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THE MINIMUM EXISTENCE OF FUNCTIONAL $\int_a^b F(x(t), y(t), \dot{x}(t), \dot{y}(t))\,dt$

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Introduction. The problem concerning the existence of the minimum of the functional in the parametric form is solved in the Achiezer's book: Lekcii po variacionnom izčisleniju, Moskva 1955, on the bases of Russel's lemma and Tonelli's theorem. A different method is used in this paper. First of all (§ 1) the generalized theorem on the existence of the minimum of the functional in the nonparametric form is proved. By means of this theorem the theorem on the existence of the minimum of the functional in the parametric form (Theorem 2.3) can be easily proved, in this theorem it is not even necessary to assume anything about the second derivatives of the function F. Another advantage of this method is the possibility of defining the function spaces, in which the minimum is searched, more generally, and at the same time to prove the equality of the functional values of the couples of the functions that express "the same curve". The theorem proved in the Achiezer's book (Theorem 2.4) is a consequence of Theorem 2.3.

§ 1. The generalized theorem on the minimum of the functional in a nonparametric form

In this paragraph we shall discuss the existence of the minimum of the functional

$$\Phi(Y) = \int_a^b F(x, Y(x), Y'(x))\,dx$$

where $F(x, Y, Z)$ is defined on $\langle a, b\rangle \times E_{2N}$ and $Y(x) = [Y_1(x), \dots, Y_N(x)]$ is the vector-function in $\langle a, b \rangle$ whose range is in E_N .

Let $C^{(0)}(\langle a, b\rangle)$ and $W_p^{(1)}(\langle a, b\rangle)$ be the usual spaces of the functions, i.e.

let $C^{(0)}(\langle a, b\rangle)$ be the space of the continuous functions on $\langle a, b\rangle$ and

let $W_p^{(1)}(\langle a, b\rangle)$ be the space of the absolutely continuous functions $y(x)$ on $\langle a, b\rangle$ whose derivatives $y'(x)$ are in $L_p(\langle a, b\rangle)$. If B is a Banach space, we denote

$$Y_n \xrightarrow{B} Y_0 \ ; \ Y_n, Y_0 \in B \ \text{ if } \ \| Y_n - Y_0 \|_B \longrightarrow 0$$

and

$$Y_n \xrightarrow{B} Y_0 \ ; \quad Y_n, Y_0 \in B \quad \text{if} \quad (Y_n, V) \longrightarrow$$
$$\longrightarrow (Y_0, V) \qquad \text{for all} \quad V \in B^*$$

Definition 1.1. We denote by $[W_p^{(1)}(\langle a, b\rangle)]^N$ the space of the vector-functions $Y(x) = [Y_1(x), \dots$ $\dots, Y_N(x)]$ where $Y_i(x) \in W_p^{(1)}(\langle a, b\rangle)$; $i = 1, \dots, N$ with the norm

$$\| Y \|_{[W_p^{(1)}]^N} = \sum_{i=1}^{N} \| Y_i \|_{W_p^{(1)}} \ .$$

Similarly we denote $[C^{(0)}(\langle a, b\rangle)]^N$ the space of the vector-functions $Y(x) = [Y_1(x), \dots, Y_N(x)]$ where $Y_i(x) \in C^{(0)}(\langle a, b\rangle)$; $i = 1, \dots, N$ with the norm

$$\|Y\|_{[C^{(0)}]^N} = \sum_{i=1}^{N} \|Y_i\|_{C^{(0)}} .$$

Theorem 1.1. Let $1 < p < \infty$; $D_1, \dots, D_s \in E_N$. Let $F(x, Y, Z) \in C^{(0)}(\langle a, b\rangle \times E_{2N})$, let $F_{Z_i}(x, Y, Z) \in C^{(0)}(\langle a, b\rangle \times E_N \times G)$; $i = 1, \dots, N$, where

$$G = \{ Z \in E_N ;\ Z \neq D_j ,\ j = 1, \dots, s \} .$$

Suppose that

(1.1) $\quad |F(x, Y, Z)| \le c_0(|Y|_{E_N})(1 + \sum_{i=1}^{N} |Z_i|^p)$

for all $[x, Y, Z] \in \langle a, b\rangle \times E_{2N}$ and

(1.2) $\quad |F_{Z_i}(x, Y, Z)| \le c_i(|Y|_{E_N})(1 + \sum_{i=1}^{N} |Z_i|^{p-1})$

for all $[x, Y, Z] \in \langle a, b\rangle \times E_N \times G$; $i = 1, \dots, N$ where $c_i(t)$, $i = 0, 1, \dots, N$ are continuous, nonnegative and nondecreasing functions for $t \in \langle 0, \infty)$.

Let for $Z^1, Z^2 \in G$; $x \in \langle a, b\rangle$; $Y \in E_N$

(1.3) $\quad \sum_{i=1}^{N} [F_{Z_i}(x, Y, Z^2) - F_{Z_i}(x, Y, Z^1)](Z_i^2 - Z_i^1) \ge 0 .$

Let $Y_n \in [W_p^{(1)}(\langle a, b\rangle)]^N$; $n = 0, 1, \dots \dots$ and

$$Y_n \xrightarrow{[W_p^{(1)}]^N} Y_0 ;\ Y_n \xrightarrow{[C^{(0)}]^N} Y_0 .$$

Then the integrals

$$\Phi(Y_n) = \int_a^b F(x, Y_n(x), Y_n'(x))\,dx ;\quad n = 0, 1, \dots \dots$$

exist and

$$\lim_{m \to \infty} \Phi(Y_m) \geq \Phi(Y_o) .$$

Proof. 1) Consider $\delta > 0$. For all $m = 0, 1, \dots$ we define

$$Y_m'(x) = 0$$

for those $x \in \langle a, b \rangle$ for which $Y'(x)$ has not been defined. Since $Y_m \xrightarrow{[C^{(0)}]^N} Y_o$, there exists $K_o > 0$ such that

(1.4) $|Y_m(x)|_{E_N} < K_o$ for all $x \in \langle a, b \rangle ; m \in \mathcal{N}$

(1.5) $|D_1|_{E_N} < K_o, \dots, |D_s|_{E_N} < K_o$.

The function $F(x, Y, Z)$ is continuous in $\langle a, b \rangle \times E_{2N}$, hence $F(x, Y, Z)$ is uniformly continuous in $\langle a, b \rangle \times \mathcal{K}$, where $\mathcal{K}$ is a compact subset in E_{2N} , i.e.

(1.6) for every $\varepsilon_1 > 0$ there exists $\varepsilon_2 > 0$ such that for any $[x, Y, Z^j] \in \langle a, b \rangle \times \mathcal{K}$; $j = 1, 2$ we have

$$|Z^1 - Z^2|_{E_N} < \varepsilon_2 \Longrightarrow |F(x, Y, Z^1) - F(x, Y, Z^2)| < \varepsilon_1 .$$

Hence, there exists a vector-function $E(x) \in [L_p]^N$ such that

(1.7) $\overline{Y}_o(x) = Y_o'(x) + E(x) \neq D_m ; m = 1, \dots, s ; x \in \langle a, b \rangle$

(1.8) $\left| \int_a^b [F(x, Y_m, \overline{Y}_o) - F(x, Y_m, Y_o')] dx \right| < \delta$; $m \in \mathcal{N}$

(1.9) $E(x) = 0$; $x \in \langle a, b \rangle$, $|Y_0'(x)|_{E_N} > K_0$

(1.10) $|E(x)|_{E_N} < \delta$; $x \in \langle a, b \rangle$

since for sufficiently small $E(x)$ (1.6) implies (1.8) and for $x \in \langle a, b \rangle$; $Y_0'(x) = D_m$ the function $E(x)$ may be taken such that $\overline{Y}_0(x) \neq D_m$.

Next, we consider

$$\Phi(Y_n) - \Phi(Y_0) = A_n + B_n + C_n \quad ,$$

where

$$A_n = \int_a^b [F(x, Y_n, Y_n') - F(x, Y_n, \overline{Y}_0)] \, dx \quad ,$$

$$B_n = \int_a^b [F(x, Y_n, \overline{Y}_0) - F(x, Y_n, Y_0')] \, dx \quad ,$$

$$C_n = \int_a^b [F(x, Y_n, Y_0') - F(x, Y_0, Y_0')] \, dx \quad .$$

The integrals are convergent by (1.1).

For all $x \in \langle a, b \rangle$ we have

$$\lim_{n \to \infty} F(x, Y_n, Y_0') = F(x, Y_0, Y_0')$$

and so

$$\lim_{n \to \infty} C_n = 0 \ .$$

By (1.8) we see that $B_n \geq -\delta$ for all $n \in \mathcal{N}$.

2) For all $n \in \mathcal{N}$, $x_0 \in \langle a, b \rangle$ we can consider the function

$$\varphi(t) = F(x_0, Y_n(x_0), \overline{Y}_0(x_0) + t[Y_n'(x_0) - \overline{Y}_0(x_0)]) \ .$$

The function $\varphi(t)$ is defined and continuous in $\langle 0, 1 \rangle$, $\varphi'(t)$ exists for all but a finite set of $t \in \langle 0, 1 \rangle$, $\varphi'(t)$ is integrable in $\langle 0, 1 \rangle$ by (1.2), hence

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\,dt .$$

Thus we have

$$A_n = \int_a^b [\int_0^1 \sum_{i=1}^N F_{zi}(x, Y_n, \overline{Y}_0 + t[Y_n' - \overline{Y}_0']) \cdot [Y_{n,i}' - \overline{Y}_{0,i}']\,dt]\,dx$$

and $A_n = \alpha_n + \beta_n + \gamma_n$, where

$$\alpha_n = \int_a^b [\int_0^1 \sum_{i=1}^N [F_{zi}(x, Y_n, \overline{Y}_0' + t[Y_n' - \overline{Y}_0']) -$$

$$- F_{zi}(x, Y_n, \overline{Y}_0')][Y_{n,i}' - \overline{Y}_{0,i}']\,dt]\,dx ,$$

$$\beta_n = \int_a^b \sum_{i=1}^N F_{zi}(x, Y_n, \overline{Y}_0') - [Y_{n,i}' - Y_{0,i}']\,dx ,$$

$$\gamma_n = \int_a^b \sum_{i=1}^N F_{zi}(x, Y_n, \overline{Y}_0')\,[Y_{0,i}' - \overline{Y}_{0,i}']\,dx .$$

3) By (1.3) we have $\alpha_n \geq 0 , \quad n \in \mathcal{N}$.

We easily see that

$$F_{zi}(x, Y_n, \overline{Y}_0') \xrightarrow{L_q} F_{zi}(x, Y_0, \overline{Y}_0') ; \quad q = \frac{p}{p-1}$$

and $Y_{n,i}' \longrightarrow Y_{0,i}'$ weakly in L_p , thus $\beta_n \longrightarrow 0$.

For γ_n we obtain

$$|\gamma_n| \leq \int_a^b \sum_{i=1}^N |F_{zi}(x, Y_n, \overline{Y}_0')| \cdot |E_i|\,dx \leq$$

$$\leq \max_{i=1,\dots,N} c_i(\|Y_n\|_{C(a)}) \cdot \int_a^b (1 + |\overline{Y}_0'|_{E_N}^{p-1}) \cdot \sum_{i=1}^N |E_i|\,dx$$

but for $x \in \langle a, b\rangle$, $|\overline{Y}_0'(x)|_{E_N} > K_0 + \delta$ we have by (1.9) that $E(x) = 0$, hence

$$\gamma_n \geq - \max_{i=1,\dots,N} c_i(K_0)\,(1 + K_0 + \delta) \cdot \delta \cdot N(b - a) .$$

Thus

$$\varliminf_{n\to\infty} [\Phi(Y_n) - \Phi(Y_0)] \geq - \delta - \max_{i=1,\dots,N} c_i(K_0)(1+K_0+\delta)\cdot\delta\cdot N(b-a)$$

but $\delta > 0$ is arbitrary.

<u>Theorem 1.2</u>. Let $1 < p < \infty$; $A, B \in E_N$; $D_1, \dots, D_k \in E_m$. Let $F(x, Y, Z) \in C^{(0)}(\langle a, b\rangle \times E_{2N})$ and $F_{zi}(x, Y, Z) \in C^{(0)}(\langle a, b\rangle \times E_N \times G)$; $i = 1, \dots, N$,

where

$$G = \{ Z \in E_N ; Z \neq D_j , j = 1, \dots, s \} .$$

Suppose that

(1.1) $$| F(x, Y, Z) | \leq c_0 (|Y|_{E_N}) (1 + \sum_{i=1}^{N} |Z_i|^p)$$

for $[x, Y, Z] \in \langle a, b \rangle \times E_{2N}$ and

(1.2) $$| F_{Z_i} (x, Y, Z) | \leq c_i (|Y|_{E_N}) (1 + \sum_{i=1}^{N} |Z_i|^{p-1})$$

for $[x, Y, Z] \in \langle a, b \rangle \times E_N \times G ; i = 1, \dots, N$, where $c_i(t)$, $i = 0, 1, \dots, N$ are continuous, nonnegative and nondecreasing functions for $t \in \langle 0, \infty)$.

Let for $Z^1, Z^2 \in G$; $x \in \langle a, b \rangle$; $Y \in E_N$

(1.3) $$\sum_{i=1}^{N} [F_{Z_i} (x, Y, Z^2) - F_{Z_i} (x, Y, Z^1)] (Z_i^2 - Z_i^1) \geq 0 .$$

Let there exist $\gamma_1 > 0$; $\gamma_2 \in E_1$ such that

(1.11) $$F(x, Y, Z) \geq \gamma_1 \cdot \sum_{i=1}^{N} |Z_i|^p + \gamma_2$$

for all $[x, Y, Z] \in \langle a, b \rangle \times E_{2N}$.

Then the functional

$$\Phi(Y) = \int_a^b F(x, Y(x), Y'(x)) \, dx$$

is defined in

$$M = \{ Y \in [W_p^{(1)}]^N ; Y(a) = A , Y(b) = B \}$$

and there exists an absolute minimum of Φ in M .

<u>Proof</u>. By (1.11) we have

$$\Phi(Y) \geq \gamma_2 (b - a)$$

for all $Y \in M$, thus there exists

$$\inf_{Y \in M} \Phi(Y) \in E_1 .$$

Next, by (1.11), there exists $K_2 > 0$ such that

$$\Phi(Y) > \Phi(0)$$

for all $Y \in M$, $\|Y\|_{[W_p^{(1)}]^N} > K_2$, hence

$$\inf_{Y \in M} \Phi(Y) = \inf_{Y \in M \cap K} \Phi(Y) = d \;, \quad .$$

where

$$K = \{ Y \in [W_p^{(1)}]^N ,\; \|Y\|_{[W_p^{(1)}]^N} \leq K_2 \} .$$

There exists a sequence

$$\{Y_n\}_{n=1}^{\infty} \;, \; Y_n \in M \cap K$$

such that $\lim_{n \to \infty} \Phi(Y_n) = d$ and at the same time

$$Y_n \xrightarrow{[C^{(0)}]^N} Y_0 \;, \quad Y_n \xrightarrow{[W_p^{(1)}]^N} Y_0 \; .$$

By Theorem 1.1 we have

$$d = \lim \Phi(Y_n) \geq \Phi(Y_0) \geq d \; .$$

§ 2. Theorems on the minimum of the functional in a parametric form

Definition 2.1. Let $(A_1, A_2), (B_1, B_2) \in E_2$; $(A_1, A_2) \neq (B_1, B_2)$. The set of pairs of functions $[x(t), y(t); \langle \alpha, \beta \rangle]$ which are defined and absolutely continuous in $\langle \alpha, \beta \rangle$ and for which we have

$$x(\alpha) = A_1 \;, \qquad x(\beta) = B_1 \;,$$
$$y(\alpha) = A_2 \;, \qquad y(\beta) = B_2 \;,$$

will be denoted by S .

Definition 2.2. For all pairs $[x, y; \langle \alpha, \beta \rangle] \in S$ we define the natural parametrisation $[\tilde{x}(s), \tilde{y}(s); \langle 0, l \rangle]$ in this way:

the function

$$s(t) = \int_{\alpha}^{t} \sqrt{\mathring{x}^2(\tau) + \mathring{y}^2(\tau)}\, d\tau$$

is absolutely continuous, nondecreasing in $\langle \alpha, \beta \rangle$, it maps $\langle \alpha, \beta \rangle$ onto $\langle 0, \ell \rangle$, where

$$\ell = \int_{\alpha}^{\beta} \sqrt{\mathring{x}^2(\tau) + \mathring{y}^2(\tau)}\, d\tau \quad .$$

Then the function $\varphi(s_0) = \min\{t \in \langle \alpha, \beta \rangle ; s(t) = s_0\}$ is defined on $\langle 0, \ell \rangle$ and we can define

$$\tilde{x}(s) = x(\varphi(s)), \quad \tilde{y}(s) = y(\varphi(s))$$

for all $s \in \langle 0, \ell \rangle$.

<u>Theorem 2.1</u>. Let the function $\varphi(u)$ be absolutely continuous and monotonous in $\langle \alpha, \beta \rangle$; $\alpha, \beta \in E_1$; $\alpha < \beta$; let the function $f(x)$ be defined in $\langle \varphi(\alpha), \varphi(\beta) \rangle$ (or $\langle \varphi(\beta), \varphi(\alpha) \rangle$), let the function $f(x)$ be Lebesgue-measurable in $\langle \varphi(\alpha), \varphi(\beta) \rangle$. Then we have

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx = \int_{\alpha}^{\beta} f(\varphi(u)) \cdot \varphi'(u)\, du$$

iff at least one of the integrals exists.

<u>Proof</u>. [1], p.434.

<u>Lemma 2.1.</u> The functions $\tilde{x}(s)$, $\tilde{y}(s)$ which are defined in Definition 2.2, are absolutely continuous on $\langle 0, \ell \rangle$ moreover, $[\tilde{x}, \tilde{y}; \langle 0, \ell \rangle]$ is the natural parametrisation of a pair $[\tilde{x}, \tilde{y}; \langle 0, \ell \rangle]$ and

$$\mathring{\tilde{x}}^2(s) + \mathring{\tilde{y}}^2(s) = 1 \quad ,$$

a.e. on $\langle 0, \ell \rangle$.

__Proof.__ 1) We denote on $\langle\alpha, \beta\rangle$:

$\Phi_1(t) = \frac{\dot{x}(t)}{\dot{s}(t)}$; $\Phi_2(t) = \frac{\dot{y}(t)}{\dot{s}(t)}$, if there exist $\dot{y}(t)$, $\dot{x}(t)$, $\dot{s}(t)$ and if $\dot{s}(t) > 0$;

$\Phi_1(t) = \dot{x}(t)$; $\Phi_2(t) = \dot{y}(t)$, if there exist $\dot{x}(t)$, $\dot{y}(t)$, $\dot{s}(t)$ and if $\dot{s}(t) = 0$;

$\Phi_1(t) = \Phi_2(t) = 0$ in the other cases.

We also denote

$P = \{t \in \langle\alpha, \beta\rangle$; there exists $\dot{x}(t)$, $\dot{y}(t)$, $\dot{s}(t)$ and $\dot{s}(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\}$.

For $t_0 \in P$ we have

a) if $\dot{s}(t_0) > 0$, then $\varphi(s(t_0)) = t_0$ and

$$\Phi_1(\varphi(s(t_0))) \cdot \dot{s}(t_0) = \frac{\dot{x}(t_0)}{\dot{s}(t_0)} \cdot \dot{s}(t_0) = \dot{x}(t_0) \;;$$

b) if $\dot{s}(t_0)$, then

$$\Phi_1(\varphi(s(t_0))) \cdot \dot{s}(t_0) = \dot{x}(t_0) = 0 \;.$$

Hence

$$\Phi_1(\varphi(s(t))) \cdot \dot{s}(t) = \dot{x}(t)$$

a.e. on $\langle\alpha, \beta\rangle$.

By Theorem 2.1 the integrals are convergent and we have

$$\int_0^{s} \Phi_1(\varphi(s))\,ds = \int_\alpha^{\varphi(s_0)} \Phi_1(\varphi(s(t))) \cdot \dot{s}(t)\,dt = \int_\alpha^{\varphi(s_0)} \dot{x}(t)\,dt =$$
$$= x(\varphi(s_0)) - x(\alpha) = \tilde{x}(s_0) - \tilde{x}(0)$$

for all $s_0 \in \langle 0, \ell\rangle$.

Hence the functions $\tilde{x}(s)$, $\tilde{y}(s)$ are absolutely continuous on $\langle 0, \ell \rangle$ and

$$\dot{\tilde{x}}(s) = \Phi_1(\varphi(s)) \; ; \quad \dot{\tilde{y}}(s) = \Phi_2(\varphi(s))$$

a.e. on $\langle 0, \ell \rangle$.

2) We want to prove that

$$\int_0^{s_0} \sqrt{\dot{\tilde{x}}^2(s) + \dot{\tilde{y}}^2(s)}\, ds = s_0$$

For $t_0 \in P$:

a) if $\dot{s}(t_0) > 0$, then $\varphi(s(t_0)) = t_0$ and

$$\sqrt{[\Phi_1(\varphi(s(t_0)))]^2 + [\Phi_2(\varphi(s(t_0)))]^2} \cdot \dot{s}(t_0) =$$

$$= \frac{1}{\dot{s}(t_0)} \sqrt{\dot{x}^2(t_0) + \dot{y}^2(t_0)} \cdot \dot{s}(t_0) = \dot{s}(t_0) \; ;$$

b) if $\dot{s}(t_0) = 0$, then

$$\sqrt{[\Phi_1(\varphi(s(t_0)))]^2 + [\Phi_2(\varphi(s(t_0)))]^2} \cdot \dot{s}(t_0) = \dot{s}(t_0) = 0 .$$

Hence by Theorem 2.1 we have

$$\int_0^{s_0} \sqrt{\dot{\tilde{x}}^2(s) + \dot{\tilde{y}}^2(s)}\, ds = \int_0^{s_0} \sqrt{[\Phi_1(\varphi(s))]^2 + [\Phi_2(\varphi(s))]^2}\, ds =$$

$$= \int_\alpha^{\varphi(s_0)} \sqrt{[\Phi_1(\varphi(s(t)))]^2 + [\Phi_2(\varphi(s(t)))]^2} \cdot \dot{s}(t)\, dt = \int_\alpha^{\varphi(s_0)} \dot{s}(t) = s_0$$

and also

$$\sqrt{\dot{\tilde{x}}^2(s) + \dot{\tilde{y}}^2(s)} = 1$$

a.e. on $\langle 0, \ell \rangle$.

<u>Definition 2.3</u>. A pair $[x_1, y_1 ; \langle a_1, b_1 \rangle] \in S$ is equivalent with a pair $[x_2, y_2 ; \langle a_2, b_2 \rangle] \in S$ if

$$\int_{a_1}^{b_1} \dot{x}_1^2(t) + \dot{y}_1^2(t)\, dt = \int_{a_2}^{b_2} \dot{x}_2^2(t) + \dot{y}_2^2(t)\, dt = \ell$$

and if for natural parametrisations we have

$$\tilde{x}_1(s) = \tilde{x}_2(s) \; ; \qquad \tilde{y}_1(s) = \tilde{y}_2(s)$$

for all $s \in \langle 0, \ell \rangle$.

<u>Definition 2.4.</u> By the concept of a curve we understand the class of equivalence in S .
We denote by ℓ the length of the curve.
We denote by T the space consisting of all the curves.

<u>Definition 2.5.</u> Let $F(x, y, u, v) \in C^{(0)}(E_4)$; let $F(x, y, u, v)$ be positive homogenous in $[u, v]$, i.e.

$$F(x, y, ku, kv) = k \cdot F(x, y, u, v)$$

for all $k > 0$; $x, y, u, v \in E_1$.

We define the functional Φ on T by

$$\Phi(C) = \int_{\alpha}^{\beta} F(x(t), y(t), \dot{x}(t), \dot{y}(t))\, dt$$

where $[x, y; \langle \alpha, \beta \rangle]$ is any pair of the class C .

<u>Theorem 2.2.</u> Let $F(x, y, u, v) \in C^{(0)}(E_4)$, let $F(x, y, u, v)$ be positive homogeneous in $[u, v]$, let $[\tilde{x}, \tilde{y}; \langle 0, \ell \rangle]$ be the natural parametrisation of $[x, y; \langle \alpha, \beta \rangle] \in S$.
Then there exist integrals

$$\int_{\alpha}^{\beta} F(x(t), y(t), \dot{x}(t), \dot{y}(t))\, dt \; ,$$

$$\int_0^{\ell} F(\tilde{x}(s), \tilde{y}(s), \dot{\tilde{x}}(s), \dot{\tilde{y}}(s))\, ds \; ,$$

values of which are equals.

<u>Proof.</u> 1) We denote

$$N_1 = \{ s \in \langle 0, \ell \rangle ;\ ex.\ \dot{\tilde{x}}(s), \dot{\tilde{y}}(s) \text{ and } \dot{\tilde{x}}^2(s) + \dot{\tilde{y}}^2(s) = 1 \} \; ,$$

$N_2 = \{t \in \langle \alpha, \beta \rangle;\ ex.\ \dot{s}(t), \dot{x}(t), \dot{y}(t)$ and $\dot{s}(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\}$.

We define

$$\Psi(s) = F(\tilde{x}(s), \tilde{y}(s), \dot{\tilde{x}}(s), \dot{\tilde{y}}(s)) \text{ if } s \in N_1,$$

$$\Psi(s) = 0 \qquad \text{if } s \in \langle 0, \ell \rangle - N_1.$$

There exists $K > 0$ such that $|\Psi(s)| \leq K$ for $s \in \langle 0, \ell \rangle$, hence we have by Theorem 2.1

$$\int_0^\ell F(\tilde{x}(s), \tilde{y}(s), \dot{\tilde{x}}(s), \dot{\tilde{y}}(s))\,ds = \int_0^\ell \Psi(s)\,ds = \int_\alpha^\beta \Psi(s(t)) \cdot \dot{s}(t)\,dt.$$

2) Let $t_0 \in N_2$:

a) if $\dot{s}(t_0) = 0$, then

$$F(x(t_0), y(t_0), \dot{x}(t_0), \dot{y}(t_0)) = \Psi(s(t)) \cdot \dot{s}(t_0) = 0;$$

b) if $\dot{s}(t_0) > 0$, then by Lemma 2.2 the function

$$\varphi(\tau) = \min\ \{t \in \langle \alpha, \beta \rangle;\ s(t) = \tau\}$$

is continuous in $t_0 = \varphi(s_0)$, hence there exists

$$\dot{\tilde{x}}(s_0) = \lim_{s_1 \to s_0} \frac{\tilde{x}(s_1) - \tilde{x}(s_0)}{s_1 - s_0} = \lim_{s_1 \to s_0} \frac{x(\varphi(s_1)) - x(\varphi(s_0))}{s(\varphi(s_1)) - s(\varphi(s_0))} =$$

$$= \lim_{s_1 \to s_0} \frac{x(\varphi(s_1)) - x(\varphi(s_0))}{\varphi(s_1) - \varphi(s_0)} \cdot \lim_{s_1 \to s_0} \frac{1}{\dfrac{s(\varphi(s_1)) - s(\varphi(s_0))}{\varphi(s_1) - \varphi(s_0)}} =$$

$$= \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0} \cdot \lim_{t_1 \to t_0} \frac{1}{\dfrac{s(t_1) - s(t_0)}{t_1 - t_0}} = \dot{x}(t_0) \cdot \frac{1}{\dot{s}(t_0)}.$$

Similarly there exists

$$\dot{\tilde{y}}(s_0) = \dot{y}(t_0) \cdot \frac{1}{\dot{s}(t_0)}$$

and then

$$\mathring{x}^2(s_0)+\mathring{y}^2(s_0)=\frac{\dot{x}^2(t_0)+\dot{y}^2(t_0)}{\dot{s}^2(t_0)} = 1 \quad ,$$

hence $s_0 \in N$ and we have

$$\Psi(s(t_0))\cdot \dot{s}(t_0) =$$

$$= F(x(t_0), y(t_0), \frac{\dot{x}(t_0)}{\dot{s}(t_0)}, \frac{\dot{y}(t_0)}{\dot{s}(t_0)})\cdot \dot{s}(t_0) =$$

$$= F(x(t_0), y(t_0), \dot{x}(t_0), \dot{y}(t_0)) \ .$$

Hence both integrals exist and we have

$$\int_\alpha^\beta \Psi(s(t))\cdot \dot{s}(t) = \int_\alpha^\beta F(x(t), y(t), \dot{x}(t), \dot{y}(t))\, dt \ .$$

<u>Lemma 2.2.</u> Let $[x, y; \langle\alpha,\beta\rangle] \in S$, $t_0 \in \langle\alpha,\beta\rangle$; $s(t)=\int_\alpha^t \sqrt{\dot{x}^2(\tau)+\dot{y}^2(\tau)}\,d\tau$, let there exist $\dot{s}(t_0)$, $\dot{x}(t_0)$, $\dot{y}(t_0)$ and $\dot{s}(t_0)=\sqrt{\dot{x}^2(t_0)+\dot{y}^2(t_0)}$, $\dot{s}(t_0) > 0$. Then the function $\varphi(s_0) = \min\{t \in \langle\alpha,\beta\rangle;\ s(t) = s_0\}$ is continuous in $s_0 = s(t_0)$.

<u>Proof.</u> There exist $\delta > 0$ such that for $t \in \langle t_0-\delta, t_0+\delta\rangle$ we have

$$(2.1) \qquad s(t) - s(t_0) \geq c\cdot(t-t_0) \ ,$$

where $c = 1/2\ \dot{s}(t_0) > 0$.

Since the function $s(t)$ is increasing in t_0 , we have $\varphi(s_0) = t_0$. If for any $s_1 \in (s_0, s_0 + c\delta)$ we have $\varphi(s_1) \geq t_0 + \delta$, then

$$s_1 = s(\varphi(s_1)) \geq s(t_0+\delta) \geq s_0 + c\delta$$

which is a contradiction.

Hence for $s_1 \in (s_0, s_0 + c\delta)$ we have

$$\varphi(s_1) \in (t_0, t_0 + \sigma) \ .$$

Similarly for $s_1 \in (s_0 - c\sigma, s_0)$ we have $\varphi(s_1) \in$ $\in (t_0 - \sigma, t_0)$ and for $s_1 \in (s_0 - c\sigma, s_0 + c\sigma)$ we have $|\varphi(s_1) - t_0| < \sigma$ and by (2.1)

$$\varphi(s_1) - \varphi(s_0) \leq \frac{\ell}{c}(s_1 - s_0) \ .$$

<u>Theorem 2.3</u>. Let $F(x, y, u, v) \in C^{(0)}(E_4)$,

$$F_u(x, y, u, v),\ F_v(x, y, u, v) \in C^{(0)}(G) \ ,$$

where

$$G = \{[x, y, u, v] \in E_4 ;\ u^2 + v^2 \neq 0\} \ .$$

Let $F(x, y, u, v)$ be positive homogeneous in $[u, v]$, let Φ be the functional defined in Definition 2.5, let $F(x, y, \cos\vartheta, \sin\vartheta) \geq \omega > 0$ for all $x, y, \vartheta \in E_1$.

Next, let the function

$$E(x, y, u, v, \alpha, \beta) = F(x, y, u, v) - u \cdot F_u(x, y, \alpha, \beta) - v \cdot F_v(x, y, \alpha, \beta)$$

be nonnegative for all $x, y, u, v, \alpha, \beta \in E_1$; $\alpha^2 + \beta^2 \neq 0$; $u^2 + v^2 \neq 0$. Then there exists the minimum of the functional Φ on T.

<u>Proof</u>. For $C \in T$ we have $\Phi(C) \geq \omega \cdot \ell > 0$, hence there exists

$$\inf_{C \in T} \Phi(C) = d \in E_1 \ .$$

Let $C_1 \in T$, we denote

$$K_1 = \frac{\Phi(C_1)}{\omega} \ ,$$

$$U = \{C \in T ;\ \ell_C < K_1\} \ .$$

If $C \in T$, $\Phi(C) \leq \Phi(C_1)$, then

$$\omega \cdot \ell_C \leq \Phi(C) \leq \Phi(C_1) ,$$

hence $C \in \mathcal{U}$. This means that

$$\inf_{C \in \mathcal{U}} \Phi(C) = \inf_{C \in T} \Phi(C) = d .$$

There exists a sequence of the curves $C_n \in \mathcal{U}$, the lengths of which we denote ℓ_n so that

$$\lim_{n \to \infty} \Phi(C_n) = d .$$

By substitution $\tau = \frac{s}{\ell_n}$ in the natural parametrisations $[\tilde{x}_n, \tilde{y}_n ; \langle 0, \ell_n \rangle]$ we obtain

$$\bar{x}_n(\tau) = \tilde{x}_n(\tau \cdot \ell_n) ,$$
$$\bar{y}_n(\tau) = \tilde{y}_n(\tau \cdot \ell_n) .$$

Then $[\bar{x}_n, \bar{y}_n ; \langle 0, 1 \rangle] \in C_n$ and

$$\left| \frac{d\bar{x}_n}{d\tau} \right| \leq \ell_n \leq K_1 ,$$
$$\left| \frac{d\bar{y}_n}{d\tau} \right| \leq \ell_n \leq K_1 ,$$

a.e. on $\langle 0, 1 \rangle$.

Hence $\bar{x}_n, \bar{y}_n \in W_\infty^{(1)}(\langle 0, 1 \rangle)$ and moreover, by $\bar{x}_n(0) = A_1$; $\bar{x}_n(1) = B_1$; $\bar{y}_n(0) = A_2$; $\bar{y}_n(1) = B_2$ for all $n \in \mathcal{N}$, we have

$$\| \bar{x}_n \|_{W_2^{(1)}(\langle 0,1 \rangle)} \leq K_2 ,$$

$$\| \bar{y}_n \|_{W_2^{(1)}(\langle 0,1 \rangle)} \leq K_2 ,$$

where K_2 is the constant independent on n.
Then there exists a subsequence $\{n_k\}$ so that

$$\bar{x}_{n_k} \xrightarrow{W_2^{(1)}} x_0 \; ; \quad \bar{y}_{n_k} \xrightarrow{W_2^{(1)}} y_0$$

and at the same time

$$\overline{x}_{n_k} \xrightarrow{C^{(0)}} x_0 \ ; \quad \overline{y}_{n_k} \xrightarrow{C^{(0)}} y_0$$

where $x_0, y_0 \in W_2^{(1)}$.

The function $F(x, y, u, v)$ is positive homogeneous in $[u, v]$, so by differentiating by k we see that

$$u \cdot F_u(x, y, u, v) + v F_v(x, y, u, v) = F(x, y, u, v),$$

hence

$$u[F_u(x, y, u, v) - F_u(x, y, \alpha, \beta)] + v[F_v(x, y, u, v) - \\ - F_v(x, y, \alpha, \beta)] \geq 0$$

and

$$[F_u(x, y, u, v) - F_u(x, y, \alpha, \beta)](u - \alpha) + \\ + [F_v(x, y, u, v) - F_v(x, y, \alpha, \beta)](v - \beta) \geq 0$$

for all $x, y, u, v, \alpha, \beta \in E_1$; $u^2 + v^2 \neq 0$; $\alpha^2 + \beta^2 \neq 0$.

The function $F(x, y, u, v)$ is positive homogeneous in $[u, v]$, hence

$$F_u(x, y, ku, kv) = F_u(x, y, u, v); \ F_v(x, y, ku, kv) = F_v(x, y, u, v)$$

for all $k > 0$; $x, y, u, v \in E_1$; $u^2 + v^2 \neq 0$. By $F(x, y, u, v) \in C^{(0)}(E_4)$; $F_u(x, y, u, v), F_v(x, y, u, v) \in C^{(0)}(G)$ and by the homogeneity of the function F there exist the functions $c_0(t), c_1(t), c_2(t)$ continuous, nonnegative and nondecreasing in $\langle 0, \infty)$, so that

$$|F(x, y, u, v)| \leq c_0(|[x, y]|_{E_2})(1 + u^2 + v^2)$$

for all $x, y, u, v \in E_1$ and

$$|F_u(x, y, u, v)| \leq c_1(|[x, y]|_{E_2})(1 + |u| + |v|),$$

$$|F_v(x, y, u, v)| \leq c_2(|[x, y]|_{E_2})(1 + |u| + |v|)$$

for $x, y, u, v \in E_1$; $u^2 + v^2 \neq 0$.

By application of Theorem 2.1 we obtain

$d = \lim_{n \to \infty} \Phi(c_u) \geq \Phi(c_o)$,

hence

$$\Phi(C_o) = d \ .$$

Lemma 2.3. Let $F(x,y,u,v) \in C^{(0)}(E_4)$; $F(x,y,u,v) \in C^{(2)}(G)$; let the function $F(x,y,u,v)$ be positive homogeneous in $[u,v]$. Then

(2.2) $F_{uu}(x,y,u,v) = v^2 \cdot F_1(x,y,u,v)$;

(2.3) $F_{vv}(x,y,u,v) = u^2 \cdot F_1(x,y,u,v)$;

(2.4) $F_{uv}(x,y,u,v) = -uv \cdot F_1(x,y,u,v)$,

where

(2.5) $F_1(x,y,u,v) = \dfrac{F_{uu}(x,y,u,v) + F_{vv}(x,y,u,v)}{u^2 + v^2}$.

Proof. The differentiation

$u \cdot F_u(x,y,u,v) + v \cdot F_v(x,y,u,v) = F(x,y,u,v)$;

by u, v leads to

(2.6) $u \cdot F_{uu}(x,y,u,v) + v \cdot F_{vu}(x,y, \ \ v) = 0$;

(2.7) $v \cdot F_{vv}(x,y,u,v) + u \cdot F_{vu}(x,y,u,v) = 0$

which together with (2.5) yield (2,2),(2.3),(2.4).

Theorem 2.4. Let $F(x,y,u,v) \in C^{(0)}(E_4)$; $F(x,y,u,v) \in C^{(2)}(G)$, let $F(x,y,u,v)$ be positive homogeneous in $[u,v]$, let

$F(x, y, \cos\vartheta, \sin\vartheta) \geq \omega > 0$

for all $x, y, \vartheta \in E_1$. Let

$F_{uu}(x, y, \cos\vartheta, \sin\vartheta)$, $F_{vv}(x, y, \cos\vartheta, \sin\vartheta) \geq 0$

for all $x, y, \vartheta \in E_1$. Then there exists the minimum of the functional Φ on $\mathcal{T}$.

<u>Proof.</u> It is sufficient to prove that

$$E(x, y, u, v, \alpha, \beta) \geq 0$$

for all $x, y, u, v, \alpha, \beta \in E$; $u^2 + v^2 \neq 0$; $\alpha^2 + \beta^2 \neq 0$.

1) Let $x, y, u, v, \alpha, \beta \in E_1$; $u^2 + v^2 = \alpha^2 + \beta^2 = 1$.

There exist $\Theta, \varphi \in E_1$ so that $\varphi \in \langle \Theta - \pi, \Theta + \pi \rangle$. and

$$\cos\Theta = \alpha \ ; \quad \sin\Theta = \beta \ ;$$

$$\cos\varphi = u \ ; \quad \sin\varphi = v \ .$$

We have

$$\frac{d}{d\tau}[F_u(x, y, \cos\tau, \sin\tau)] = -\sin\tau \cdot F_{uu}(x, y, \cos\tau, \sin\tau) +$$

$$+ \cos\tau \cdot F_{uv}(x, y, \cos\tau, \sin\tau) =$$

$$= [-\sin\tau \cdot \sin^2\tau - \sin\tau \cdot \cos^2\tau] \cdot F_1(x, y, \cos\tau,$$

$$\sin\tau) = -\sin\tau \cdot F_1(x, y, \cos\tau, \sin\tau) \ .$$

Hence

$$F_u(x, y, \cos\varphi, \sin\varphi) - F_u(x, y, \cos\Theta, \sin\Theta) =$$

$$= -\int_\Theta^\varphi \sin\tau \cdot F_1(x, y, \cos\tau, \sin\tau)\, d\tau \ .$$

Similarly

$$F_v(x, y, \cos\varphi, \sin\varphi) - F_v(x, y, \cos\Theta, \sin\Theta) =$$

$$= \int_\Theta^\varphi \cos\tau \cdot F_1(x, y, \cos\tau, \sin\tau)\, d\tau \ .$$

Hence

$$E_1(x, y, u, v, \alpha, \beta) =$$

$$= \int_{\Theta}^{\varphi} [-\cos\varphi \sin\tau + \sin\varphi \cos\tau] \cdot F_1(x, y, \cos\tau, \sin\tau)\, d\tau =$$

$$= \int_{\Theta}^{\varphi} \sin(\varphi - \tau) \cdot F_1(x, y, \cos\tau, \sin\tau)\, d\tau \ .$$

Since $F_1(x, y, \cos\tau, \sin\tau) \geq 0$, we have:

a) if $\Theta \leq \varphi$, then for $\tau \in \langle \Theta, \varphi \rangle$ we have

$$\sin(\varphi - \tau) \geq 0$$

and $E_1(x, y, u, v, \alpha, \beta) \geq 0$.

b) if $\varphi \leq \Theta$, then for $\tau \in \langle \varphi, \Theta \rangle$ we have

$$\sin(\varphi - \tau) \leq 0$$

and

$$F(x, y, u, v, \alpha, \beta) =$$

$$= \int_{\varphi}^{\Theta} [-\sin(\varphi - \tau)] \cdot F_1(x, y, \cos\tau, \sin\tau)\, d\tau \geq 0 \ .$$

2) Let $x, y, u, v, \alpha, \beta \in E_1 ; u^2 + v^2 \neq 0 ; \alpha^2 + \beta^2 \neq 0$.
There exist $k, \ell \in (0, \infty)$ so that

$$k^2(u^2 + v^2) = \ell^2(\alpha^2 + \beta^2) = 1 \ .$$

Then

$$F_u(x, y, ku, kv) = F_u(x, y, u, v) \quad ,$$

$$F_u(x, y, \ell\alpha, \ell\beta) = F_u(x, y, \alpha, \beta) \quad ,$$

$$F_v(x, y, ku, kv) = F_v(x, y, u, v) \quad ,$$

$$F_v(x, y, \ell\alpha, \ell\beta) = F_v(x, y, \alpha, \beta) \quad ,$$

and

$$E(x, y, u, v, \alpha, \beta) =$$

$$= \frac{1}{k} \{ ku [F_u(x, y, ku, kv) - F_u(x, y, l\alpha, l\beta)] + \\ + kv [F_v(x, y, ku, kv) - F_v(x, y, l\alpha, l\beta)] \} \geq 0 .$$

R e f e r e n c e s

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