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REMARKS ON DOUBLY SUBSTOCHASTIC RECTANGULAR MATRICES

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In the present paper we show applications of doubly stochastic unit matrix E of the type (m, n) introduced in [1], to doubly substochastic matrices. We obtain a generalization of some results due to L. Mirsky for the case $m = n$ (see [2]).

Contents:

1. Orderings and doubly substochastic matrices

Suppose m, n are two positive integers.

Let R_n be the euclidean space of the dimension n.

Define $b \cdot y = \sum_{k=1}^{n} b_k y_k$ for $b, y \in R_n$,

$a \leq c$ iff $a_1 \leq c_1$, $a_2 \leq c_2, \ldots, a_n \leq c_n$ for $a, c \in R_n$,

$x^+ = (x_1^+, x_2^+, \ldots, x_n^+)$ and $x^- = (x_1^-, x_2^-, \ldots, x_n^-)$ for $x \in R_n$.

If Z is a subset of R_n then we put:

$Z^+ = \{x^+ ; x \in Z\}$ $\overset{\circ}{Z} = \{\xi \in R_n ; \xi \cdot x \geq 0$ for all $x \in Z\}$

and $\overset{\circ}{Z}{}^{+} = \{\xi \in R_m ;\ \xi \cdot x \geq 0$ for all $x \in Z^+\}$.

(1.1) Proposition. Let $V_n = \{b \in R_m ; b_1 \geq b_2 \geq \ldots \geq b_m\}$. Suppose $c^1 = (1,-1,0,\ldots;0)$, $c^2 = (0,1,-1,0,\ldots,0),\ldots, c^{n-1} = (0,\ldots,0,1,-1)$ and $c^n = (0,\ldots,0,1)$ are elements of R_m. Then the convex cone $\overset{\circ}{V}_n$ is generated by the elements $c^1, c^2, \ldots$ $\ldots, c^{n-1}$ and the convex cone $\overset{\circ}{V}{}_n^+$ is generated by the elements $c^1, c^2, \ldots, c^{n-1}, c^n$.

Proof. According to [1] $d \in \overset{\circ}{V}{}_n^+$ ($d \in \overset{\circ}{V}_n$ resp.) if and only if $d_1 + d_2 + \ldots + d_\varkappa \geq 0$ for $\varkappa = 1, 2, \ldots$ $\ldots, m$ (for $\varkappa = 1, 2, \ldots, m-1$ and $d_1 + d_2 + \ldots +$ $+ d_m = 0$ resp.). Hence $d \in \overset{\circ}{V}{}_m^+$ ($d \in \overset{\circ}{V}_m$ resp.) if and only if there are nonnegative numbers $\gamma_1, \gamma_2, \ldots$ $\ldots \gamma_m$ (nonnegative numbers $\gamma_1, \gamma_2, \ldots, \gamma_m$, where $\gamma_m = 0$ resp.) such that

$$d = \gamma_1 c^1 + \gamma_2 c^2 + \ldots + \gamma_{n-1} c^{n-1} + \gamma_n c^n .$$

(1.2) Definition. A matrix $S = (s_{jk})_{j,k}$ of the type (m, n) will be called doubly substochastic iff

$$s_{jk} \geq 0,\ \sum_{k=1}^{n} s_{jk} \leq 1 \quad \text{and} \quad \frac{1}{m}\sum_{j=1}^{m} s_{jk} \leq \frac{1}{n} \quad \text{for}$$

$j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$.

The set of all doubly substochastic matrices of the type (m, n) will be denoted by $\tilde{D}_{m,n}$.

(1.3) Definition. Suppose $S = (s_{jk})_{j,k} \in \tilde{D}_{m,n}$ and $T = (t_{jk})_{j,k} \in \tilde{D}_{m,n}$. Then we define the following orderings on the set $\tilde{D}_{m,n}$:

$$\leqq\ ,\ \leq \ \text{and}\ \overset{+}{\leqq}\ , \text{ where}$$

1° $S \leqq T$ iff $s_{jk} \leq t_{jk}$ for all j and k .

2° $S \leqq T$ iff $S b . x \leqq T b . x$ for all $x \in V_m$ and $b \in V_n$.

3° $S \overset{+}{\leqq} T$ iff $S b . x \leqq T b . x$ for all $x \in V_m^+$ and $b \in V_n^+$.

(1.4) Lemma. If $S \leqq T$ ($S \leqq T$ resp.) then $S \overset{+}{\leqq} T$.

(1.5) Lemma. Let $S \in \tilde{\mathfrak{D}}_{m,n}$. Then there is a doubly stochastic matrix $Q \in D_{m,n}$ (see [1]) such that $S \leqq Q$.

Proof. This lemma follows from Lemma 9.1 in [3].

(1.6) Theorem. Let E be the doubly stochastic unit matrix of the type (m, n) (see [1], 5). Then $S \overset{+}{\leqq} E$ for all $S \in \tilde{\mathfrak{D}}_{m,n}$.

Proof. Let $S \in \tilde{\mathfrak{D}}_{m,n}$. Then there is a doubly stochastic matrix $Q \in D_{m,n}$ such that $S \leqq Q$ by (1.5). According to Theorem (5.3) in [1] $Q \leqq E$. Hence $S \overset{+}{\leqq} \overset{+}{\leqq} Q \overset{+}{\leqq} E$ by (1.4) and $S \overset{+}{\leqq} E$.

(1.7) Corollary. Let $U = \{x \otimes b;\ x \in R_m,\ x_1 > x_2 > \dots > x_m > 0,\ b \in R_n,\ b_1 > b_2 > \dots > b_n > 0\}$ (see [1], 5.5). Then E is a U-exposed element of the set $\tilde{\mathfrak{D}}_{m,n}$, i,e. $E b . x > S b . x$ for all $x \in R_m$, $b \in R_n$ and $S \in \tilde{\mathfrak{D}}_{m,n}$ such that

$x_1 > x_2 > \dots > x_m > 0,\ b_1 > b_2 > \dots > b_n > 0$ and $S \neq E$.

2. Doubly stochastic and substochastic maps

Suppose a is an element of the set V_m and b is an element of the set V_n .

Let E be the doubly stochastic unit matrix of the type (m, m).

(2.1) Lemma. 1° The following inclusion holds: $R_m^+ \cup \mathring{V}_m \subset \mathring{V}_m^+$.

2° $E b^+ - a \in \mathring{V}_m^+ \iff E b^+ - a^+ \in \mathring{V}_m^+$.

(2.2) Theorem. The following conditions are equivalent:

1° There is a doubly stochastic matrix $Q \in D_{m,n}$ such that $a \leq Qb$.

2° $Eb - a \in \mathring{V}_m^+$.

3° $a \cdot v^k \leq b \cdot w^k$ for $k = 1, 2, \dots, m$, where the vectors v^k and w^k are defined in [1].

Proof. 1° $\Longrightarrow$ 2°: If $a \leq Qb$, where $Q \in D_{m,m}$, put $d = Qb$. Then $d - a \in R_m^+$ and $Eb - d \in \mathring{V}_m$ by (5.3) in [1]. Using Lemma (2.1) we obtain Property 2°.
2° $\Longrightarrow$ 1°: Let $Eb - a \in \mathring{V}_m^+$. Put $c = Eb$ and $\check{c} = (c_1, c_2, \dots, c_{m-1}, \check{c}_m)$, where $\check{c}_m = a_1 + a_2 + \dots + a_m - c_1 - c_2 - \dots - c_{m-1}$. Then $\check{c} \in V_m$; further $\check{c} \leq c$ and $\check{c} - a \in \mathring{V}_m$. Therefore there is a doubly stochastic matrix $R \in D_{m,m}$ such that $a = R\check{c}$ by (6.3) in [1].
Hence $a = R\check{c} \leq Rc = REb$ and $Q = RE \in D_{m,m}$ by (2.2) in [1].
2° $\iff$ 3°: $Eb - a \in \mathring{V}_m^+ \iff a \cdot x \leq Eb \cdot x$ for all $x \in V_m^+$. Since $Eb \cdot v^k = b \cdot E^* v^k = b \cdot w^k$ for $k = 1, 2, \dots, m$ and since the conus V_m^+ is generated by the vectors $v^1, v^2, \dots, v^m$ we obtain

the equivalence of the conditions 2^o and 3^o.

(2.3) <u>Theorem.</u> There is a matrix $S \in \tilde{D}_{m,n}$ such that $a \leq Sb$ if and only if $Eb^+ - a \in \mathring{V}_m^+$.

<u>Proof.</u> 1^o Let $a \leq Sb$, where $S \in \tilde{D}_{m,n}$. Then there is a doubly stochastic matrix $Q \in D_{m,n}$ such that $S \leq Q$ by (1.5). We obtain the following inequalities: $a \leq Sb \leq Sb^+ \leq Qb^+$.
Hence $Eb^+ - a \in \mathring{V}_m^+$ by (2.2).

2^o Let $Eb^+ - a \in \mathring{V}_m^+$. Then according to Theorem (2.2) there is a doubly stochastic matrix $Q \in D_{m,n}$ such that $a \leq Qb^+$. Define $S = (s_{jk})_{j,k} \in \tilde{D}_{m,n}$:

$s_{jk} = q_{jk}$ ($s_{jk} = 0$ resp.) if $j \in \{1, 2, \dots, m\}$, $k \in \{1, 2, \dots, n\}$ and $b_k > 0$ ($b_k \leq 0$ resp.). Then $Qb^+ = Sb$. Hence $a \leq Sb$.

(2.4) <u>Note.</u> If $a \in V_m^+$, $b \in V_n$ and $Eb^+ - a \in \mathring{V}_m^+$ then there is a doubly substochastic matrix

$S = (s_{jk})_{j,k} \in \tilde{D}_{m,n}$ such that $a = Sb$, where $s_{jk} = 0$ if $j \in \{1, 2, \dots, m\}$, $k \in \{1, 2, \dots, n\}$ and $b_k \leq 0$ or $a_j = 0$.

<u>Proof.</u> Suppose that $a \in V_m^+$, $b \in V_n$ and $Eb^+ - a \in \mathring{V}_m^+$. Then there is a matrix $R = (r_{jk})_{j,k} \in \tilde{D}_{m,n}$ such that $a \leq Rb$ by (2.3). We can suppose that $r_{jk} = 0$ if $b_k \leq 0$ for all j and k .

Put $\lambda_j = \dfrac{a_j}{\sum_{k=1}^{n} r_{jk} b_k}$ ($\lambda_j = 0$ if $j \in \{1, 2, \dots, m\}$ and $a_j = 0$), $s_{jk} = \lambda_j r_{jk}$ and $S = (s_{jk})_{j,k}$ for all j and k. Clearly, $0 \leq \lambda_j \leq 1$, $S \in \tilde{D}_{m,n}$ and

$a = S b$.

(2.5) Corollary. Suppose $a \in V_m$ and $b \in V_n$. Then there is a doubly substochastic matrix $S \in \tilde{D}_{m,n}$ such that $a = S b$ if and only if $E b^+ - a \in \mathring{V}_m^+$ and $E'^+_m (E b^- + a) \in \mathring{V}_m^+$, where E'_m is the converse-permutation matrix of the set $D_{m,m}$ (see [1],4).

Proof. 1^o If $a = S b$, where $S \in \tilde{D}_{m,n}$ then $E'_m(-a) = E'_m(-a) = E'_m S E'_n (E'_n(-b)), E'_m(-a) \in V_m$ and $E'_n(-b) \in V_n$. According to our theorem (2.3) and to Theorem (5.8) in [1] $E b^+ - a \in \mathring{V}_m^+$ and $E'_m(E b^- + a) = E E'_n(-b)^+ - E'_m(-a) \in \mathring{V}_m^+$.

2^o Suppose that $E b^+ - a \in \mathring{V}_m^+$ and $E'_m(E b^- + a) \in \mathring{V}_m^+$. Then $E b^+ - a^+ \in \mathring{V}_m^+$ and $E(E'_n(-b))^+ - E'_m a^- \in \mathring{V}_m^+$ by (2.1). Using the note (2.4) we obtain substochastic matrices $R = (r_{jk})_{j,k}$ and $T = (t_{jk})_{j,k}$ in $\tilde{D}_{m,n}$ such that $a^+ = R b$, $a^- = T(-b)$, where $r_{jk} = 0$ ($t_{jk} = 0$ resp.) if $j \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$ and $b_k \leq 0$ or $a_j \leq 0$ ($b_k \geq 0$ or $a_j \geq 0$ resp.).

Put $S = R + T$. Then $S \in \tilde{D}_{m,n}$ and

$a = a^+ - a^- = (R + T) b = S b$.

The proof is complete.

References

[1] P. ČIHÁK: On an exposed element of a set of doubly stochastic rectangular matrices, Comment.Math.Univ.Carolinae 11(1970),99-113.

[2] L. MIRSKY: On a convex set of matrices, Archiv der Mathematik,Vol.10(1952),88-92.

[3] R. PHELPS: Lectures on Choquet's theorem, New York 1966.

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