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0. Introduction. The notion of the measure of non-compactness was introduced by G. Darbo [7] and Sadovskii [13]. By means of this notion G. Darbo defined a k-set-contraction and Sadovskii a concentrative mapping. Darbo and Sadovskii proved for their classes of mappings the fixed point theorems. We observe that the Sadovskii's class of mappings is broader than the Darbo's one. But the sum of a completely continuous mapping and a k-contraction is the k-set-contraction of Darbo [7]. Hence the most important case is already covered by Darbo (implicitly). Let us note that the classes $C_3(k)$ $(0 \leqq k < 1)$ and C_4 of Frum-Ketkov [8] are near to that of Darbo.

Further development of concentrative mappings (resp. k-set-contractions) is contained in Badoev, Sadovskii [2], Borisovič, Sapronov [3], Daneš [4,5,6], and Nussbaum [12]. Index and rotation notions for this class of mappings are developed in [3, 12]. The notion of a generalized concentrative mapping was introduced by Lifšic - Sadovskii in [11], and in more general fashion in our report [6] [1].

The purpose of the present paper is to introduce α -generalized concentrative mappings in a topological space and to prove fixed point theorems for such mappings. Our point of departure is [5,Th.1] which hypotheses are very near to the definition of a generalized concentrative mapping.

In Section 1 we introduce notation and definitions. Some simple and well-known lemmas are given. Section 2 deals with α -generalized concentrative mappings. Fixed point theorems are contained in Section 3.

1. <u>Notations and definitions</u>. $\mathbb{R}$ and $\mathbb{C}$ denotes the field of real and complex numbers, resp. For X a set, we denote by $\exp X$ the set of all subsets of X and by 2^X the set of all non-empty subsets of X.

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1) (January 9,1970): Further results are contained in "Problemy matematičeskogo analiza složnych sistem", vyp. 2, 1968, Voronež which was sent to me by B.N. Sadovskii:

V.A. Bondarenko: On the existence of the universal measure of non-compactness, pp.18-21;

G.M. Vainikko, B.N. Sadovskii: On rotation of concentrative vector fields,pp.84-88;

B.N. Sadovskii: On measures of non-compactness and concentrative operators, pp.89-119.

If X is a topological space, then $cl\, M$ and $\overline{M}$ denote the closure of M in X. If X is a linear space (over $\mathbb{R}$ or $\mathbb{C}$), then $co\, M$, $con\, M$, $aff\, M$, $sp\, M$ denote the convex, cone, affine, linear hull of the subset M of X, resp. For X a topological linear space (over $\mathbb{R}$ or $\mathbb{C}$), the operations $\overline{co}$, $\overline{con}$, $\overline{aff}$, $\overline{sp}$ are defined by $\overline{co} = cl\, co$, $\overline{con} = cl\, con$, $\overline{aff} = cl\, aff$, $\overline{sp} = cl\, sp$.

Let (X, d) be a pseudometric space and M a subset of X. Let $B(M, \varepsilon)$ denote the closed ε-ball at the set M, i.e. $B(M, \varepsilon) = \{x \in X : \sup\{d(x, y) : y \in M\} \leq \varepsilon\}$. The measure of non-compactness of the set M in X is defined by

$$\chi(M) = \inf Q(M) \quad (\inf \emptyset = +\infty),$$

where

$Q(M) = \{\varepsilon \in \mathbb{R} : \varepsilon > 0$ and there is a finite ε-net for M in X, i.e. $B(\sigma, \varepsilon) \supset M$ for some finite subset σ of $X\}$.

If M, N are subsets of X and M is bounded and N non-empty, we define

$$\vartheta(M, N) = \inf\{\varepsilon \in \mathbb{R} : \varepsilon > 0, B(N, \varepsilon) \supset M\}.$$

If M and N are both bounded and non-empty, let

$$d_H(M, N) = \max\{\vartheta(M, N), \vartheta(N, M)\}$$

be the Hausdorff distance between M and N.

The following lemmas are easy to prove:

Lemma 1. Let (X, d) be a pseudometric space and M a subset of X. Then

(1) $\chi(M) = \inf\{\varepsilon \in \mathbb{R} : \varepsilon > 0$ and there is a compact subset K of X with $B(K, \varepsilon) \supset M\} =$

$= \inf\{\varepsilon \in \mathbb{R} : \varepsilon > 0$ and there is a precompact subset P of X with $B(P, \varepsilon) \supset M\} =$

$= \inf\{\vartheta(M, \sigma) : \sigma$ a finite subset of $X\}$,

(2) M is precompact if and only if $\chi(M) = 0$;

(3) M is bounded if and only if $\chi(M) < +\infty$;

(4) if $\mathfrak{M}$ is a subset of $\exp X$, then $\chi(\bigcup \mathfrak{M}) \geqq \sup \chi(\mathfrak{M})$;

(5) if $\mathfrak{M}$ is a finite subset of $\exp X$, then $\chi(\bigcup \mathfrak{M}) = \sup \chi(\mathfrak{M})$;

(6) if $M \subset N \in \exp X$, then $\chi(M) \leqq \chi(N)$;

(7) $\chi(M) \leqq \vartheta(M, N) + \chi(N)$ for all $M, N \in \exp X$;

(8) $|\chi(M) - \chi(N)| \leqq d_H(M, N)$ for all bounded non-empty subsets M, N of X;

(9) the measure of non-compactness $\chi(\cdot)$ is continuous on the pseudometric space $(B(X), d_H)$ of all non-empty bounded subsets of X.

Lemma 2. Let $(X, \|\cdot\|)$ be a pseudonormed space (over $\mathbb{R}$ or $\mathbb{C}$) and M and N subsets of X. Then

(1) $\chi(\lambda M) = |\lambda| \chi(M)$ for all $\lambda \in \mathbb{R}$ (resp. $\lambda \in \mathbb{C}$), $\lambda \neq 0$;

(2) $\max\{\chi(M), \chi(N)\} \leqq (M + N) \leqq \chi(M) + \chi(N)$;

(3) $\chi(M) = \chi(\overline{co}\, M)$.

2. Generalized concentrative mappings

Definition 1. Let (X, d) and (Y, e) be pseudometric spaces and C a subset of X. Then a mapping $f: C \to Y$ is called concentrative, if f satisfies the following two conditions:

(1) f is continuous;

(2) if M is a bounded non-precompact subset of X, i.e.

$0 < \chi_d(M) < +\infty$, then $\chi_e(f(M)) < \chi_d(M)$.

The following two lemmas are obvious.

Lemma 3. Let (X, d) and (Y, e) be pseudometric spaces, C a subset of X and $f: C \to Y$ a mapping. Suppose that one of the following conditions is satisfied:

(1) f is continuous and maps bounded subsets of C onto pre-compact subsets of Y;

(1') f is continuous and $f(C)$ is precompact;

(2) f is a k-contraction $(0 \leqq k < 1)$, i.e. $e(f(x), f(y)) \leqq k d(x, y)$ for $x, y \in C$;

(1-2) $C = C_1 \cup C_2$, f is continous on C, maps bounded subsets of C_1 onto precompact subsets of Y and is a k-contraction on C_2 $(0 \leqq k < 1)$.

Then f is a concentrative mapping (even, k-concentrative, i.e. $\chi_e(f(M)) \leqq k \chi_d(M)$ for bounded subsets M of C).

Lemma 4. Let (X, d) be a pseudometric space, $(Y, \|\cdot\|)$ a pseudonormed space and C a subset of X. Suppose that f and g are mappings on C into

Y such that

(1) f is continuous and maps bounded subsets of C onto precompact subsets of Y;

(2) g is a k-contraction $(0 \leq k < 1)$, i.e. $\|g(x)-g(y)\| \leq k\,d(x,y)$ for $x, y \in C$.

Then $(f+g): C \to Y$ is a concentrative mapping of C into Y (even, a k-concentrative mapping). (This was pointed out by Sadovskii [13].)

Definition 2. Let X be a set and α a set-to-set mapping defined on all subsets of X, i.e. $exp\, X \longrightarrow exp\, X$, such that:

(1) $i \subset \alpha$ (α is extensive), i.e. $iM \equiv M \subset \alpha M$ for all $M \in exp\, X$;

(2) $\alpha\alpha = \alpha$ (α is idempotent), i.e. $\alpha(\alpha M) = \alpha M$ for all $M \in exp\, X$;

(3) α is monotone, i.e. $\alpha M \subset \alpha N$ for all $M \subset \subset N \subset X$.

Then α is called a c-closure on the set X. A subset M of X is called α-closed, if $\alpha M = M$.

Examples. (1) Let X be a set and $\mathcal{S}$ a system of subsets of X with $X \in \mathcal{S}$. For $M \in exp\, X$, let

$$\alpha_{\mathcal{S}}(M) = \bigcap \{ S \in \mathcal{S} : S \supset M \} .$$

Clearly, $\alpha_{\mathcal{S}}$ is extensive and monotone. If $S_1 \supset M$, $S_1 \in \mathcal{S}$, then $\alpha_{\mathcal{S}}(M) \subset S_1$, since $S_1 \in \{S \in \mathcal{S} : S \supset M\}$. Hence

$$\alpha_{\mathcal{S}}(M) \supset \alpha_{\mathcal{S}}(\alpha_{\mathcal{S}}(M)) .$$

Since the inverse inclusion follows from the extensivity and the monotonicity of $\alpha_{\mathcal{S}}$, $\alpha_{\mathcal{S}}$ is idempotent.

Thus, α_y is a $\mathcal{C}$-closure on X.

(2) Let X be a set, P a subset of X and β a $\mathcal{C}$-closure on X. For $M \in \exp X$, let

$$\alpha M = \beta(P \cup M).$$

Then we have:

$\alpha M = \beta(P \cup M) \supset P \cup M \supset M$ for all $M \in \exp X$,

$\alpha(\alpha M) = \beta(P \cup \beta(P \cup M)) = \beta(\beta(P \cup M)) = \beta(P \cup M) = \alpha M$

for all $M \in \exp X$.

If $M, N \in \exp X$ and $M \subset N$, then $M \cup P \subset N \cup P$, and hence

$$\alpha M = \beta(P \cup M) \subset \beta(N \cup P) = \alpha N.$$

Thus, α is a $\mathcal{C}$-closure on X.

(3) Let X and Y be sets, β a $\mathcal{C}$-closure on Y and f a mapping on X into Y such that $ff^{-1} = id_Y$. For $M \in \exp X$, let

$$\alpha M = f^{-1}(\beta(f(M))).$$

Then, for $M, N \in \exp X$, $M \subset N$, we have successively,

$\beta(f(M)) \supset f(M)$, $\alpha M = f^{-1}(\beta(f(M))) \supset f^{-1}(f(M)) \supset M$,

$$\alpha(\alpha M) = f^{-1}(\beta(f(f^{-1}(\beta(f(M)))))) = f^{-1}(\beta(\beta(f(M)))) =$$

$$= f^{-1}(\beta(f(M))) = \alpha M,$$

$$f(M) \subset f(N),\ \beta(f(M)) \subset \beta(f(N)),\ \alpha M = f^{-1}(\beta(f(M))) \subset$$

$$\subset f^{-1}(\beta(f(N))) = \alpha N.$$

Thus, α is a $\mathcal{C}$-closure on X.

(4) Let X be a set and let β and γ be $\mathcal{C}$-closures on X such that $\gamma\beta\gamma = \beta\gamma$, i.e. $\gamma(\beta(\gamma(M)))=$

$= \beta(\gamma(M))$ for all $M \in exp\, X$. It is easy to see that $\alpha = \beta\gamma$ is a c-closure on X (the idempotency of α follows from $\gamma\beta\gamma = \beta\gamma$).

(5) Let X be a linear space. For $M \in exp\, X$ we define:

$co\, M$ = the convex hull of M ;

$con\, M = \{x \in X : x = tm, m \in M, t \in \mathbb{R}, t \geq 0\}$;

$aff\, M$ = the affine hull of M ;

$sp\, M$ = the linear hull (span) of M .

Then co, con, aff, sp are c-closures on X .

(6) Let X be a topological space. Then its closure operation cl is a c-closure on X .

(7) Let X be a linear topological space. By (4-6), $\overline{co} = cl\, co$ = the closed convex hull, $\overline{con} = cl\, con$ = the closed cone hull, $\overline{aff} = cl\, aff$ = the closed affine hull, $\overline{sp} = cl\, sp$ = the closed linear hull, are c-closures on X .

(8) Let X be a pseudometric space and, for $M \in exp\, X$, let

$\alpha M = \bigcap\{B : B$ a ball (closed) in $X, B \supset M\}$.

By (1), α is a c-closure on X .

Lemma 5. Let X be a set and α a c-closure on X . Then:

(1) If $\mathfrak{M}$ is a non-empty subset of $exp\, X$, then $\bigcap \alpha\mathfrak{M}$ $(\equiv \bigcap\{\alpha M : M \in \mathfrak{M}\})$ is α-closed;

(2) if M is a subset of X, then $\alpha M = \bigcap\{N \in exp\, X : N \supset M,\ N$ is α-closed$\}$.

Proof. (1) Since $\bigcap \alpha \mathfrak{M} \subset \alpha M$ for all $M \in \mathfrak{M}$, we have, by the monotonicity and the idempotency of α,

$\alpha(\bigcap \alpha \mathfrak{M}) \subset \alpha(\alpha M) = \alpha M$ for all $M \in \mathfrak{M}$.

Thus,

$$\alpha(\bigcap \alpha \mathfrak{M}) \subset \bigcap \alpha \mathfrak{M} .$$

By the extensivity of α, $\bigcap \alpha \mathfrak{M} \subset \alpha(\bigcap \alpha \mathfrak{M})$, i.e. $\alpha(\bigcap \alpha \mathfrak{M}) = \bigcap \alpha \mathfrak{M}$, and $\bigcap \alpha \mathfrak{M}$ is α-closed.

(2) Let $\mathfrak{M} = \{N \in \exp X : N \supset M ; N$ is α-closed$\}$. If $N \in \mathfrak{M}$, then $N = \alpha N \supset \alpha M$ (α monotone), and $\bigcap \mathfrak{M} \supset \alpha M$. Further, $\alpha M \supset M$ and αM is α-closed (α extensive and idempotent), i.e. $\alpha M \in \mathfrak{M}$. Therefore, $\alpha M = \bigcap \mathfrak{M}$.

Definition 3. Let X be a topological space, C a subset of X, α a c-closure on X and $f : C \to X$ a mapping. The mapping f is called α-generalized concentrative if the following three conditions are satisfied:

(1) f is continuous;

(2) if M is a subset of C and $M = \alpha f(M)$, then M is compact;

(3) if M is a subset of C such that $f(M) \subset M$ and $\mathrm{card}(M \setminus \overline{f(M)}) \leqq 1$, then $\overline{M}$ is compact.

If X is a linear topological space and $\alpha = \overline{co}$, then f, if it is $\overline{co}$-generalized concentrative, it is called generalized concentrative (on C).

Remark. The notion of the generalized concentrative mapping was recently introduced by Lifšic and Sadovskii in [11] in the following sense: A continuous mapping from a subset C of a locally convex space X into X is called "generalized concentrative" if it satisfies the condition:

if M is a subset of C such that $f(M) \subset M$ and $M \setminus \overline{co}\, f(M)$ is compact, then $\overline{M}$ is compact.

It is easy to see that this notion is a special case of our definition (even in the case of locally convex space X).

Proposition 1. Let X be a non-empty topological space and $f: X \to X$ a mapping satisfying the condition:

If M is a subset of X such that $f(M) \subset M$ and $\mathrm{card}\,(M \setminus \overline{f(M)}) \leq 1$, then $\overline{M}$ is compact, i.e. f satisfies the condition (3) of Definition 3 for $C = X$.

Then there exists a non-empty subset K of X such that

$$\overline{f(K)} \supset K .$$

Proof. The first part of the proof is very similar to that of [11]. Let O be the class of all ordinal numbers, O_1 the class of all ordinal numbers of the first kind, i.e. which have predecessor, and O_2 is the class of all ordinal numbers of the second kind. For each x_0 in X and δ in O we construct a directed net $\{x_\alpha\}_{\alpha<\delta}$ such that:

(1) $\alpha \in O_1$ and $\alpha < \delta$ implies $x_\alpha = f(x_{\alpha-1})$;

(2) $\alpha \in O_2$ and $0 < \alpha < \delta$ implies that x_α is a limit point of the directed net $\{x_\beta\}_{\beta < \alpha}$.

Let $x_0 \in X$ and $\delta \in O$ be given. The proof proceeds by transfinite induction. Suppose we have constructed a directed net $\{x_\alpha\}_{\alpha<\gamma}$ for some $\gamma < \delta$ such that 1) and 2) are satisfied with γ instead of δ. If $\gamma \in O_1$, we set $x_\gamma = f(x_{\gamma-1})$. Now, suppose that $0 < \gamma < \delta$, $\gamma \in O_2$. Let $M = \{x_\alpha : \alpha < \gamma\}$. Clearly, $f(M) = \{x_\alpha : \alpha < \gamma,\ \alpha \in O_1\} \subset M$. Denote by S the set of all ordinals $\alpha' \in O_2$ such that $0 < \alpha' < \gamma$ and $x_{\alpha'} \notin \overline{f(M)}$. If $S \neq \emptyset$, then there exists $\alpha'' = \min S$. From the definition of α'' it follows that $\{x_\beta\}_{0<\beta<\alpha''} \subset \overline{f(M)}$. By the inductive hypothesis, $x_{\alpha''}$ is a limit point of the directed net $\{x_\beta\}_{\beta<\alpha''}$. Thus $x_{\alpha''} \in \overline{\{x_\beta\}_{0<\beta<\alpha''}} \subset \overline{f(M)}$, a contradiction with the definition of α''. Therefore, $S = \emptyset$ and we have:

$$\{x_\beta\}_{0<\beta<\gamma} = M \setminus \{x_0\} \subset \overline{f(M)} \ ,$$

i.e.

$$M \setminus \overline{f(M)} < \{x_0\} .$$

Hence $card(M \setminus \overline{f(M)} \leqq 1$. By the hypothesis, $\overline{M}$ is compact. The directed net $\{x_\alpha\}_{\alpha<\gamma}$ has limit points in $\overline{M}$. One of this limit points we denote by x_γ. Hence we have the directed net $\{x_\alpha\}_{\alpha<\gamma+1}$ which satisfies the conditions 1) and 2) with $\gamma + 1$ instead of δ. By the principle of transfinite induction, there is a directed net $\{x_\alpha\}_{\alpha<\delta}$ which satisfies the conditions 1) and 2).

Let δ be an ordinal number such that $card\ \delta >$ $> card\ X$. Let $\{x_\alpha\}_{\alpha<\delta}$ be a directed net which satisfies the conditions 1) and 2). Since $card\ \delta >$ $> card\ X$,there are distinct ordinal numbers α' and β' such that $x_{\alpha'} = x_{\beta'}$. Let α be the smallest ordinal number such that $x_\alpha = x_{\beta'}$ for some ordinal number $\beta' > \alpha$ and let β be the smallest ordinal number greater than α such that $x_\alpha = x_\beta$. Put

$$K = \{x_\gamma : \alpha < \gamma \leq \beta\} .$$

We shall prove that $\overline{f(K)} \supset K$. If $\gamma \in O_2$ and $\alpha < \gamma \leq$ $\leq \beta$ then x_γ is a limit point of the directed net $\{x_\eta\}_{\eta<\gamma}$, and of $\{x_\eta\}_{\alpha<\eta<\gamma} \subset K$, too. Thus, $x_\gamma \in$ $\in \overline{f(K)}$ (the proof is similar to the construction of the element x_γ given above). If $\gamma \in O_1$ and $\alpha <$ $< \gamma \leq \beta$, then $x_\gamma = f(x_{\gamma-1})$, where $\alpha \leq \gamma - 1 < \beta$ and hence $x_\gamma \in f(K) \subset \overline{f(K)}$ (because $x_{\gamma-1} = x_\beta \in K$ in the case $\gamma - 1 = \alpha$). Thus, we have $K \subset \overline{f(K)}$ and K is non-empty.

Proposition 2. Let (X, d) be a bounded complete pseudometric space and $f : X \to X$ a concentrative mapping. Then f is a cl -generalized concentrative mapping.

Proof. Since f is concentrative, it is continuous. Let M be a subset of X such that $M = cl\ f(M) \equiv$ $\equiv \overline{f(M)}$. Then $\chi(f(M)) = \chi(\overline{f(M)}) = \chi(M)$; therefore, $\chi(M) = 0$ and M is precompact. Since M is closed in X and X is complete, M is compact. Now, let M be a subset of X such that $f(M) \subset M$ and $\mathrm{card}\,(M \setminus cl\, f(M)) \leqq 1$.

Then $M \setminus \overline{f(M)} = A$, where A is empty or a singleton of M. Hence

$$\chi(M) \leqq \chi(\overline{f(M)} \cup A) = \chi(f(M)) \leqq \chi(M) .$$

Since f is concentrative, M is precompact; the completeness of X implies the compactness of $\overline{M}$. Thus, f is a cl-generalized concentrative mapping in X.

Corollary 1. Let (X, d) be a pseudometric space, $f: X \to X$ a concentrative mapping, $\overline{f^m(X)}$ bounded and complete subset of X for some positive integer m ($f^0 = id$). Then the mapping f, considered as a mapping of $\overline{f^m(X)}$ into $\overline{f^m(X)}$, is a cl-generalized concentrative mapping.[1)]

Proposition 3. Let $(X, \|\cdot\|)$ be a pseudonormed space, C a non-empty convex bounded complete subset of X, $f: C \to C$ a concentrative mapping. Then f is a generalized concentrative mapping.

Proof is similar to that of Proposition 2 (we use the equality $\chi(M) = \chi(\overline{co}\, M)$; note that $\overline{co} : \exp C \longrightarrow \exp C$).

Corollary 2. Let $(X, \| \cdot \|)$ be a pseudonormed space, C a convex complete non-empty subset of X

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1) The inclusion $f(\overline{f^m(X)}) \subset \overline{f^m(X)}$ follows from the continuity of f:

$$f(\overline{f^m(X)}) \subset \overline{f(f^m(X))} = \overline{f^{m+1}(X)} \subset \overline{f^m(X)} .$$

and $f : C \longrightarrow C$ a generalized concentrative mapping. Suppose that, for some positive integer m, $f^{m}(C)$ is bounded. Then f, considered as a mapping of $\overline{co}\, f^{m}(C)$ into itself, is a generalized concentrative mapping.

3. Fixed point theorems

The following proposition is crucial in the following exposition.

Proposition 4. Let C be a set, α a c-closure on C and $f : C \longrightarrow 2^{C}$ a mapping such that:

(1) there exists a non-empty subset K of C such that

$$\alpha f(K) \supset K .$$

Then there exists a non-empty subset C_0 of C such that

$$\alpha f(C_0) = C_0 .$$

Proof. Let

$$\mathfrak{M} = \{ M \in \exp C : K \subset M = \alpha M,\ f(M) \subset M \} .$$

Clearly, $\mathfrak{M} \neq \emptyset$ since: $C \supset K, \alpha C = C$ (this follows from the extensivity of α), $f(C) \subset C$ and we have $C \in \mathfrak{M}$.

The system $\mathfrak{M}$ has the following property:

(P) if $M \in \mathfrak{M}$, then $\alpha f(M) \in \mathfrak{M}$.

Indeed, let $M \in \mathfrak{M}$ and $M_1 = \alpha f(M)$. From the idempotency of α it follows that $\alpha M_1 = \alpha\alpha f(M) = \alpha f(M) = M_1$. By the hypothesis (i) and the monotonicity of α, we have

$$K \subset \alpha f(K) \subset \alpha f(M) = M_1 .$$

Since $f(M) \subset M$, the monotonicity of α implies that

$$M_1 = \alpha f(M) \subset \alpha M = M ;$$

therefore,

$$f(M_1) \subset f(M) \subset \alpha f(M) = M_1 \text{ (the extensivity of } \alpha \text{)}.$$

Thus, $M_1 = \alpha f(M) \in \mathfrak{M}$, and the property (P) is proved.

Put

$$C_0 = \cap \mathfrak{M} = \cap \{M : M \in \mathfrak{M}\} .$$

First of all, $C_0 \in \mathfrak{M}$ since: $K \subset \cap \mathfrak{M} = C_0$, $f(C_0) = f(\cap \mathfrak{M}) \subset \cap f(\mathfrak{M}) \subset \cap \mathfrak{M} = C_0$, and, by Lemma 5, C_0 is α-closed, i.e. $\alpha C_0 = C_0$ (the second inclusion follows from the fact that $f(M) \subset M$ for all $M \in \mathfrak{M}$).

Now, by the property (P), we have $\alpha f(C_0) \in \mathfrak{M}$, and hence $C_0 \subset \alpha f(C_0)$. Since $f(C_0) \subset C_0$, the monotonicity of α implies that $\alpha f(C_0) \subset \alpha C_0 = C_0$, i.e. we have

$$\alpha f(C_0) = C_0 .$$

As a consequence of Proposition 4 we obtain

<u>Theorem 1</u>. (A special case of Theorem 1 in [5].) Let X be a locally convex (Hausdorff) linear topological space (over $\mathbb{R}$ or $\mathbb{C}$), C a non-empty convex subset of X, and $f: C \longrightarrow C$ a continuous mapping. Suppose that f satisfies the following conditions:

(i) there is a non-empty subset K of C such that $\overline{co}\, f(K) \supset K$;

(ii) if M is a subset of C with $\overline{co}\, f(M) = M$, then M is compact.

Then f has a fixed point in C.

Proof. See the proof of Theorem 1 in [5]. The first part of that proof is contained in Proposition 4 if we set $\alpha = \overline{co}$. The second parts of both proofs are the same.

Theorem 2. Let X be a locally convex (Hausdorff) linear topological space, C a non-empty convex closed subset of X and $f : C \longrightarrow C$ a generalized concentrative mapping. Then f has a fixed point in C.

Proof. Since f is generalized concentrative, it is continuous and satisfies the condition (ii) of the hypotheses of Theorem 1. The condition (i) of Theorem 1 is a consequence of Proposition 4. Now, it suffices to apply Theorem 1.

Corollary 3 (Sadovskii [13]). Any concentrative mapping of a non-empty convex bounded closed subset of a Banach space into itself has a fixed point.

Proof. See Proposition 3 and Theorem 2.

Remark. Further fixed point theorem can be obtained at once from Corollary 2.

Corollary 4. Let X be a Banach space, C a non-empty convex bounded closed subset of X and $f : C \longrightarrow C$ a mapping. Suppose that f is the sum of a completely continuous mapping $g : C \longrightarrow X$ and a k-concentrative mapping $(0 \leq k < 1)$ $h: C \to X$. Then f has a fixed point in C.

Proof. See Lemma 4 and the preceding Corollary 3.

The following theorem is Theorem 3 in [5], but we remove a superfluous hypothesis on the set C.

Theorem 3. Let X be a locally convex (Hausdorff) linear topological space and C a non-empty complete bounded convex subset of X. Let P be a defining system of pseudonorms on X (i.e., the collection $\{p^{-1}(\langle 0, \varepsilon)) : p \in P, 0 < \varepsilon < 1\}$ is a base for neighborhoods of the origin in X) and $f: C \to C$ a P-concentrative mapping in the sense that f is continuous and satisfies the following condition:

(C) if $p \in P$ and M is a bounded non-p-precompact (i.e. M is not precompact in the pseudonormed space (X, p)) subset of X, then

$$\chi_p(f(M)) < \chi_p(M),$$

where $\chi_p(\cdot)$ denotes the measure of non-compactness in the pseudonormed space (X, p).

Then the mapping f has a fixed point in C.

Proof. We shall show that f is a generalized concentrative mapping on C.

Let M be a subset of C such that $\overline{co}\, f(M) = M$. Then $\chi_p(f(M)) = \chi_p(M)$ for all $p \in P$. Hence M is precompact in X. Since M is closed, precompact and C is complete, the set M is compact.

Now, let M be a subset of C such that $f(M) \subset M$

and $card(M \setminus \overline{f(M)}) \leq 1$. Then $M \setminus \overline{f(M)} = A$ for some subset A of M with $card\, A \leqq 1$. Hence $\chi_p(f(M)) \leqq \chi_p(M) \leqq \chi_p(\overline{f(M)} \cup A) = \chi_p(f(M))$ for all $p \in P$. As before, it follows that $\overline{M}$ is compact.

Since f is also continuous, it is a generalized concentrative mapping on C. Theorem 2 assures the existence of a fixed point of f in C.

Theorem 4. Let X be a non-empty complete metric space and $f: X \to X$ a concentrative mapping. Let $d: X \times X \to \langle 0, +\infty)$ be a lower semi-continuous function such that the two conditions are satisfied:

(1) $d^{-1}(0) = \Delta = \{(x,x): x \in X\}$ (= the diagonal in $X \times X$), i.e. $d(x,y) = 0$ iff $x = y$;

(2) $d \circ (f \times f) < d$ on $X \times X \setminus \Delta$, i.e. $x, y \in X$, $x \neq y$ implies $d(f(x), f(y)) < d(x,y)$.

Suppose that $f^m(X)$ is bounded for some non-negative integer m ($f^0(X) = X$).

Then f has a unique fixed point in X.

Proof. Let $C = \overline{f^m(X)}$. By continuity of f,

$$f(C) = f(\overline{f^m(X)}) \subset \overline{f(f^m(X))} = \overline{f^{m+1}(X)} \subset \overline{f^m(X)} = C.$$

Hence f is a concentrative mapping of the bounded complete metric space C into itself. By Proposition 2, f is cl-generalized concentrative on C. By Propositions 1 and 4, there exists some non-empty subset C_0 of C such that $cl\, f(C_0) = C_0$. The cl-generalized concentrativeness of f on C implies the compactness of C_0. Define on C_0 a function φ by

$$\varphi(x) = d(x, f(x)) \quad \text{for } x \in C_o .$$

Let $\delta : C_o \to C_o \times C_o$ be defined by

$$\delta(x) = (x, x), \quad x \in C_o .$$

Since d is lower semi-continuous on $C_o \times C_o$ to $\mathbb{R}$, $id \times f$ is continuous on $C_o \times C_o$ to $C_o \times C_o$ and δ is continuous on C_o to $C_o \times C_o$, their composite $\varphi = d \circ (id \times f) \circ \delta$ is lower semi-continuous on C_o to $\mathbb{R}$. Hence $\varphi(x)$ attains its minimum at some point x_o in the compact set C_o. Suppose that $\varphi(x_o) \neq 0$. Then $d(x_o, f(x_o)) > 0$. Therefore, $x_o \neq f(x_o)$, and, by (2),

$$\varphi(f(x_o)) = d(f(x_o), f(f(x_o))) < d(x_o, f(x_o)) = \varphi(x_o) ,$$

a contradiction with the minimality of the function $\varphi(x)$ at x_o. Thus, $\varphi(x_o) = 0$, i.e. $d(x_o, f(x_o)) = 0$. Hence (cf. (1)), $x_o = f(x_o)$. The uniqueness of the fixed point x_o follows at once from (1) and (2).

Remark. If the function d is the metric of the metric space X the preceding theorem can be deduced from Edelstein's theorem [9]. Edelstein's theorem was generalized by Ang and Daykin [1,Th.1] to topological spaces with a family of continuous pseudometrics. From Ang-Daykin's theorem we can derive

Theorem 5. Let X be a non-empty topological space, D a family of continuous pseudometrics on X and α a c-closure on X. Let $f : X \to X$ be an α-generalized concentrative mapping such that both following conditions are satisfied:

(1) $d(f(x),f(y)) \leqq d(x,y)$ for all $x, y \in X$ and $d \in D$, i.e. $d \circ (f \times f) \leqq d$ on $X \times X$;

(2) for each $x, y \in X$, $x \neq y$ there exists $d \in D$ such that $d(f(x),f(y)) < d(x,y)$,

i.e. the function $\{d(x,y) - d(f(x),f(y)) : d \in D\} = s(x,y)$ is positive on $X \times X \setminus \Delta$, where $\Delta = \{(x,x) : x \in X\}$ is the diagonal in $X \times X$.

Then the mapping f has a unique fixed point p in X. Furthermore, $f^n(x) \longrightarrow p$ in the D-topology on X for each $x \in X$, i.e. $d(f^n(x),p) \longrightarrow 0$ for each $d \in D$, $x \in X$.

Proof. Let $x \in X$ be arbitrary. Let $M = \{f^n(x) : n = 0, 1, 2, \dots\}$. Then $f(M) \subset M$, and $card(M \setminus \overline{f(M)}) \leqq card(M \setminus f(M)) \leqq card\{x\} \leqq 1$. Hence $\overline{M}$ is compact since f is α-generalized concentrative. Thus, the sequence $\{f^n(x)\}$ has a limit point $p(x)$. Now, we can apply Ang-Daykin's theorem to obtain $d(f^n(x), p(x)) \longrightarrow 0$ for each $d \in D$, and $p(x)$ is the unique fixed point of f i.e. $p(x) = const = p$.

R e f e r e n c e s

[1] D.D. ANG - D.E. DAYKIN: Some fixed point theorems and convolution equations. Proc.Amer.Math. Soc.19(1968),1187-1194.

[2] A.L. BADOEV - B.N. SADOVSKII: An example of concentrative operator in the theory of differen-

tial equations with deviating argument of neutral type. Doklady AN SSSR 186(1969), 1239-1242.

[3] Ju.G. BORISOVIČ - Ju.I. SAPRONOV: On topological theory of concentrative mappings. Doklady AN SSSR 183(1969),18-20.

[4] J. DANEŠ: Nonlinear operators and functionals(Thesis). Charles University,Faculty of Mathematics and Physics,Prague,1968(in Czech).

[5] J. DANEŠ: Some fixed point theorems.Comment.Math. Univ.Carolinae 9(1968),223-235.

[6] J. DANEŠ: Generalized concentrative mappings. Summer school on Fixed Points,Krkonoše,Czechoslovakia,Aug.31-Sept.6,1969(in Czech).

[7] G. DARBO: Punti uniti in transformazioni a condominio non compatto. Rend.Sem.Mat.Univ.Padova 24(1955),84-92.

[8] R.L. FRUM-KETKOV: On mappings into the sphere of a Banach space. Doklady AN SSSR 175(1967), 1229-1231.

[9] M. EDELSTEIN: On fixed and periodic points under concentrative mappings. J.London Math. Soc.37(1962),74-79.

[10] K. KURATOWSKI: Topologie I. Warszaw,1958.

[11] E.A. LIFŠIC - B.N. SADOVSKIJ: A fixed point theorem for generalized concentrative mappings, Doklady AN SSSR 183(1968),278-279.

[12] R.D. NUSSBAUM: The fixed point index and asymptotic fixed point theorems for k-set-contractions. Bull.Amer.Mat'..Soc.75(1969), 490-495.

[13] B.N. SADOVSKII: ON a principle of fixed points. Funkcional.analiz i ego prilož.1(1967), 74-76.

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