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Jaroslav Ježek

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ON ATOMS IN LATTICES OF PRIMITIVE CLASSES

Jaroslav JEŽEK, Praha

This paper is a continuation of my papers [2] and [3] on lattices $\mathcal{L}_\Delta$ (of all primitive classes of algebras of type Δ). For the terminology see [3] . We shall be concerned with atoms in $\mathcal{L}_\Delta$. It is well-known (see [1]) that every $\mathcal{L}_\Delta$ is atomic.

In § 1, Theorem 1, a complete answer to the following question (Grätzer's problem 33 in [1] is given: find the number of atoms in $\mathcal{L}_\Delta$, for all types Δ .

For any complete atomic lattice L we can define, in a natural way, an element of L : the supremum of the set of all atoms of L . If $L = \mathcal{L}_\Delta$, then every element of L determines a primitive class of algebras of type Δ and we may ask to describe the primitive class determined by the supremum of atoms. The description depends on whether Δ contains or does not contain at least binary operations. The description is found in Theorems 2 and 3.

For the terminology and notation see § 1 of [3].

As in [3], we fix an infinitely countable set X and for each type Δ an absolutely free algebra W_Δ of type Δ . If A is an algebra of type $\Delta = (m_i)_{i \in I}$

and $i \in I$, then the i-th fundamental operation of A is denoted by $f_i^{(A)}$; the i-th fundamental operation of W_Δ is denoted by f_i. If $m_i = 0$, then f_i is an element of W_Δ.

Elements of W_Δ are called Δ-terms. A Δ-term w is called constant if $X \cap S(w)$ is empty (the set $S(w)$ is the set of all subwords of w, defined in [3]). A Δ-term is evidently constant, if and only if it belongs to the subalgebra of W_Δ generated by the empty set.

A Δ-equation $\langle w_1, w_2 \rangle$ is called constant if w_1 and w_2 are constant Δ-terms.

Let a type $\Delta = (m_i)_{i \in I}$ be given. Elements $i \in I$ such that $m_i = 1$ are called unary symbols (of Δ). A finite (not necessarily non-empty) sequence of unary symbols is called unary sequence. If A is an algebra of type Δ, $a \in A$ and $s = s_1, \dots, s_m$ is a unary sequence, then a^s is defined in this way: $a^s = a$ if s is empty; $a^{s_1,\dots,s_m} = f_{s_m}^{(A)}(a^{s_1,\dots,s_{m-1}})$.

If $s = s_1, \dots, s_m$ and $t = t_1, \dots, t_m$ are two unary sequences, then st is the unary sequence $s_1, \dots, s_m, t_1, \dots, t_m$.

If Δ is a type, then $\mathcal{L}_\Delta$ is the dual of the lattice of all FI-congruence relations of W_Δ. Let us denote the greatest element of $\mathcal{L}_\Delta$ by $1_{\mathcal{L}_\Delta}$ and the smallest by $0_{\mathcal{L}_\Delta}$.

A Δ-theory E is called consistent if

$Cn(E) \neq 0_{\mathcal{L}_\Delta}$, i.e.if E has a non-trivial model; "inconsistent" means "not consistent".

§ 1. The number of atoms in lattices $\mathcal{L}_\Delta$

Given a type Δ, denote by $AT(\Delta)$ the cardinality of the set of all atoms in $\mathcal{L}_\Delta$.

Lemma 1. Let $\Delta = (m_i)_{i \in I}$ where $I = \{i_1, i_2\}$, $i_1 \neq i_2$ and $m_{i_1} = m_{i_2} = 1$. Then $AT(\Delta) = 2^{\aleph_0}$.

Proof. It is sufficient to prove $AT(\Delta) \geq 2^{\aleph_0}$. Denote i_1 by $|$ and i_2 by $+$. If A is an algebra of type Δ and $a \in A$, then $a' = f_{i_1}^{(A)}(a)$ and $a^+ = f_{i_2}^{(A)}(a)$. Let x and y be two different elements of X. Denote by M the set of all infinite sequences $e = \langle e_1, e_2, e_3, \ldots \rangle$ of numbers 0 and 1, so that M has $2^{\aleph_0}$ elements. For each $e \in M$ define a Δ-theory E_e: it contains all equations $\langle x^{+|\overset{n}{+}|}, y^{+|\overset{n}{+}||} \rangle$ where n is such that $e_n = 0$ and all equations $\langle x, x^{+|\overset{n}{+}|} \rangle$ where n is such that $e_n = 1$. (Here $\overset{n}{+}$ denotes the sequence containing n symbols $+$.) If e_1 and e_2 are two different elements of M, then $E_{e_1} \cup E_{e_2}$ is evidently inconsistent; as $\mathcal{L}_\Delta$ is an atomic lattice, it is sufficient to prove that every E_e is consistent. Let $e \in M$.

Denote by A the set of all ordered pairs $\langle \ell, r \rangle$ where $r \geq 1$ is a rational number and ℓ

is either 0 or 1. Let us fix a one-to-one mapping φ of the set of all rational numbers $r \geq 1$ onto the set of all rational numbers q such that $1 \leq q < 2$. Define an algebra A_e with the underlying set A in this way:

(i) $\langle 0, r\rangle^{+} = \langle 1, \varphi(r)\rangle$;

(ii) $\langle 1, r\rangle^{+} = \langle 1, r+1\rangle$;

(iii) $\langle 1, r\rangle^{|} = \langle 0, r\rangle$;

(iv) If $m \leq r < m+1$ and $e_m = 0$, then $\langle 0, r\rangle^{|} = \langle 0, m\rangle$;

(v) Let $m \leq r < m+1$ and $e_m = 1$. If $\varphi^{-1}(r - m + 1) < 2$, put $\langle 0, r\rangle^{|} = \langle 0, \varphi^{-1}(\varphi^{-1}(r - n + 1))\rangle$. If $\varphi^{-1}(r - m + 1) \geq 2$, put $\langle 0, r\rangle^{|} = \langle 1, \varphi^{-1}(r - m + 1) - 1\rangle$.

We shall prove that A_e is a model of E_e. Let an integer $n \geq 1$ be given.

Let $e_n = 0$. Let $a \in A$. There exists an $r < 2$ such that $a^{+|+} = \langle 1, r\rangle$. We have $a^{+|++^{n-1}} = \langle 1, r + n - 1\rangle$ where $n \leq r + n - 1 < n + 1$, so that $a^{+|\overset{n}{+}||} = \langle 0, n\rangle$. Hence, $\langle x^{+|\overset{n}{+}||}, y^{+|\overset{n}{+}||}\rangle$ is valid in A_e.

Let $e_n = 1$. Let $a \in A$. If $a = \langle 0, r\rangle$, then $a^{+|\overset{n}{+}|} = \langle 0, \varphi(\varphi(r)) + n - 1\rangle$; as $n \leq \varphi(\varphi(r)) + n - 1 < n + 1$ and $\varphi^{-1}(\varphi(\varphi(r)) + n - 1 - n + 1) = \varphi(r) < 2$, we get $a^{+|\overset{n}{+}||} = \langle 0, \varphi^{-1}(\varphi^{-1}(\varphi(\varphi(r)) + n - 1 + 1))\rangle = \langle 0, r\rangle = a$. If $a = \langle 1, r\rangle$, then $a^{+|\overset{n}{+}|} = \langle 0, \varphi(r+1) + n - 1\rangle$; as

$n \leq \varphi(\kappa+1)+m-1 < m+1$ and $\varphi^{-1}(\varphi(\kappa+1)+m-1-m+1) = \kappa+1 \geq 2$, we get $a^{+|\overset{m}{+}\|} = \langle 1, \varphi^{-1}(\varphi(\kappa+1)+m-1-m+1)-1 \rangle = \langle 1, \kappa \rangle = a$. Hence, $\langle x, x^{+|\overset{m}{+}\|} \rangle$ is valid in A_e.

<u>Lemma 2</u>. Let $\Delta = (n_i)_{i \in I}$ where $n_i \leq 1$ for all $i \in I$. If α is a constant Δ-equation and A an atom in $\mathcal{L}_\Delta$, then $\alpha \in A$.

<u>Proof.</u> Let C be the set of all $w \in W_\Delta$ such that $\langle w, \bar{w} \rangle \in A$ for some constant Δ-term $\bar{w}$. It is easy to prove that $A \cup (C \times C)$ is a FI-congruence relation of W_Δ and $A \cup (C \times C) \neq Q_{\mathcal{L}_\Delta}$. As A is an atom, we set $A = A \cup (C \times C)$, i.e. $C \times C \subseteq A$. Each constant Δ-equation belongs to $C \times C$.

<u>Lemma 3</u>. Let $\Delta = (m_i)_{i \in I}$ where $m_i \geq 1$ for all $i \in I$. If I is infinite, then $AT(\Delta) = 2^{\mathrm{Card}\, I}$.

<u>Proof</u>. It is sufficient to prove $AT(\Delta) \geq 2^{\mathrm{Card}\, I}$. Let x and y be two different elements of X. For each subset M of I define a Δ-theory E_M in this way: it contains all equations $\langle x, f_i(x, \ldots, x) \rangle$ where $i \in M$ and all equations $\langle f_i(x, \ldots, x), f_i(y, \ldots, y) \rangle$ where $i \in I - M$. Evidently, each E_M is consistent, so that there exists an atom A_M in $\mathcal{L}_\Delta$ such that $A_M \vdash E_M$. If M_1 and M_2 are two different subsets of I, then $E_{M_1} \cup E_{M_2}$ is evidently inconsistent, so that $A_{M_1} \neq A_{M_2}$. There are $2^{\mathrm{Card}\, I}$ different subsets of I.

Lemma 4. Let $\Delta = (n_i)_{i \in I}$; let there exist an $i_0 \in I$ such that $n_{i_0} = 1$ and $n_i = 0$ for all $i \in I - \{i_0\}$. Then $AT(\Delta) = 2$. If C is the set of all constant Δ-equations and x, y two different elements of X, then the two atoms of $\mathcal{L}_\Delta$ are just $C_n(C \cup \{\langle x, f_{i_0}(x)\rangle\})$ and $C_n(C \cup \{\langle f_{i_0}(x), f_{i_0}(y)\rangle\})$.

Proof is easy; for the complete description of $\mathcal{L}_\Delta$ in this case see [2].

Theorem 1. Let a type $\Delta = (n_i)_{i \in I}$ be given.
(i) Let $n_i \leq 1$ for all $i \in I$; put $m = Card\{i \in I;\ n_i = 1\}$. If $m = 0$, then $AT(\Delta) = 1$. If $m = 1$, then $AT(\Delta) = 2$. If $2 \leq m < \aleph_0$, then $AT(\Delta) = 2^{\aleph_0}$. If m is infinite, then $AT(\Delta) = 2^{m}$.
(ii) Let there exist an $i_0 \in I$ such that $n_{i_0} \geq 2$. If I is finite, then $AT(\Delta) = 2^{\aleph_0}$. If I is infinite, then $AT(\Delta) = 2^{Card\, I}$.

Proof. Let $n_i \leq 1$ for all $i \in I$. If $m = 0$, the assertion is easy, and if $m = 1$, it follows from Lemma 4. Let $m \geq 2$. By Lemma 2, if $i, j \in I$ and $n_i = n_j = 0$, then $\langle f_i, f_j \rangle$ belongs to every atom of $\mathcal{L}_\Delta$. Thus, the atoms in $\mathcal{L}_\Delta$ are in a one-to-one correspondence with some primitive classes of algebras with one nullary and m unary operations; we get $AT(\Delta) \leq 2^{\aleph_0}$ if m is finite and $AT(\Delta) \leq 2^{m}$, if m is infinite. The converse inequalities follow from Lemmas 1 and 3.

Let there exist an $i_0 \in I$ such that $m_{i_0} \geq 2$. If I is finite, then the assertion follows from Kalicki [4] ; see also Grätzer [1], Theorem 2 in § 27. Let I be infinite. It is sufficient to prove $AT(\Delta) \geq$ $\geq 2^{\mathrm{Card}\, I}$. At least one of the two sets $\{i \in I ; m_i \geq 1\}$ and $\{i \in I ; m_i = 0\}$ has the same cardinality as I. If $\mathrm{Card}\, \{i \in I ; m_i \geq 1\} = \mathrm{Card}\, I$, the assertion follows easily from Lemma 3. Let $\mathrm{Card} \{i \in I ; m_i = 0\} = \mathrm{Card}\, I$. Let x and y be two different elements of X. For every subset M of $\{i \in I ; m_i = 0\}$ define a Δ-theory E_M : it contains all equations $\langle x, f_{i_0}(x, f_j, \dots, f_j)\rangle$ where $j \in M$ and all equations $\langle f_{i_0}(x, f_j, \dots, f_j)$, $f_{i_0}(y, f_j, \dots, f_j)\rangle$ where $j \in \{i \in I ; m_i = 0\} - M$. The proof can be finished as in Lemma 3.

§ 2. Supremum of the set of atoms in $\mathcal{L}_\Delta$: the case $m_i \leq 1$ for all $i \in I$.

Let $\Delta = (m_i)_{i \in I}$ be a type such that $m_i \leq 1$ for all $i \in I$. We shall describe the supremum $\mathcal{S}$ of the set of all atoms in $\mathcal{L}_\Delta$.

Firstly, let $m_i = 0$ for all $i \in I$. As there is exactly one atom in $\mathcal{L}_\Delta$, $\mathcal{S}$ is just the atom, i.e., the set of all Δ-equations that are either constant or trivial.

Secondly, let $\{i \in I ; m_i = 1\}$ have exactly one element i_0 . $\mathcal{L}_\Delta$ has exactly two atoms; they are described in Lemma 4. It is easy to see that the supremum $\mathcal{S}$ of these two atoms is just $C_m(C \cup \{\langle f_{i_0}(x),\ f_{i_0}(f_{i_0}(x))\rangle\})$ where x and C are as in Lemma 4.

It remains to consider the case $Card\,\{i \in I\ ;\ m_i = 1\} \geq 2$.

<u>Lemma 5</u>. Let $m_i \leq 1$ for all $i \in I$ and $Card\,\{i \in I; m_i = 1\} \geq 2$. Let $x \in X$; let s and $\bar{s}$ be two different unary sequences (of Δ). Then there exists a consistent Δ -theory E such that $E \cup \{\langle x^s, x^{\bar{s}}\rangle\}$ is inconsistent.

<u>Proof</u>. Let us fix two different unary symbols $|$ and $+$ (of type Δ). We may suppose that if either $\bar{s} = st$ or $s = \bar{s}t$ for some unary sequence t , then the first symbol in t is not $|$.(If this were not true, we could exchange the role of $|$ and $+$.) Denote by t_1 the longest common beginning of s and $\bar{s}$; we may write $\bar{s} = t_1 t_2$ for some unary sequence t_2 . Denote by c the length of s, by d_1 the length of t_1 and by d_2 the length of t_2 .

If r and $\bar{r}$ are two rational numbers, then $[r, \bar{r}]$ denotes the set of all rational numbers q such that $r < q < \bar{r}$. Put $A = [0, 1]$. It is evidently possible to choose subsets $A_0, \dots, A_c$ of A so that the following be true: A_0 is an infinite subset of $[\frac{1}{2}, 1]$ and its complement in

$[\frac{1}{2}, 1]$ is infinite, too; if $0 < k \leq c$ and if the k-th symbol in s is $+$, then $A_k = \{\frac{1}{2}x ; x \in A_{k-1}\}$; if $0 < k \leq c$ and if the k-th symbol in s is different from $+$, then A_k is an infinite subset of $[\frac{1}{2}, 1] - (A_0 \cup \ldots \cup A_{k-1})$ and its complement in $[\frac{1}{2}, 1] - (A_0 \cup \ldots \cup A_{k-1})$ is infinite, too. It is evidently possible to choose sets $\overline{A}_0, \ldots, \overline{A}_{d_2}$ so that the following be true: $\overline{A}_0 = A_{d_1}$; if $0 < k \leq d_2$ and if the k-th symbol in t_2 is $+$, then $\overline{A}_k = \{\frac{1}{2}x ; x \in \overline{A}_{k-1}\}$; if $0 < k \leq d_2$ and if the k-th symbol in t_2 is different from $+$, then $\overline{A}_k$ is an infinite subset of $[\frac{1}{2}, 1] - (A_0 \cup \ldots \cup A_c \cup \overline{A}_0 \cup \ldots \cup \overline{A}_{k-1})$ and its complement in $[\frac{1}{2}, 1] - (A_0 \cup \ldots \cup A_c \cup \overline{A}_0 \cup \ldots \cup \overline{A}_{k-1})$ is infinite, too.

Let us fix an integer $n \geq 1$ such that neither s nor $\overline{s}$ contains $\overset{n}{+}$ (the unary sequance, consisting of n symbols $+$) as a connected subsequnce. The sets $[0, \frac{1}{2^n}], A_0, \ldots, A_c, \overline{A}_1, \ldots, \overline{A}_{d_2}$ are evidently pairwise disjoint.

We shall make A algebra of type Δ. For all $a \in A$ put $a^+ = \frac{1}{2}a$; for all $a \in [0, \frac{1}{2^n}]$ put $a' = \varphi(a)$ where φ is a fixed one-to-one mapping of $[0, \frac{1}{2^n}]$ onto A_0; if $0 < k \leq c$ and if the k-th symbol in s is $i \neq +$, then for all

$a \in A_{k-1}$ put $f_i^{(A)}(a) = \varphi_k(a)$ where φ_k is a fixed one-to-one mapping of A_{k-1} onto A_k; if $0 < k \le d_2$ and if k-th symbol in t_2 is $i \ne +$, then for all $a \in \bar{A}_{k-1}$ put $f_i^{(A)}(a) = \psi_k(a)$ where ψ_k is a fixed one-to-one mapping of $\bar{A}_{k-1}$ onto $\bar{A}_k$. The definition of the algebra A is not yet completed, but realize this: $a^{\overset{n}{+}1s}$ is already defined for all $a \in A$ and $a \longrightarrow a^{\overset{n}{+}1s}$ is a one-to-one mapping of A onto A_c; similarly, $a^{\overset{n}{+}1\bar{s}}$ is already defined for all $a \in A$ and $a \longrightarrow a^{\overset{n}{+}1\bar{s}}$ is a one-to-one mapping of A onto $\bar{A}_{d_2}$; by the assumption stated at the beginning of this proof, b' is not yet defined for any $b \in A_c$ and for any $b \in \bar{A}_{d_2}$. Let us fix an element $\alpha \in A$. We can complete the definition of the algebra A in this way: if $b \in A_c$, then b' is the uniquely determined $a \in A$ such that $a^{\overset{n}{+}1s} = b$; if $b \in \bar{A}_{d_2}$, then $b' = \alpha$; in all other cases the operations are defined arbitrarily.

In this algebra A, the equations $\langle x, x^{\overset{n}{+}1s1} \rangle$ and $\langle x^{\overset{n}{+}1\bar{s}1}, y^{\overset{n}{+}1\bar{s}1} \rangle$ ($y \in X$ being different from x) are valid and thus the theory $E = \{ \langle x, x^{\overset{n}{+}1s1} \rangle, \langle x^{\overset{n}{+}1\bar{s}1}, y^{\overset{n}{+}1\bar{s}1} \rangle \}$ is consistent; $E \cup \{ \langle x^{s}, x^{\bar{s}} \rangle \}$ is evidently inconsistent.

<u>Theorem 2</u>. Let $\Delta = (n_i)_{i \in I}$ where $n_i \le 1$ for all $i \in I$ and $Card\, \{ i \in I ; n_i = 1 \} \ge 2$. The supremum of the set of all atoms in $\mathcal{L}_\Delta$ is just the

set of all Δ-equations that are either constant or trivial.

<u>Proof</u>. Denote the supremum by $\mathcal{S}$ and the set of all Δ-equations that are either constant or trivial by C. By Lemma 2, we have $C \subseteq \mathcal{S}$. Let $\langle w_1, w_2 \rangle \notin C$. Then $w_1 \neq w_2$ and either w_1 or w_2 is not a constant Δ-term, so that it is equal to x^{s} for some $x \in X$ and some unary sequence s. There exists evidently a unary sequence $\bar{s} \neq s$ such that $\langle w_1, w_2 \rangle \vdash \langle x^{s}, x^{\bar{s}} \rangle$. By Lemma 5 there exists a consistent theory and hence an atom E in $\mathcal{L}_\Delta$ such that $E \cup \{\langle x^{s}, x^{\bar{s}} \rangle\}$ is inconsistent. As $\langle x^{s}, x^{\bar{s}} \rangle \notin E$, we have $\langle x^{s}, x^{\bar{s}} \rangle \notin \mathcal{S}$ and consequently, $\langle w_1, w_2 \rangle \notin \mathcal{S}$. We get $\mathcal{S} = C$.

§ 3. <u>Supremum of the set of atoms in $\mathcal{L}_\Delta$: the case $n_{i_0} \geq 2$ for some $i_0 \in I$</u>

Let $\Delta = (n_i)_{i \in I}$ be a type such that there exists an $i_0 \in I$ satisfying $n_{i_0} \geq 2$; let us fix such an i_0.

For all $w \in W_\Delta$ and $n = 1, 2, 3, \dots$ define w^{n} in this way: $w^1 = w$; $w^{n+1} = f_{i_0}(w^{n}, \dots, w^{n})$.

<u>Lemma 6</u>. Let w_1 and w_2 be two different elements of W_Δ and x, y two different elements of $X \cap (S(w_1) \cup S(w_2))$. Then there exist two different elements $\overline{w}_1, \overline{w}_2 \in W_\Delta$ such that

$\langle w_1, w_2 \rangle \vdash \langle \overline{w}_1, \overline{w}_2 \rangle$ and $X \cap (S(\overline{w}_1) \cup S(\overline{w}_2)) \subseteq$
$\subseteq (X - \{y\}) \cap (S(w_1) \cup S(w_2))$.

<u>Proof</u>. For each $n = 1, 2, 3, \dots$ let η_n be the endomorphism of W_Δ defined by $\eta_n(y) = x^n$ and $\eta_n(x) = x$ for all $x \in X - \{y\}$. Evidently, we have $\langle w_1, w_2 \rangle \vdash \langle \eta_n(w_1), \eta_n(w_2) \rangle$ and $X \cap (S(\eta_n(w_1)) \cup S(\eta_n(w_2))) \subseteq (X - \{y\}) \cap (S(w_1) \cup S(w_2))$. There exists an integer $n \geq 1$ such that $x^n \notin S(w_1)$ and $x^n \notin S(w_2)$. It is sufficient to prove the following assertion for all $t_1, t_2 \in W_\Delta$: whenever $n \geq 1$ is an integer such that $x^n \notin S(t_1)$, $x^n \notin S(t_2)$ and $\eta_n(t_1) = \eta_n(t_2)$, then $t_1 = t_2$. We shall prove by the induction on t_1 that the assertion holds for this t_1 and for all $t_2 \in W_\Delta$.

Let $t_1 \in X$. If $t_1 \in X - \{y\}$, then $\eta_n(t_1) = t_1$, so that (if $\eta_n(t_1) = \eta_n(t_2)$) $\eta_n(t_2) \in X$, so that evidently, $\eta_n(t_2) = t_2$ and consequently, $t_1 = t_2$. If $t_1 = y$, then $\eta_n(t_1) = x^n$, so that (if $\eta_n(t_1) = \eta_n(t_2)$) $\eta_n(t_2) = x^n$ and thus either $t_2 = x^n$ or $t_2 = y$; in the first case we would get a contradiction with $x^n \notin S(t_2)$, so that $t_2 = y = t_1$.

Let $t_1 = f_i(t_1^{(1)}, \dots, t_1^{(m_i)})$ and let the assertion hold for $t_1^{(1)}, \dots, t_1^{(m_i)}$. If $t_2 \in X$, then the proof is similar to the proof in the case $t_1 \in X$.

Let $t_2 \notin X$, so that $t_2 = f_j(t_2^{(1)}, \dots, t_2^{(m_j)})$ for some $j \in I$ and $t_2^{(1)}, \dots, t_2^{(m_j)} \in W_\Delta$. If $\eta_m(t_1) = \eta_m(t_2)$, then

$$f_i(\eta_m(t_1^{(1)}), \dots, \eta_m(t_1^{(m_i)})) = \eta_m(f_i(t_1^{(1)}, \dots, t_1^{(m_i)})) = \eta_m(t_1) = \eta_m(t_2) = f_j(\eta_m(t_2^{(1)}), \dots, \eta_m(t_2^{(m_j)})) ,$$

so that $i = j$ and $\eta_m(t_1^{(1)}) = \eta_m(t_2^{(1)}), \dots, \eta_m(t_1^{(1)}) = \eta_m(t_2^{(m_i)})$. By the induction assumption (as $x^{\Omega} \notin S(t_1) \cup S(t_2)$ evidently implies $x^{\Omega} \notin S(t_1^{(1)}) \cup S(t_2^{(2)}), \dots$), we get $t_1^{(1)} = t_2^{(1)}, \dots, t_1^{(m_i)} = t_2^{(m_i)}$, so that $t_1 = t_2$.

<u>Lemma 7</u>. Let w_1 and w_2 be two different elements of W_Δ. Then there exist two different elements $\overline{w}_1, \overline{w}_2 \in W_\Delta$ and an $x \in X$ such that $\langle w_1, w_2 \rangle \vdash \langle \overline{w}_1, \overline{w}_2 \rangle$ and $X \cap S(\overline{w}_1) = X \cap S(\overline{w}_2) = \{x\}$.

<u>Proof</u>. As every $S(w)$ is a finite set, the finite number of applications of Lemma 6 gives the existence of different elements $v_1, v_2 \in W_\Delta$ satisfying $\langle w_1, w_2 \rangle \vdash \langle v_1, v_2 \rangle$ and $\mathrm{Card}\,(X \cap (S(v_1) \cup S(v_2))) \leq 1$. If $X \cap (S(v_1) \cup S(v_2))$ is non-empty, let x be its (only) element; if it is empty, let x be an arbitrary element of X. It is sufficient to put $\overline{w}_1 = f_{i_0}(v_1, x, \dots, x)$ and $\overline{w}_2 = f_{i_0}(v_2, x, \dots, x)$.

<u>Lemma 8</u>. Let a non-trivial Δ-equation $\langle w_1, w_2 \rangle$ and an element $x \in X$ be given; let $X \cap S(w_1) =$

$= X \cap S(w_2) = \{x\}$. Then there exists a consistent Δ -theory E such that $E \cup \{\langle w_1, w_2 \rangle\}$ is inconsistent.

Proof. Put $B = S(w_1) \cup S(w_2)$. Let D be the set of all u^n where $u \in B$ and $n = 1, 2, 3, \ldots$. Let K be the set of all constant Δ -terms belonging to D ; put $V = D - K$. Let R be the set of all rational numbers. Put $A = (V \times R) \cup K$. We shall suppose that no element of K is an ordered pair; in the contrary case we would use (instead of W_Δ) some algebra isomorphic to W_Δ . If a is an ordered pair, denote by $\overleftarrow{a}$ its first and by $\overrightarrow{a}$ its second member. If a is not an ordered pair, we put $\overleftarrow{a} = a$ and we do not define $\overrightarrow{a}$. Let us fix a one-to-one mapping η of A onto R . Let us fix an integer $c \geq 1$ such that $u^c \notin B$ for all $u \in W_\Delta$; the existence of such a c is evident, and $c + 1$ has the same property. Let us fix an element $\alpha \in A$.

We shall make A algebra of type Δ . Let $i \in I$, $n_i = 0$. If $f_i \in K$, put $f_i^{(A)} = f_i$; if $f_i \notin K$, define $f_i^{(A)} \in A$ arbitrarily. Let $i \in I$, $n_i \neq 0$, and $a_1, \ldots, a_{n_i} \in A$. Evidently, at most one of the following six cases can take place:

(i) $f_i(\overleftarrow{a}_1, \ldots, \overleftarrow{a}_{n_i}) \in D$; there exists a j $(1 \leq j \leq n_i)$ such that $a_j \in V \times R$; there exists an $r \in R$ such that whenever $1 \leq j \leq n_i$ and $a_j \in V \times R$, then $\overrightarrow{a}_j = r$;

(ii) $f_i(\overleftarrow{a}_1,\dots,\overleftarrow{a}_{m_i}) \in D$ and $a_1,\dots,a_{m_i} \in K$;

(iii) $i = i_0$; there exists a $v \in V$ and an $\kappa \in R$ such that $a_1 = \langle v, \kappa \rangle$ and $a_2 = \dots = a_{m_i} = \langle v^{\ell}, \kappa \rangle$;

(iv) $i = i_0$; $a_1 \in K$; $a_2 = \dots = a_{m_i} = a_1^{\ell}$;

(v) $i = i_0$; there exists an $\kappa \in R$ such that $a_1 = \langle w_1, \kappa \rangle$ and $a_2 = \dots = a_{m_i} = \langle w_1^{\ell+1}, \kappa \rangle$;

(vi) $i = i_0$; there exists an $\kappa \in R$ such that $a_1 = \langle w_2, \kappa \rangle$ and $a_2 = \dots = a_{m_i} = \langle w_2^{\ell+1}, \kappa \rangle$.

In these cases we define $f_i^{(A)}(a_1,\dots,a_{m_i})$, successively, in this way:

(i) $= \langle f_i(\overleftarrow{a}_1,\dots,\overleftarrow{a}_{m_i}), \kappa \rangle$;

(ii) $= f_i(a_1,\dots,a_{m_i})$;

(iii) $= \langle x, \eta(a_1) \rangle$;

(iv) $= \langle x, \eta(a_1) \rangle$;

(v) $= \eta^{-1}(\kappa)$;

(vi) $= \alpha$.

In all other cases we define $f_i^{(A)}(a_1,\dots,a_{m_i})$ arbitrarily.

Let us define an endomorphism ϑ of W_A by $\vartheta(x) = f_{i_0}(x, x^{\ell}, \dots, x^{\ell})$ and $\vartheta(z) = z$ for all $z \in X - \{x\}$.

Let φ be an arbitrary homomorphism of W_Δ into A. Put $a = \varphi(x)$. Evidently, $\varphi(\nu(x)) = \langle x, \eta(a)\rangle$. For all $u \in K$ we have $\varphi(\nu(u)) = u$; by induction on v it is easy to prove for all $v \in D$ that if $X \cap S(v) = \{x\}$, then $\varphi(\nu(v)) = \langle v, \eta(a)\rangle$. We get

$$\varphi(f_{i_0}(\nu(w_1), \nu(w_1^{s+1}), \ldots, \nu(w_1^{s+1}))) =$$
$$= f_{i_0}^{(A)}(\langle w_1, \eta(a)\rangle, \langle w_1^{s+1}, \eta(a)\rangle, \ldots, \langle w_1^{s+1}, \eta(a)\rangle) =$$
$$= \eta^{-1}(\eta(a)) = a = \varphi(x)$$

and similarly,

$$\varphi(f_{i_0}(\nu(w_2), \nu(w_2^{s+1}), \ldots, \nu(w_2^{s+1}))) = x .$$

As this holds for all homomorphisms φ, A is a model of the theory E composed of $\langle x, f_{i_0}(\nu(w_1), \nu(w_1^{s+1}), \ldots$
$\ldots, \nu(w_1^{s+1}))\rangle$ and $\langle f_{i_0}(\nu(w_2), \nu(w_2^{s+1}), \ldots$
$\ldots, \nu(w_2^{s+1})), f_{i_0}(\nu(\overline{w}_2), \nu(\overline{w}_2^{s+1}), \ldots, \nu(\overline{w}_2^{s+1}))\rangle$

where $\overline{w}_2$ arises from w_2 by exchanging x with some element of $X - \{x\}$. Thus, E is consistent, and $E \cup \cup \{\langle w_1, w_2\rangle\}$ is evidently inconsistent.

<u>Theorem 3</u>. Let $\Delta = (n_i)_{i\in I}$ where $n_{i_0} \geq 2$ for some $i_0 \in I$. The supremum of the set of all atoms in $\mathcal{L}_\Delta$ is just $1_{\mathcal{L}_\Delta}$, the greatest element of $\mathcal{L}_\Delta$.

<u>Proof</u>. Let $\mathcal{S}$ be the supremum. Suppose $\mathcal{S} \neq 1_{\mathcal{L}_\Delta}$, so that some non-trivial equation belongs to $\mathcal{S}$. By Lemma 7, $\mathcal{S}$ contains a non-trivial equation $\langle w_1, w_2\rangle$ satisfying the assumptions of Lemma 8; by Lemma 8 there

exists a consistent $E \in \mathcal{L}_\Delta$ such that $E \cup \{\langle w_1, w_2 \rangle\}$ is inconsistent. As E is consistent, there exists an atom A in $\mathcal{L}_\Delta$ such that $E \subseteq A$. We have $E \cup \{\langle w_1, w_2 \rangle\} \subseteq A \cup \mathcal{S} = A$, so that A is inconsistent - a contradiction.

Let us give a re-formulation of Theorem 3. A class $\mathcal{O}\!\mathcal{L}$ of algebras of type Δ is called non-trivial if it contains at least two-element algebras; it is called non-extreme if it is non-trivial and does not contain all algebras of type Δ.

<u>Theorem 4</u>. Let $\Delta = (m_i)_{i \in I}$ where $m_{i_0} \geq 2$ for some $i_0 \in I$. For every non-extreme primitive class $\mathcal{O}\!\mathcal{L}$ of algebras of type Δ there exists a non-extreme primitive class $\mathcal{L}$ of algebras of type Δ such that $\mathcal{O}\!\mathcal{L} \cap \mathcal{L}$ is trivial.

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Matematicko-fyzikální fakulta
Karlova Universita
Praha 8 Karlín
Sokolovská 83, Czechoslovakia

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