Commentationes Mathematicae Universitatis Carolinae

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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 11 (1970), No. 2, 365--378

Persistent URL: `http://dml.cz/dmlcz/105282`

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11,2 (1970)

# WHEN THE PRODUCT-PRESERVING FUNCTORS PRESERVE LIMITS

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Let $\Phi$ be a product-preserving functor from a category $K$ with products. It is well known ([1],[3]) that $\Phi$ preserves all limits existing in $K$ whenever it preserves a pull-back-diagram

$$\begin{tikzcd} & {} \arrow[dr, "\beta"] & \\ {} \arrow[ur] \arrow[dr] & & {} \\ & {} \arrow[ur, "\beta'"'] & \end{tikzcd}$$

for every pair of morphisms $\beta$, $\beta'$, for which it exists, or whenever $\Phi$ preserves an equalizer of every pair of morphisms, for which it exists. But the latter property may be satisfied by all product-preserving functors for some categories $K$. The aim of the present note is to study such categories $K$. A characterization of them by various equivalent assertions is given in Theorem 1. One simple necessary and one simple sufficient condition are shown. They make it possible to decide in many concrete cases whether every product-preserving functor from $K$ preserve limits or not. Some examples of categories with a different behaviour in preservative properties are given.

I.

<u>Conventions.</u> A class of all objects (or a class of all morphisms) of a category $K$ by $K^{\sigma}$ (or $K^{m}$, respectively) will be denoted. As usual, the fact $\alpha$ to be a morphism of $K$ from $a$ to $b$ by $\alpha \in K(a, b)$ will be written. The identity morphism from $K(a, a)$ by $id_a$ will be denoted. If $\alpha \in K(a, b)$, $\beta \in K(b, a)$, $\beta \circ \alpha = id_a$, then $a$ is called to be a <u>retract of</u> $b$, $\beta$ is called a <u>retraction (of</u> $\alpha$ ), $\alpha$ is called a <u>coretraction (of</u> $\beta$ ). (This is shorter than the expression "splitting monomorphism".) An equalizer of morphisms $\gamma$, $\gamma'$ by $eq(\gamma, \gamma')$ is denoted. <u>Sets</u> denotes the category of all sets and all their mappings. If $K$ is a category, $a \in K^{\sigma}$, then $K(a, -)$ means the covariant hom-functor from $K$ to Sets.

Let $D$ be a small category, $\mathcal{D} : D \longrightarrow K$ be a diagram in $K$ (no matter in the present note whether the void index category $D$ is or is not included), $\Phi : K \to \longrightarrow H$ be a functor. We recall that $\Phi$ preserves a limit of $\mathcal{D}$ if, whenever $\mathcal{D}$ has a limit in $K$, namely $\langle a ; \{\lambda_d ; d \in D^{\sigma}\}\rangle$ (where $a \in K^{\sigma}$, $\lambda_d \in K(a, \mathcal{D}(d)), \ldots$ ), then $\Phi \circ \mathcal{D}$ has a limit in $H$ and $\langle \Phi(a) ; \{\Phi(\lambda_d) ; d \in D^{\sigma}\}\rangle$ is this limit. A functor $\Phi : K \longrightarrow Sets$ is said to be <u>product-covering</u> if, whenever $\langle a ; \{\pi_\iota ; \iota \in \mathcal{J}\}\rangle$ is a product of a collection $\{a_\iota ; \iota \in \mathcal{J}\}$ in $K$, then for every collection $\{x_\iota ; \iota \in \mathcal{J}\}$, where

$x_\iota \in \Phi(a_\iota)$, there exists at least one $x \in \Phi(a)$ with $[\Phi(\pi_\iota)](x) = x_\iota$ for all $\iota \in \mathcal{J}$.

<u>Lemma 1</u>. Let $\Phi : K \longrightarrow \mathbf{Sets}$ be a functor, $R = \{R_t \,;\, t \in K^\sigma\}$ be a collection of binary relations, every $R_t$ be a relation on $\Phi(t)$. Then there exists a functor $\Phi/_R : K \longrightarrow \mathbf{Sets}$ and an epitransformation $\nu : \Phi \longrightarrow \Phi/_R$ such that

1) $\nu_t(x) = \nu_t(y)$ whenever $x R_t y$ ;

2) every transformation $\mu : \Phi \longrightarrow \Psi$ with $\mu_t(x) = \mu_t(y)$ whenever $x R_t y$ factorizes uniquely through $\nu$.

<u>Proof</u>. Denote by $S_t$ the smallest equivalence on the set $\Phi(t)$ such that $[\Phi(\xi)](x) S_t [\Phi(\xi)](y)$ whenever $x R_s y$, $\xi \in K(s,t)$. Put $(\Phi/_R)(t) = \Phi(t)/_{S_t}$, let $\nu_t : \Phi(t) \longrightarrow \Phi(t)/_{S_t}$ be the factor-mapping. Then $\nu = \{\nu_t \,;\, t \in K^\sigma\}$ and $\Phi/_R$ have the required properties.

<u>Convention</u>. Let $K$ be a category, $\gamma, \gamma' \in K(s,p)$. Then by $K(s,-)/_{\gamma = \gamma'}$ will be denoted the functor $K(s,-)/_R$, where $R = \{R_t ; t \in K^\sigma\}$ is the collection such that $R_p = \{\langle \gamma, \gamma' \rangle\}$, $R_t = \emptyset$ for $t \neq p$.

<u>Lemma 2.</u> Let $\Phi : K \longrightarrow \mathbf{Sets}$ be a product-covering functor. Then there exists a product-preserving functor $\Phi_\pi : K \longrightarrow \mathbf{Sets}$ and an epitransformation $\varepsilon : \Phi \longrightarrow \Phi_\pi$ such that every transformation $\mu : \Phi \longrightarrow \Psi$, where $\Psi$ is a product-pre-

serving functor, factorizes uniquely through $\varepsilon$.

<u>Proof.</u> We shall prove it only for $K$ not small. Let $\prec$ be a well-order for the class $K^\sigma$ such that every $K^\sigma_a = \{ b \in K^\sigma ; b \prec a \}$ is a set. Denote by $K_a$ the full subcategory of $K$ such that $K^\sigma_a$ is its class of all objects. Let $a \in K^\sigma$ and let a collection $\mathbb{R}^b = \{ R^b_c ; c \in K^\sigma_b \}$ be defined for all $b \in K^\sigma_a$, $R^b_c$ being a binary relation on $\Phi(c)$. We define the collection $\mathbb{R}^a = \{ R^a_c ; c \in K^\sigma_a \}$ as follows: put $x \, S^a_c \, y$ if and only if either $x = y$ or a collection $\Pi = \{ a_\iota ; \iota \in \mathcal{J} \}$ of objects of $K$ exists such that $\operatorname{card} \mathcal{J} \leqq \operatorname{card} K^\sigma_a$, for every $\iota \in \mathcal{J}$ there exists $b_\iota \prec a$ with $a_\iota \in K^\sigma_{b_\iota}$ and $[\Phi(\pi_\iota)](x) \, R^{b_\iota}_{a_\iota} \, [\Phi(\pi_\iota)](y)$, where $\langle c ; \{ \pi_\iota ; \iota \in \mathcal{J} \} \rangle$ is a product of $\Pi$ in $K$; now, let $R^a_c$ be the smallest equivalence on $\Phi(c)$, for which $[\Phi(\varrho)](x) \, R^a_c \, [\Phi(\varrho)](y)$ whenever $x \, S^a_b \, y$, $b \in K^\sigma_a$, $\varrho \in K(b, c)$.

Put $R_a = \bigcup_{c \leqq a} R^c_a$, $R = \{ R_a ; a \in K^\sigma \}$, $\Phi_\pi = \Phi / R$, let $\varepsilon : \Phi \longrightarrow \Phi_\pi$ be the factor-transformation. Then $\Phi_\pi$ and $\varepsilon$ have the required properties.

## II.

<u>Lemma 3</u>. Let $K$ be a category, $\gamma, \gamma' \in K(s,t)$, let $eq(\gamma, \gamma')$ exist. If either $K(s,-)/_{\gamma = \gamma'}$ or

$(K(s,-)/_{\gamma=\gamma'})_{\pi}$ preserves $eq(\gamma, \gamma')$, then $eq(\gamma, \gamma')$ is a coretraction.

Proof: Put either $\Phi = K(s,-)/_{\gamma=\gamma'}$ or $\Phi = (K(s,-)/_{\gamma=\gamma'})_{\pi}$. Let $\nu : K(s,-) \longrightarrow \Phi$ be the factor-transformation. Put $\sigma = eq(\gamma, \gamma')$, $\sigma \in K(p, s)$. If $\varsigma, \varsigma' \in K(s, q)$ are morphisms such that $\nu_q(\varsigma) = \nu_q(\varsigma')$, then necessarily $\varsigma \circ \sigma = \varsigma' \circ \sigma$ (it follows from the construction of $\Phi$ ). If $\Phi$ preserves $eq(\gamma, \gamma')$, then $\nu_s(id_s) = \nu(\sigma \circ \tau)$ for some $\tau \in K(s, p)$. Thus $\sigma = \sigma \circ \tau \circ \sigma$, consequently $\tau \circ \sigma = id_p$.

Proposition 1. Let $K$ be a category, $\gamma, \gamma' \in K(s, t)$, let $eq(\gamma, \gamma')$ exist.

If $K(s,-)/_{\gamma=\gamma'}$ preserves $eq(\gamma, \gamma')$, then every functor from $K$ to any category preserves it.

If $(K(s,-)/_{\gamma=\gamma'})_{\pi}$ preserves $eq(\gamma, \gamma')$, then every product-preserving functor from $K$ to any category preserves it.

Proof. We shall prove the second assertion only. Put $\Phi = (K(s,-)/_{\gamma=\gamma'})_{\pi}$, let $\nu : K(s,-) \rightarrow \Phi$ be the factor-transformation. Let $\Psi : K \longrightarrow H$ be a product-preserving functor. If $\Phi$ preserves $eq(\gamma, \gamma') = \sigma$, then $\sigma$ is a coretraction, consequently $\Psi(\sigma)$ is a monomorphism. If $\varsigma \in H(g$ ,

$\Psi(s))$ is a morphism such that $\Psi(\gamma)\circ\xi = \Psi(\gamma')\circ\xi$, then consider the transformation $\mu : \Phi \longrightarrow H(q, \Psi(-))$ with $\mu_s(\nu_s(id_s)) = \xi$. Since there exists $\tau$ such that $\nu_s(\sigma\circ\tau) = \nu_s(id)$, then $\xi$ factorizes through $\Psi(\sigma)$.

Corollary. Let $\mathbf{K}$ be a category with products, $\mathcal{D}$ a diagram in $\mathbf{K}$. If every product-preserving functor from $\mathbf{K}$ to Sets preserves a limit of $\mathcal{D}$, then every product-preserving functor from $\mathbf{K}$ to any category preserves it.

Convention. For the sake of shortness $\mathcal{P}$ always denotes a diagram of the following form and description

$$\begin{tikzcd} & a \arrow[dr, "\beta"] & \\ d \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & b \\ & a' \arrow[ur, "\beta'"'] & \end{tikzcd}$$

and $\langle c ; \{\pi, \pi'\}\rangle$ is reserved for a product of the collection $\{a, a'\}$ in the rest of the present note.

Proposition 2. Let $\mathbf{K}$ be a category, $\mathcal{P}$ be a pull-back and let there exist a product of the collection $\{a, a'\}$. If $(\mathbf{K}(c,-)/_{\beta\circ\pi = \beta'\circ\pi'})_{\pi}$ preserves $\mathcal{P}$, then every product-preserving functor from $\mathbf{K}$ to any category preserves $\mathcal{P}$.

Proof. It follows immediately from Proposition 1.

Lemma 4. Let $\mathbf{K}$ be a category with finite products, let $\gamma, \gamma' : s \longrightarrow t$ and $\sigma$ be morphisms of

$\mathbf{K}$ . Then there exist coretractions $\tilde{\gamma}$, $\tilde{\gamma}'$ with a common retraction and such that the following assertions are equivalent:

(i) $\delta = eq(\gamma, \gamma')$ ;

(ii) $\delta = eq(\tilde{\gamma}, \tilde{\gamma}')$ ;

(iii) $$\begin{tikzcd} & \cdot \arrow[dr, "\tilde{\gamma}"] & \\ \cdot \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & & \cdot \\ & \cdot \arrow[ur, "\tilde{\gamma}'"'] & \end{tikzcd}$$ is a pull-back-diagram.

Proof. Let $\langle s \times t ; \{\pi_s , \pi_t\} \rangle$ be a product of $\{s, t\}$ . It is sufficient to consider the morphisms $\tilde{\gamma}, \tilde{\gamma}' : s \longrightarrow s \times t$ with $\pi_s \circ \tilde{\gamma} = \pi_s \circ \tilde{\gamma}' = id_s$ , $\pi_t \circ \tilde{\gamma} = \gamma$, $\pi_t \circ \tilde{\gamma}' = \gamma'$ .

Lemma 5. Let $\mathbf{K}$ be a category with products, let $\Phi : \mathbf{K} \longrightarrow \mathbf{H}$ be a product-preserving functor. Then the following assertions are equivalent:

(i) $\Phi$ preserves equalizers of pairs of coretractions (with a common retraction);

(ii) $\Phi$ preserves limits;

(iii) $\Phi$ preserves all pull-backs $\mathcal{P}$ whenever $\beta$, $\beta'$ are coretractions (with a common retraction).

Proof. Use the well-known construction of limits from products and equalizers ([1],[2]) and then use Lemma 4.

Theorem 1. Let $\mathbf{K}$ be a category with products. Then the following assertions are equivalent:

(i) Every product-preserving functor from $\mathbf{K}$ to any category preserves limits.

(ii) Every product-preserving functor from $K$ to Sets preserves limits.

(iii) For every pair of coretractions $\gamma, \gamma' : s \longrightarrow t$ with a common retraction the functor $(K(s,-)/_{\gamma = \gamma'})_{\pi}$ preserves $eq(\gamma, \gamma')$ .

(iv) For every pull-back $\mathcal{P}$ such that $\beta$, $\beta'$ are coretractions with a common retraction, the functor $(K(c,-)/_{\beta \circ \pi = \beta' \circ \pi'})_{\pi}$ preserves $\mathcal{P}$ .

<u>Proof</u>. It follows easily from Lemma 4, Proposition 1 and 2.

## III.

<u>Definition.</u> We shall say that a pull-back $\mathcal{P}$ satisfies a <u>condition N</u> if $\alpha$ is a coretraction whenever $\beta'$ is a coretraction.

<u>Proposition 3</u>. Let $\mathcal{P}$ be a pull-back in a category $K$ . If every product-preserving functor from $K$ preserves $\mathcal{P}$, then $\mathcal{P}$ satisfies the condition N.

<u>Proof</u>. Consider the functor $\Phi : K \longrightarrow$ ***Sets*** which is a subfunctor of $K(c,-)$ and $\gamma \in \Phi(x)$ if and only if $\gamma \in K(c,x)$ factorizes through $\alpha$. Let $\beta'$ be a coretraction. Then $\alpha' \in \Phi(a')$ and if $\Phi$ preserves $\mathcal{P}$, then $id_c \in \Phi(c)$ , i.e. it factorizes through $\alpha$ .

<u>Examples</u>.

1) If an intersection of two retracts is not a retract again, then $N$ is not satisfied in a category. This

situation occurs in many familiar categories, even in Sets (the intersection of non-empty sets may be empty).
2) The condition N is not sufficient for a preservation of $\mathcal{P}$ by all product-preserving functors. We give an example now:

Let $\mathbb{K}$ be the category of all non-empty sets with an equivalence and all their compactible mappings. Then every pull-back in $\mathbb{K}$ satisfies $\mathcal{N}$. Let $\mathcal{P}$ be the following pull-back in $\mathbb{K}$: $d = \langle D, R_D \rangle$, $a = \langle A, R_A \rangle$, $a' = \langle A', R_{A'} \rangle$, $b = \langle B, R_B \rangle$, $B = \{0,1,2\}$, $A = \{0,1\}$, $A' = \{0,2\}$, $D = \{0\}$, $R_B = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}$, $R_Z = R_B \cap (Z \times Z)$ for $Z \in \{A, A', D\}$.

Let $\alpha, \beta, \alpha', \beta'$ be embeddings. Now, we shall prove that the functor $(\mathbb{K}(c,-)/_{\beta \circ \pi = \beta' \circ \pi'})_{\pi}$ does not preserve $\mathcal{P}$. Let $\nu : \mathbb{K}(c,-) \longrightarrow \Phi$ be a factor-transformation. One can see from the construction of $\Phi$: if

$s \in \mathbb{K}^{\sigma}$, $s = \langle S, R_S \rangle$, $\varphi, \varphi' \in \mathbb{K}(c,s)$, $\nu_s(\varphi) = \nu_s(\varphi')$,

then necessarily $\varphi(\langle 1,2\rangle) R_S \varphi'(\langle 1,2\rangle)$, $\langle 1,2\rangle \in c$.
But $\alpha \circ \tau(\langle 1,2\rangle) R_A \pi(\langle 1,2\rangle)$ for no $\tau \in \mathbb{K}(c,d)$.

<u>Definition</u>. We shall say that a pull-back $\mathcal{P}$, where $\beta, \beta'$ are coretractions, satisfies a <u>condition S</u> if

either there exists $\rho$ such that $\rho \circ \beta = id_a$ and $\rho \circ \beta'$ factorizes through $\alpha$

or there exists $\rho'$ such that $\rho' \circ \beta' = id_{a'}$ and $\rho' \circ \beta$ factorizes through $\alpha'$ .

Proposition 4. Let $\mathcal{P}$ be a pull-back in a category $K$ , let $\beta$, $\beta'$ be coretractions. If $\mathcal{P}$ satisfies $S$ , then every functor from $K$ preserves $\mathcal{P}$ .

Proof. For example, let there exist $\rho$ with $\rho \circ \beta = id_a$ , $\rho \circ \beta' = \alpha \circ \sigma$ for some $\sigma$ . Then $\alpha \circ \sigma \circ \alpha' = \rho \circ \beta' \circ \alpha' = \rho \circ \beta \circ \alpha = \alpha$, consequently $\sigma \circ \alpha' = id_{\alpha}$ . If $\Phi : K \longrightarrow H$ is a functor, then $\Phi(\alpha')$ is a monomorphism. Thus it is sufficient to prove: if $\gamma, \gamma' \in H^m$ such that $\Phi(\beta) \circ \gamma = \Phi(\beta') \circ \gamma'$, then there exists $\delta \in H^m$ with $\Phi(\alpha) \circ \delta = \gamma$, $\Phi(\alpha') \circ \delta = \gamma'$ . If we put $\delta = \Phi(\sigma) \circ \gamma'$, then $\delta$ has the required properties.

Example. The condition S is not necessary for a preservation of $\mathcal{P}$ by all product-preserving functors. We give an example now:

Let $K$ be the category of graphs, i.e. the category of all sets with one binary relation and all their compactible mappings. Let $\mathcal{P}$ be the following pull-back in $K$:

$d = \langle D, R_D \rangle$, $a = \langle A, R_A \rangle$, $a' = \langle A', R_{A'} \rangle$, $b = \langle B, R_B \rangle$, $B = \{0,1,2\}$, $A = \{0,1\}$, $A' = \{0,2\}$, $D = \{0\}$, $R_B = \{\langle 0,0 \rangle, \langle 1,1 \rangle, \langle 2,2 \rangle, \langle 1,2 \rangle\}$, $R_Z = R_B \cap (Z \times Z)$

for $Z \in \{A, A', D\}$ . Let $\alpha, \beta, \alpha', \beta'$ be embeddings. It is easy to see that $\mathcal{P}$ does not satisfy the condition S. Now, we shall prove that the functor $\Phi = (K(c,-)/_{\beta \circ \pi = \beta' \circ \pi'})_{\pi}$ preserves $\mathcal{P}$ . For every natural number $m$ denote by $\mathscr{g}_m = \langle G_m , T_m \rangle$ the following object of $K$ : $G_m = \{0, 1, 2, \dots, 2m\}$ ,

$$T_m = \{\langle i, i \rangle ; \; i = 0, 1, \dots, 2m\}$$
$$\cup \{\langle 2i-1, 2i \rangle ; \; i = 1, 2, \dots, m\}$$
$$\cup \{\langle 2i+1, 2i \rangle ; \; i = 1, 2, \dots, m-1\} \; .$$

Let $\langle \mathscr{g} ; \{\psi_m ; m$ natural number $\} \rangle$ be a product of the collection $\{\mathscr{g}_m ; m\}$, $\mathscr{g} = \langle G, T \rangle$. Denote by $x_0, x_1, x_2, x_1', x_2'$ the points of $G$ with $\psi_m(x_0) = 0$, $\psi_m(x_1) = 1$, $\psi_m(x_2) = 2$, $\psi_m(x_1') = 2m-1$, $\psi_m(x_2') = 2m$ for all $m$ .

Let $\gamma, \gamma' : \mathscr{b} \longrightarrow \mathscr{g}$ be the morphisms with $\gamma(0) = \gamma'(0) = x_0$, $\gamma(1) = x_1$, $\gamma(2) = x_2$, $\gamma'(1) = x_1'$, $\gamma'(2) = x_2'$.
Let $\nu : K(c, -) \longrightarrow \Phi$ be a factor-transformation.
Then $\nu_{\mathscr{g}}(\gamma \circ \beta \circ \pi) = \nu_{\mathscr{g}}(\gamma \circ \beta' \circ \pi') = \nu_{\mathscr{g}}(\gamma' \circ \beta' \circ \pi') = \nu_{\mathscr{g}}(\gamma' \circ \beta \circ \pi)$, as it follows from the construction of $\Phi$ . Since $x_1$ and $x_1'$ belong to different components of the graph $T$ , then there exists a morphism $\delta : \mathscr{g} \longrightarrow c$ such that

$\delta(x_0) = \delta(x_1) = \delta(x_2) = \langle 0, 0 \rangle$, $\delta(x_1') = \delta(x_2') = \langle 1, 2 \rangle$ .

Denote by $\tau$ the morphism from $c$ to $d$ . Then

$$\pi = \pi \circ \delta \circ \gamma' \circ \beta \circ \pi, \quad \pi' = \pi' \circ \delta \circ \gamma' \circ \beta' \circ \pi' ,$$

$$\pi \circ \delta \circ \gamma \circ \beta \circ \pi = \alpha \circ \tau , \quad \pi' \circ \delta \circ \gamma \circ \beta' \circ \pi' = \alpha' \circ \tau ,$$

consequently $\nu_a(\pi) = \nu_a(\alpha \circ \tau)$, $\nu_{a'}(\pi') = \nu_{a'}(\alpha' \circ \tau)$.
If $\varphi \in \Phi(a)$, $\varphi' \in \Phi(a')$ such that $[\Phi(\beta)](\varphi) = [\Phi(\beta')](\varphi')$ , then there exists $\sigma \in K(c, c)$ such that

$\nu_a(\pi \circ \sigma) = \varphi$, $\nu_{a'}(\pi' \circ \sigma) = \varphi'$ and then

$\varphi = [\Phi(\alpha)](\nu_d(\tau \circ \sigma))$, $\varphi' = [\Phi(\alpha')](\nu_d(\tau \circ \sigma))$ .

Thus every product-preserving functor preserves $\mathcal{P}$ .

Theorem 2. Let $K$ be a category with products. If every pull-back $\mathcal{P}$ , where $\beta$ , $\beta'$ are coretractions, satisfies the condition S, then every product-preserving functor from $K$ to any category preserves limits.

Proof. It follows from Theorem 1 and Proposition 4.

Examples. The condition S is satisfied for every pull-back $\mathcal{P}$ , where $\beta$ , $\beta'$ are coretractions, in the following categories:

A) the category of all non-empty sets and all their mappings;

B) the category of all pointed sets and all point-preserving mappings;

C) the category of all sets and all inclusions;

D) the category of all vector spaces over a field and all linear mappings;

E) the category of all non-empty (or pointed) topological $T_1$ -spaces and all their closed (or, moreover, point-preserving, respectively) mappings;

F) the category of all $(X, p, R)$, where $X$ is a non-empty set, $p \in X$, $R = \{R_\iota ; \iota \in J\}$, $p \in R_\iota \subset X$ , and all point-preserving compactible mappings;

G) the category of all non-empty unary universal algebras, the (unary) operations $u_1, u_2, \dots$ of which satisfy the identity

$$u_{\alpha} u_1(x) = u_1(y) ,$$

and all homomorphisms.

Convention. Let $\mathcal{J}$ be a directed set, considered as a thin category. If $H$ is a category, then by $H^{\mathcal{J}}$ the category of all functors from $\mathcal{J}$ to $\mathrm{H}$ and all their transformations is denoted. If $h : \mathcal{J} \longrightarrow \mathrm{H}$ is such a functor, then by $h_{\iota}$ is denoted the object of $\mathrm{H}$ corresponding to $\iota \in \mathcal{J}^{\sigma}$; by $h_{\iota}^{\iota'}$ is denoted the morphism of $\mathrm{H}$, corresponding to the couple $\langle \iota , \iota' \rangle$ whenever $\iota \leq \iota'$ .

Theorem 3. Let $\mathcal{J}$ be a directed set, $\mathrm{K}$ be the category Sets$^{\mathcal{J}}$ or (Pointed sets)$^{\mathcal{J}}$ . Let $\mathcal{P}$ be a pull-back in $\mathrm{K}$, $\beta$, $\beta'$ be coretractions. Then the following assertions are equivalent:

(i) $\mathcal{P}$ satisfies the condition S;

(ii) every functor from $\mathrm{K}$ to any category preserves $\mathcal{P}$;

(iii) every product-preserving functor from $\mathrm{K}$ to any category preserves $\mathcal{P}$;

(iv) $\mathcal{P}$ satisfies the condition N.

Proof. The implication (iv) $\Longrightarrow$ (i)has to be proved only. We may suppose that $\alpha_{\iota}$, $\alpha'_{\iota}$, $\beta_{\iota}$, $\beta'_{\iota}$ are inclusions and $d_{\iota} = a_{\iota} \cap a'_{\iota}$ for every $\iota \in \mathcal{J}^{\sigma}$. Let $\lambda : b \longrightarrow a$ be a retraction of $\beta$, $\tau : a \longrightarrow d$ be a retraction of $\alpha$ . We define $\varphi : b \longrightarrow a$ as follows: denote $p_{\iota} = \bigcup_{\iota' \geq \iota} (b_{\iota}^{\iota'})^{-1} (a_{\iota'} - a'_{\iota'})$ ;

put $\varphi_{\iota}(x) = \lambda_{\iota}(x)$ for $x \in p_{\iota}$ ,

$\varphi_{\iota}(x) = (\tau_{\iota} \circ \lambda_{\iota})(x)$ for $x \in b_{\iota} - p_{\iota}$ .

We prove that $\varphi$ is a transformation, i.e.

$$(*) \qquad \varphi_{\iota'} \circ b_{\iota}^{\iota'} = a_{\iota}^{\iota'} \circ \varphi_{\iota}$$

for every $\iota \leq \iota'$. If $x \in b_{\iota} - p_{\iota}$, then $b_{\iota}^{\iota'}(x) \in$ $\in b_{\iota'} - p_{\iota'}$ and then $(*)$ holds. Let $x \in p_{\iota}$; if $b_{\iota}^{\iota'}(x) \in p_{\iota'}$, then $(*)$ is evident; if $b_{\iota}^{\iota'}(x) \notin$ $\notin p_{\iota'}$, then necessarily $b_{\iota}^{\iota'}(x) \in a_{\iota'} \cap a'_{\iota'}$, which implies $(*)$ again.

One can see easily that $\varphi \circ \beta = id_a$ and $\varphi \circ \beta'$ factorizes through $\alpha$.

R e f e r e n c e s


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(Oblatum 11.12.1969)