 \in Y$ such that $Q(a, y_0, c) = y_0$ is analogical. Now let $Q(a, x, c) = a$, $Q(b, y, c) = b$ for every $x \in X$ and $y \in Y$. If $z \in V$, then by (9)
$$Q(c, Q(b, c, x), z) = Q(Q(a, b, c), Q(b, c, x), z) =$$
$$= Q(Q(a, x, z), b, c) \begin{cases} = c, \text{ if } Q(a, x, z) = a. \\ = Q(b, c, x), \text{ if } Q(a, x, z) = x. \end{cases}$$
Thus, $(c, Q(b, c, x)) \in E$; analogically, $(c, Q(a, c, y)) \in E$. Because $Q(b, c, x) \neq c \neq$

$\neq Q(a,c,y)$ and $Q(Q(b,c,x),c,Q(a,c,y)) = Q(Q(Q(b,c,x),c,a),c,y) = Q(Q(Q(a,c,x),c,b),c,y) = Q(Q(a,c,x),c,Q(b,c,y)) = Q(a,b,c) = c$, then $(Q(b,c,x),c)$, $(c,Q(a,c,y))$ is an arc. If there exists $x_0 \in X$ such that $Q(b,c,x_0) = b$, then $c \in D_G(b,Q(a,c,y) \subset D_G(a,b)$, which is a contradiction. Now, let for every $x \in X$ hold that $Q(b,x,c) \neq b$. Obviously, $Q(Q(a,c,y),a,Q(b,x,c)) = Q(Q(a,y,Q(c,a,Q(b,x,c))) = Q(Q(a,y,Q(c,x,Q(a,b,c))) = Q(a,c,y)$. Because $d_G(a,Q(b,x,c)) < d_G(a,b)$, then $Q(a,y,c) \in D_G(a,Q(b,x,c))$. This means that $c \in D_G(a,Q(b,x,c)) \subset D_G(a,b)$, which is a contradiction and the lemma is proved.

From Lemmas 4 and 5 we conclude:

<u>Theorem 5</u>. Let A be a simple graphic algebra, G its graph. When G has a finite diameter, then D_G is the localizator of A.

<u>Corollary</u>. Let A_1 and A_2 be simple graphic algebras with the common graph G having a finite diameter. Then $A_1 = A_2$.

<u>Example.</u> Let a, b, c, d, e be different from each other; let us denote $U = \{a,b,c,d\}$, $W = U \cup \{e\}$. Let $(U,Q_1),(U,Q_2),(W,Q_3),(W,Q_4)$ be graphic algebras such that

$Q_1(b,c,d)=a,\ Q_1(c,d,a)=b,\ Q_1(d,a,b)=c,\ Q_1(a,b,c)=d$;

$Q_2(a,b,c)=a,\ Q_2(a,b,d)=b,\ Q_2(b,c,d)=c,\ Q_2(a,c,d)=d$;

$Q_3(a,b,d)=a,\ Q_3(a,b,c)=b,\ Q_3(b,c,d)=c,\ Q_3(c,d,a)=d$;

$Q_3(a,b,e)=b=Q_3(b,e,c),\ Q_3(a,d,e)=Q_3(c,d,e)=d$;

$Q_3(b,e,d)=Q_3(a,e,c)=e$;

$Q_4(a,b,c)=Q_4(a,b,d)=Q_4(a,b,e)=b,\ Q_4(b,c,d)=$
$=Q_4(b,c,e)=Q_4(a,c,d)=c,\ Q_4(a,d,e)=Q_4(b,d,e)=$
$=Q_4(c,d,e)=d,\ Q_4(a,c,e)=a$.

Then (U, Q_1) does not have the localizator and its graph is non-connected; (U, Q_2) has the localizator and its graph is non-connected; (W, Q_3) does not have the localizator and its graph is connected; (W, Q_4) has the localizator and its graph is connected but it is not normal; moreover, (W, Q_4) has the same graphs as the simple graphic algebra (W, Q_5) such that $Q_5(a,c,e)=c$, and if $\{a,c,e\} \neq \{x,y,z\} \subset W$, then $Q_5(x,y,z)=Q_4(x,y,z)$.

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