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ON NECESSARY CONDITIONS OF OPTIMALITY IN LINEAR SPACES

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1. Introduction. The approach of the present communication to necessary conditions of optimality is based on the fact that the optimality of an element can be expressed by stating that certain suitable sets have an empty intersection. Therefore, the following scheme is adopted.

Let G be a set, let ω be a subset of G and let R be a reflexive and transitive binary relation on G. An element x of G will be called optimal with respect to ω and R - or more briefly (ω, R)-optimal - if

(a) $x \in \omega$,

(b) $y \in \omega$ and $yRx \Rightarrow xRy$.

This scheme is clearly general enough to include both the problems of constrained optimization under scalar-valued criteria and the problems of constrained optimization under vector-valued criteria.

For $x \in G$ let G_x^R mean $\{y \in G: yRx$ and $\neg\, xRy\}$. We notice that

(a) if $G_x^R \cap \omega \neq \emptyset$ then x is not (ω, R)-optimal,

(b) if $G_x^R \cap \omega = \emptyset$ and if $x \in \omega$ then x is (ω, R)-optimal.

If we are interested only in necessary conditions of the optimality of a point x of ω then it is sufficient to have at our disposal an "approximation" $A(x, \omega)$ of the set ω at the point x and an "approximation" $B(x, G_x^R)$ of the set G_x^R at the point x possessing the property

$$\omega \cap G_x^R = \emptyset \Rightarrow A(x,\omega) \cap B(x, G_x^R) = \emptyset \ .$$

If we are interested also in sufficient conditions of the optimality,then, of course, also the converse implication is important. Without additional mathematical structures we can hardly construct suitable "approximations" different from the trivial ones, as for example $A(x, \omega) = \omega$ or perhaps $A(x, \omega) \subset \omega$.

In what follows we are going to present a relatively simple realization of the idea in real linear spaces. Nevertheless, a number of the known necessary conditions of the optimality are consequences of theorems, obtained by this approach. A rederivation of both classical and more recent necessary conditions will be the subject of another paper. Here, only simple illustrations are included for the reader's convenience. Moreover, the reader will find in [1] a suitable background for the present communication.

Finally, for the reader's convenience and to avoid misunderstandings, we recall some definitions and results from the theory of linear spaces. Let L be a real linear space. A surrounded point of the set $A \subset L$ is a point $x \in A$ possessing the property that whatever the element $u \in L$, some segment $[x, x + \alpha u]$, where $\alpha > 0$, is contained in A. Here the segment $[x, x + \alpha u]$ is the set of all points $y \in L$ which are representable in the form $y = (1 - \lambda)x + \lambda(x + \alpha u)$, where $0 \leqq \lambda \leqq 1$. A cone in L is any set $A \subset L$ satisfying the condition

$$x \in A, \; \alpha > 0 \implies \alpha x \in A .$$

A linear functional is an additive and homogeneous functional. The set of all linear functionals on L will be denoted by L^{*}. If L is a real linear topological space then the set of all continuous linear functionals on L will be denoted by L^{*}.

Lemma 1 ([2], Chapter 3, Section 3.2, Theorem 1). Let M, N be convex subsets of a real linear space L and assume that N has a surrounded point but no point of M is a surrounded point of N. Then M, N can be separated, that is, there is a nontrivial linear functional f and a real number λ such that

$$x \in M \implies f(x) \geqq \lambda ,$$
$$x \in N \implies f(x) \leqq \lambda .$$

Lemma 2 ([3], Chapter I, Paragraph 4). Let L be a real linear topological space. If $f \in L^{*}$ and if there is a non-empty open subset U of L and a real number t such that $f(x) \neq t$ for all $x \in U$, then $f \in L^{*}$

2. <u>Conical approximations</u>. Let L be a real linear space, let Q be a subset of L and let x be a point of L. We define approximations $C(x, Q)$ and $K(x, Q)$ of the set Q at the point x as follows:

$$u \in C(x,Q) \Longleftrightarrow u \in L \text{ and } \underset{\varepsilon>0}{\forall}\ \underset{\alpha\in(0,\varepsilon)}{\exists} : x+\alpha u \in Q ,$$

$$u \in K(x,Q) \Longleftrightarrow u \in L \text{ and } \underset{\varepsilon>0}{\exists}\ \underset{\alpha\in(0,\varepsilon)}{\forall} : x+\alpha u \in Q .$$

We notice the following properties of $C(x, Q)$ and $K(x,Q)$:

(a) $K(x,Q) \subset C(x,Q)$,

(b) $C(x,Q)$ and $K(x,Q)$ are cones in L,

(c) $K(x,Q_1 \cap Q_2) = K(x,Q_1) \cap K(x,Q_2)$,

(d) $C(x,Q_1 \cap Q_2) \supset K(x,Q_1) \cap C(x,Q_2)$.

<u>Theorem 1.</u> Let G be a real linear space. Then

$$\omega \cap G_x^R = \emptyset \Longrightarrow [C(x,\omega) \cap K(x, G_x^R)] \cup$$
$$\cup [K(x,\omega) \cap C(x, G_x^R)] = \emptyset .$$

<u>Proof</u>. If $u \in C(x,\omega) \cap K(x, G_x^R)$ then $\underset{\varepsilon>0}{\exists}\ \underset{\alpha\in(0,\varepsilon)}{\forall} : x + \alpha u \in G_x^R$. At the same time there is $\alpha' \in (0,\varepsilon)$ such that $x + \alpha' u \in \omega$. Thus $y = x + \alpha' u$ belongs to $\omega \cap G_x^R$. If $u \in K(x,\omega) \cap$

$\cap C(x, G_x^R)$ then the similar argumentation completes the proof.

Illustration. Let $x R y$ mean $f(x) \leq f(y)$, where f is a functional defined on a real linear space G and let $\omega \subset G$.

(a) If there is an element $u \in K(x, \omega)$ such that the limit

$$f'(x, u) = \lim_{\alpha \to 0+} \frac{f(x + \alpha u) - f(x)}{\alpha}$$

exists and is negative, then x is not (ω, R)-optimal (since there is $\varepsilon > 0$ such that $f(x + \alpha u) < f(x)$ for all $\alpha \in (0, \varepsilon)$, so that u belongs also to $K(x, G_x^R) \subset C(x, G_x^R))$.

(b) If $f'(x, u)$ is defined ($f'(x, \cdot)$ need not be linear) for all $u \in K(x, \omega)$ and if $\inf_{u \in K(x,\omega)} f'(x, u) \neq 0$ then x is not (ω, R) -optimal $(x \in \omega \implies \theta \in K(x, \omega);\ f'(x, \theta) = 0)$.

(c) (See [4],p.77) Let the functional f be defined on a region ω of a real Banach space G and let x_0 be an interior point of ω, at which f has a linear Gâteaux differential. Then in order that the point x_0 be (ω, R)-optimal, it is necessary that $\operatorname{grad} f(x_0) = \theta$ $(\operatorname{grad} f(x_0) \neq \theta \implies \operatorname{grad} f(x_0)(u) < 0$ for some $u \in G$; $K(x_0, \omega) = G$).

The approximations $K(x, Q)$ and $C(x, Q)$ are insufficient in some important cases. However, more accurate approximations require beside the linearity also a convenient topology, as a rule. Thus, it is natural to consider now real linear topological spaces. For the sake of simplicity we consider first the case of real normed spaces.

Let L be a real normed space, let Q be a subset of L and let x be a point of L. We define approximations $M(x, Q)$ and $N(x, Q)$ of the set Q at the point x as follows:

$$u \in M(x,Q) \iff u \in L \quad \text{and} \underset{\varepsilon>0}{\forall} \underset{\substack{\alpha\in(0,\varepsilon)\\ v\in O(u,\varepsilon)}}{\exists} : x+\alpha v \in Q ,$$

$$u \in N(x,Q) \iff u \in L \quad \text{and} \underset{\varepsilon>0}{\exists} \underset{\substack{\alpha\in(0,\varepsilon)\\ v\in O(u,\varepsilon)}}{\forall} : x+\alpha v \in Q .$$

Here $O(u, \varepsilon)$ denotes the set $\{v \in L : \|v-u\| < \varepsilon\}$. Let us again notice a few useful properties of $M(x, Q)$ and $N(x, Q)$:

(a) $N(x,Q) \subset \mathring{K}(x,Q)$, $\overline{C(x,Q)} \subset M(x,Q)$;

(b) $M(x,Q)$ is a closed cone in L ;
$N(x,Q)$ is an open cone in L ;

(c) $N(x,Q_1 \cap Q_2) = N(x,Q_1) \cap N(x,Q_2)$;

(d) $M(x,Q_1 \cap Q_2) \supset N(x,Q_1) \cap M(x,Q_2)$.

<u>Theorem 2.</u> Let G be a real normed space. Then

$$\omega \cap G_x^R = \emptyset \implies [M(x,\omega) \cap N(x,G_x^R)] \cup$$
$$\cup [N(x,\omega) \cap M(x,G_x^R)] = \emptyset .$$

Proof. If $u \in M(x,\omega) \cap N(x, G_x^R)$ then there is $\varepsilon_0 > 0$ such that

$$\alpha \in (0, \varepsilon_0),\ \|v-u\| < \varepsilon_0 \Longrightarrow x + \alpha v \in G_x^R \ .$$

Since $u \in M(x,\omega)$ for every $\varepsilon > 0$ there are $\beta_\varepsilon \in \in (0, \varepsilon)$ and $w_\varepsilon \in O(u, \varepsilon)$ such that $x + \beta_\varepsilon w_\varepsilon \in \omega$. Consequently, $x + \beta_{\varepsilon_0} w_{\varepsilon_0} \in \omega \cap G_x^R$. If $u \in N(x,\omega) \cap \cap M(x, G_x^R)$ then the similar argumentation completes the proof.

Remark. It is not difficult to verify that

$$\omega \cap G_x^R = \emptyset \Longrightarrow \overline{K(x,\omega)} \cap \overset{*}{\mathring{K}}(x, G_x^R) = \emptyset \ .$$

It is obvious that

$$\omega \cap G_x^R = \emptyset \Longrightarrow M(x,\omega) \cap N(x, G_x^R) = \emptyset \ .$$

However, there are G, ω, R and x such that

$$\omega \cap G_x^R = \emptyset \quad \text{and} \quad M(x,\omega) \cap \mathring{K}(x, G_x^R) \neq \emptyset \ .$$

Illustration. Let xRy again mean $f(x) \leq f(y)$, where f is a functional on G. If at a point $x \in \in \omega$ $f'(x,u)$ exists for all u and if $f'(x,\cdot)$ is not only continuous but the functions $o_{x,u}(\alpha)$, where

$$f(x+\alpha u) = f(x) + \alpha f'(x,u) + o_{x,u}(\alpha) \ ,$$

$$\lim_{\alpha \to 0+} \frac{o_{x,u}(\alpha)}{\alpha} = 0 \ ,$$

satisfy also the condition

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{\substack{\alpha \in (0,\delta) \\ v \in O(u,\delta)}}{\forall} : \left| \frac{o_{x,u}(\alpha)}{\alpha} \right| < \varepsilon$$

then

$\inf_{u \in M(x,\omega)} f'(x, u) \neq 0 \Longrightarrow x$ is not (ω, R)-optimal.

For other examples in real normed spaces we refer to [1], [5] and [6].

3. Convexity. Since the necessary conditions of the optimality assert that an intersection of some sets is empty, it is possible, in the case that the sets ω and G_x^R or their suitable approximations are convex, to use various separation theorems. With regard to the properties (c) and (d) of the approximations $C(x, Q)$, $K(x, Q)$, $M(x, Q)$ and $N(x, Q)$ and with regard to the fact that the set ω is often given as an intersection of some sets (for example, as the set of all solutions to a given system of equations and/or inequalities), it is desirable to develop separation theorems for the case of finite families of convex sets. Some theorems of this sort have been stated in [1] and [7]. These results are straightforward consequences of the following theorems.

Theorem 3. Let A_0 be a non-empty convex set of a real linear space L. Let $A_1, A_2, \dots, A_n$ be convex sets of L such that every A_i $(i = 1, 2, \dots, n)$ has a surrounded point. If $\bigcap_{i=0}^{n} A_i = \emptyset$ then there are linear functionals $f_0, f_1, \dots, f_n$ and real numbers λ_0, $\lambda_1, \dots, \lambda_n$ such that

(1) $\sum_{i=0}^{m} f_i = \theta$ and not all f_i are trivial,

(2) $\sum_{i=0}^{m} \lambda_i \leqq 0$,

(3) $x \in A_i \Longrightarrow f_i(x) \leqq \lambda_i$ for $i = 0, 1, \dots, m$.

Proof. Let us define

$$N = A_1 \times A_2 \times \dots \times A_m \quad ,$$
$$M = \{(x, x, \dots, x) : x \in A_0\} \quad .$$

Since M, N are disjoint non-empty convex subsets of $L \times L \times \dots \times L$ and since N has a surrounded point, it follows from Lemma 1 that there is $f \in (L \times L \times \dots \times L)^*$ and a real number λ such that $f \neq \theta$ and

(a) $x \in N \Longrightarrow f(x) \leqq \lambda$,

(b) $x \in M \Longrightarrow f(x) \geqq \lambda$.

Since $(L \times L \times \dots \times L)^*$ is isomorphic to $L^* \times L^* \times \dots \times L^*$, there are $f_1, f_2, \dots, f_m \in L^*$ such that $f = f_1 + f_2 + \dots + f_m$. Denoting $\lambda_i = \sup_{x \in A_i} f_i(x)$ $(i = 1, 2, \dots, m)$ we obtain $\sum_{i=1}^{m} \lambda_i \leqq \lambda$ and $x \in A_i \Longrightarrow f_i(x) \leqq \lambda_i$. We complete the proof by defining $f_0 = -f$, $\lambda_0 = -\lambda$.

Theorem 4. Let A_0 be a non-empty convex set of a real linear space L . Let $A_1, A_2, \dots, A_m$ be non-empty algebraically open convex sets of L . If there are $f_0, f_1, \dots, f_m \in L^*$ and real numbers

$\lambda_0, \lambda_1, \dots, \lambda_m$ such that (1),(2) and (3) from Theorem 3 are valid, then $\bigcap_{i=0}^{m} A_i = \emptyset$.

Proof. If $x \in \bigcap_{i=0}^{m} A_i$ then $0 = \sum_{i=0}^{m} f_i(x) \leq$ $\leq \sum_{i=0}^{m} \lambda_i \leq 0$ and thus $f_i(x) = \lambda_i$ for $i = 0, 1, \dots, m$. However, this is impossible, since there is $i_0 \in \{1, 2, \dots, m\}$ such that $f_{i_0} \neq \theta$ and since A_{i_0} is a non-empty algebraically open set.

Theorem 3′. Let A_0 be a non-empty convex set of a real linear topological space L. Let $A_1, A_2, \dots, A_m$ be convex sets of L such that every A_i $(i = 1, 2, \dots, m)$ has an inner point. If $\bigcap_{i=0}^{m} A_i = \emptyset$ then there are $f_0, f_1, \dots, f_m \in L^*$ and real numbers $\lambda_0, \lambda_1, \dots, \lambda_m$ such that (1),(2) and (3) from Theorem 3 are valid.

Proof. Every inner point of A_i is a surrounded point of A_i. Hence there are $f_0, f_1, \dots, f_m \in L^*$ and $\lambda_0, \lambda_1, \dots, \lambda_m$ satisfying (1),(2) and (3). Since $f_i(x) \leq \lambda_i$ for $x \in A_i$ $(i = 1, 2, \dots, m)$ there are non-empty open sets U_i and real numbers t_i such that

$$y \in U_i \Longrightarrow f_i(y) \neq t_i, \quad i = 1, 2, \dots, m.$$

Consequently, by Lemma 2, f_i is continuous for $i = 1, 2, \dots, m$. The functional f_0 is also continuous, since $f_0 = -\sum_{i=1}^{m} f_i$.

Theorem 4′. Let A_0 be a non-empty convex set of a real linear topological space L. Let $A_1, A_2, \dots, A_m$ be non-empty open convex sets of L. If there are such

$f_0, f_1, \dots, f_m \in L^*$ and such real numbers $\lambda_0, \lambda_1, \dots, \lambda_m$ that (1),(2) and (3) from Theorem 3 are valid, then $\bigcap_{i=0}^{m} A_i = \emptyset$.

Proof. The sets $A_1, A_2, \dots, A_m$ are algebraically open.

Remark. We have already mentioned that the corresponding assertions from [1] and [7] easily follow from the theorems of this section. During preparation of this text some other results of this sort were published ([8]). Also these assertions are straightforward consequences of the theorems presented here.

Corollary 1 [8]. Let A_0 be a non-empty convex cone of a real linear space L. Let $A_1, A_2, \dots, A_m$ be non-empty algebraically open convex cones in L. The intersection $\bigcap_{i=0}^{m} A_i$ is empty if and only if there are additive functionals $f_0, f_1, \dots, f_m$ (not all trivial) such that $\sum_{i=0}^{m} f_i = \Theta$ and $x \in A_i \Rightarrow f_i(x) \leq 0$ for $i = 0, 1, \dots, m$.

Proof. We use Theorem 3 and notice that Θ is not a surrounded point of $L \setminus A_i$. Hence every λ_i is non-negative and thus $\lambda_i = 0$ for all i .

Corollary 2 [8]. Let A_0 be a non-empty convex cone of a real linear topological space L . Let $A_1, A_2, \dots, A_m$ be non-empty open convex cones in L . The intersection $\bigcap_{i=0}^{m} A_i$ is empty if and only if there are continuous additive functionals $f_0, f_1, \dots, f_m$ (not all

trivial) such that $\sum_{i=0}^{m} f_i = \theta$ and $x \in A_i \Longrightarrow$
$\Longrightarrow f_i(x) \leq 0$.

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