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GEOMETRIC OBJECTS OF SUBMANIFOLDS OF A SPACE WITH FUNDAMENTAL LIE PSEUDOGROUP

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G.F. Laptěv [4] and Vasiljev [7] have established some computational procedures for finding of geometric objects of submanifolds of a space with fundamental Lie group or pseudogroup. Their papers give an extension of the methods by É. Cartan and are also written in an analogous form, which is generally considered as unsatisfactory nowadays. In particular, geometric objects of submanifolds are defined by some algorithms which are not deeper justified. In this paper, we present an invariant and exact definition of these concepts based on the theory of jets. Our approach also gives a true picture of the following specific property of geometric objects of submanifolds. Every such object expresses a geometric construction which determines a geometric object field on every submanifold of the corresponding dimension, or at least on every submanifold of some type (as hyperbolic or elliptic surface for instance). That's why we define a geometric m^k-object on the space of all contact m^k-elements, or on some its invariant subspace. To be quite invariant, we use a space with a groupoid of operators in our definition, but we also deduce the equivalent form for fibre bundles. We show in § 1 that the problem is reduced to the construction of covariant mappings of the standard fibres. For the sake of simplicity, we treat only transitive Lie pseudogroups. At the end, we describe the case of a localization of a Lie group.

Our considerations are in the category C^∞.

1. Let Φ be a Lie groupoid over B with projections a, b and let G_x denote its isotropic group over $x \in B$, [5]. We shall use frequently the following relation between Lie groupoids and principal fibre bundles. For every $c \in B$, $\Phi_c = \{\theta \in \Phi, a(\theta) = c\}$ is a principal fibre bundle over B with structure group G_c and projection b; conversely, if $P(B, G)$ is a principal fibre bundle, then the groupoid PP^{-1} associated with P is a Lie groupoid over B. Moreover, if Φ is a groupoid of operators on a fibred manifold (E, p, B), then E is a fibre bundle associated with Φ_c with standard fibre $E_c = p^{-1}(c)$; conversely, if an associated fibre bundle $E(B, F, G, P)$ is given, then PP^{-1} is a groupoid of operators on E.

Let $\bar{G}$ be a group and let $\varphi: G \to \bar{G}$ be a homomorphism, so that G acts on $\bar{G}$ on the left by $(g, \bar{g}) \mapsto \varphi(g)\bar{g}$, $g \in G$, $\bar{g} \in \bar{G}$. Construct the associated fibre bundle $\bar{P} = P \times \bar{G} / G$ and define a right action of $\overline{G}$ on $\overline{P}$ by $\{(u,\overline{g})\}\overline{g}' = \{(u,\overline{g}\overline{g}')\}$, $u \in P$, $\overline{g}$, $\overline{g}' \in \overline{G}$, then we get a principal fibre bundle $\overline{P}(B,\overline{G})$ which will be called the φ - image of P. Further, φ is extended to a mapping $\varphi_1 : P \to \overline{P}$ given by $\varphi_1(u) = \{(u,e)\}$ and $(\varphi_1,\varphi) : P(B,G) \to \overline{P}(B,\overline{G})$ is a homomorphism of principal fibre bundles. Conversely, let $P(B,G)$ and $\overline{P}(B,\overline{G})$ be two principal fibre bundles and let $(\varphi_1,\varphi) : P(B,G) \to \overline{P}(B,\overline{G})$ be a base-preserving homomorphism, then every $\overline{u} \in \overline{P}_x$ is of the form $\overline{u} = \varphi_1(u)\overline{g}$, $u \in P_x$, $\overline{g} \in \overline{G}$, and it holds $\overline{u} = \varphi_1(ug^{-1})\varphi(g)\overline{g}$, $g \in G$, so that $\overline{P}$ coincides with the φ-image of P. Moreover, let Φ be a Lie groupoid over B, let $c \in B$, let $\overline{G}$ be a group and let $\varphi : G_c \to \overline{G}$ be a homomorphism. Construct the φ -image $\overline{P}_c$ of Φ_c and denote by $\tilde{\varphi} : \overline{P}_c \to \Phi_c$ the extension of φ, then the groupoid $\overline{\Phi} = \overline{P}_c\,\overline{P}_c^{-1}$ will be said the φ -image of Φ. The mapping $\varphi_1 : \Phi \to \overline{\Phi}$, $\varphi_1(u'u^{-1}) = \tilde{\varphi}(u')\tilde{\varphi}^{-1}(u)$ is a functor which will be called the extension of φ. Conversely, let Φ, $\overline{\Phi}$ be two Lie groupoids over the same base, let $\varphi_1 : \Phi \to \overline{\Phi}$ be a base-preserving functor and denote $\varphi = \varphi_1 | G_c$, then $\overline{\Phi}$ coincides with the φ -image of Φ.

The concept of a geometric object field is used in two equivalent forms which will be called the indirect or the direct form respectively. Let $P(B, G)$ be a principal fibre bundle and let G act on the left on F. A geometric object field in indirect form is a mapping $\gamma : P \to F$ satisfying $\gamma(ug^{-1}) = g\gamma(u)$ for every $u \in P$, $g \in G$; while a geometric object field in direct form is a cross-section of $E(B, F, G, P)$. Let $\bar{F}$ be a $\bar{G}$-space, then a pair of mappings $(\varphi, \varphi_o) : (G, F) \to (\bar{G}, \bar{F})$ is called a covariant mapping, if $\varphi : G \to \bar{G}$ is a homomorphism and $\varphi_o : F \to \bar{F}$ satisfies $\varphi_o(gs) = \varphi(g)\varphi_o(s)$ for every $g \in G$, $s \in F$. Further, let $\bar{P}(B, \bar{G})$ be the φ-image of P and let φ_1 be the extension of φ to P, then (φ, φ_o) is extended to a mapping $\varphi_2 : E(B, F, G, P) \to \bar{E}(B, \bar{F}, \bar{G}, \bar{P})$ given by $\varphi_2(\{(u, s)\}) = \{(\varphi_1(u), \varphi_o(s))\}$. The mapping φ_2 carries every cross section of E into a cross section of $\bar{E}$. In indirect form, let $\gamma : P \to F$ be a geometric object field, then its image $\bar{\gamma} : \bar{P} \to \bar{F}$ is given by $\bar{\gamma}(\{(u, \bar{g})\}) = \bar{g}^{-1}\varphi_o(\gamma(u))$. Dealing with a Lie groupoid Φ operating on a fibred manifold (E, p, B), we define a geometric object field as a cross section of E. Let $\bar{\Phi}$ be another groupoid of operators on a fibred manifold $(\bar{E}, \bar{p}, B)$ over the same base, then a pair of mappings $(\varphi_1, \varphi_2) : (\Phi, E) \to (\bar{\Phi}, \bar{E})$ is called a covariant mapping, if $\varphi_1 : \Phi \to \bar{\Phi}$ is a base-preserving functor and $\varphi_2 : E \to \bar{E}$ is a morphism satisfying $\varphi_2(\theta \cdot x) = \varphi_1(\theta) \cdot \varphi_2(x)$ for all composable $\theta \in \Phi$, $x \in E$. Naturally, φ_2 carries every cross section of E into a cross section of $\overline{E}$. Now, suppose (Φ, E) is given; let $\overline{F}$ be a $\overline{G}$-space and let $(\varphi, \varphi_0): (G_c, E_c) \to \to (\overline{G}, \overline{F})$ be a covariant mapping. Construct the φ-image $\overline{\Phi}$ of Φ as well as the associated fibre bundle $\overline{E}(B, \overline{F}, \overline{G}, \overline{\Phi}_c)$, then (φ, φ_0) is extended to a covariant mapping $(\varphi_1, \varphi_2): (\Phi, E) \to (\overline{\Phi}, \overline{E})$. Thus, the construction of covariant mappings of a space with a Lie groupoid of operators is reduced to the corresponding problem for a G-space.

2. Let Γ be a transitive Lie pseudogroup on a manifold M, $\dim M = n$. Let $\Pi^{k}(\Gamma)$ denote the groupoid of all k-jets of the transformations of Γ, see [1], and let $K^{k}_{m}(M)$ be the space of all regular contact m^{k}-elements on M, $m < n$, [3], then $\Pi^{k}(\Gamma)$ is a Lie groupoid of operators on $K^{k}_{m}(M)$. Let Φ be a Lie groupoid of operators on a fibred manifold (E, p, M) over M.

<u>Definition</u>. A geometric m^{k}-object (ψ, σ) on M with values in E is a covariant mapping of $(\Pi^{k}(\Gamma), K^{k}_{m}(M))$ into (Φ, E). More generally, let W be an invariant subspace of $K^{k}_{m}(M)$, then a geometric m^{k}-object on M of type W with values in E is a covariant mapping of $(\Pi^{k}(\Gamma), W)$ into (Φ, E).

Let V be an m-dimensional submanifold of M, then V determines at every point $x \in V$ a contact m^r-element $k_x^r V$, [3]. The mapping $x \mapsto k_x^r V$ is a cross section of $K_m^r(M) | V$ which can be said the fundamental field of order r on V, cf.[4]. If (ψ, σ) is a geometric m^r-object on M, then $x \mapsto \sigma(k_x^r V)$ is a section of $E | V$ which will be called the field of (ψ, σ) on V. (In classical terminology it is said that $x \mapsto \sigma(k_x^r V)$ is a concomitant of the fundamental field of order r on V.) Thus, we may also say that (ψ, σ) is a geometric object of order r for m-dimensional submanifolds of M. More generally, a submanifold V can be said of type W at $x \in V$, if $k_x^r V \in W$, so that a geometric m^r-object of type W is a geometric object of order r for m-dimensional submanifolds of type W.

<u>Example.</u> In particular, an invariant of order r for m-dimensional submanifolds of M is a covariant mapping of $(\Pi^r(\Gamma), K_m^r(M))$ into $(Id, M \times \mathbb{R})$, where Id means the groupoid of identities on $M \times \mathbb{R}$. These invariants were studied from another point of view by Vanžura [6].

If we take an r-frame h on M, $\beta h = c$, then $H^r(\Gamma, h) = \{\theta h,\ \theta \in \Pi_c^r(\Gamma)\}$ is a reduction of $H^r(M)$ to a subgroup G of L_m^r. Then $K_m^r(M)$ can be considered as the associated fibre bundle with $H^r(\Gamma, h)$ with standard fibre $K_{n,m}^r$ = the space of all regular contact m^r-elements on $\mathbb{R}^n$ at 0, i.e. $K_m^r(M)$ has the symbol $(M, K_{n,m}^r$,

$G, H^k(\Gamma, h))$. To construct geometric m^k-objects on M, one have to construct covariant mappings of $(G, K^k_{n,m})$. For this purpose, the algorithms of G.F. Laptěv can be applied. But we cannot explain it in details here, since it requires many auxiliary considerations.

3. We conclude with some remarks to the case if Γ is a localization of a Lie group G. Suppose G acts effectively on M, then the groupoid $\Pi^\lambda(\Gamma)$ of all germs of the transformations of Γ coincides with $G \times M$ provided that the projections a, b are defined by $a(g, x) = x$, $b(g, x) = gx$, $g \in G$, $x \in M$, and the multiplication is given by $(g', x')(g, x) = (g'g, x)$. We have canonical functor $j^k : \Pi^\lambda(\Gamma) \to \Pi^k(\Gamma)$. If a geometric m^k-object (ψ, σ) on M is given, then $(\psi j^k, \sigma)$ is a covariant mapping of $(\Pi^\lambda(\Gamma), K^k_m(M))$. Let $c \in M$ and let H be the isotropic group of c, then $\Pi^\lambda_c(\Gamma)$ coincides with the principal fibre bundle $G/H = G/H(M, H)$ and every geometric object field in indirect form on a submanifold V can also be considered as a mapping of the restriction of G/H to V which is nothing but the space of all frames associated with V in the classical terminology. - These remarks also give a comparison with our direct approach to homogeneous spaces in [3].

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