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NOWHERE DENSE SET WHICH IS FINITELY OPEN

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1. Introduction. The notion of finitely open set plays an important role in the theory of analytic operators and in the differentiability of mappings which act in complex Banach spaces (see e.g. [1],chapt.26). A point set $A \subset X$, where X is a real or complex normed linear space, is said to be finitely open (in X) if and only if its intersection $A \cap P$ with an arbitrary finite-dimensional normed linear subspace $P \in X$ is open in P.

Clearly, every open set $A \subset X$ is also finitely open. But, if the dimension of X is infinite, there is a finitely open set in X which is not open. Such a simple example was given, e.g., in [1], 1.10. It is the purpose of this note to show that a (non-void) finitely open set $A \subset X$ not only need not be open in X, but also it may be nowhere dense in X (i.e. not even its closure $\bar{A}^X$ has any interior point with respect to X).

2. Example. Let X be the linear normed space of all sequences of real numbers having only a finite

number of non-zero terms. The algebraic operations are defined in a customary way "over coordinates" and the norm of the point $x = (\xi_1, \xi_2, \dots)$ is $\|x\| = \max\limits_{n=1,2,\dots} |\xi_n|$.

Taking the points

$$p_1 = (1, 0, 0, \dots) ,$$

$$p_2 = (0, 1, 0, \dots) ,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots ,$$

we construct, for every natural number n, the n-dimensional subspace $P_n = \mathcal{L}in\{p_1, \dots, p_n\}$. Then

$$P_1 \subset P_2 \subset \dots , \quad X = \bigcup_{n=1}^{\infty} P_n .$$

Let M_n be a point set in P_n, ε a positive number. Then we shall define a point set $\sigma_\varepsilon(M_n)$ in P_{n+1} as follows:

$\sigma_\varepsilon(M_n) = \{x \in P_{n+1} : x = m + \lambda p_{n+1}$, where $m \in M_n$ and $|\lambda| < \varepsilon\}$.

It is easy to see that such an expression $x = m + \lambda p_{n+1}$ (where $m \in P_n$) is unique. Obviously, if M_n is open in P_n, then $\sigma_\varepsilon(M_n)$ is open in P_{n+1}.

Denote by K_ε an open spherical neighbourhood of the origin with a radius $\varepsilon > 0$, i.e. $K_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$.

We shall construct a sequence of sets $A_1, A_2, \dots$. Let $A_1 = P_1 \cap K_1$ and, if for some n the set A_n

has been already constructed, we put $A_{n+1} = \sigma_{\frac{1}{n+1}}(A_n)$. Thus, for every natural number n, A_n is an open subset of the space P_n and we have, as one sees easily, $A_1 \subset A_2 \subset \dots$, $A_{n+1} \cap P_n = A_n$. Eventually, we may define $A = \bigcup_{n=1}^{\infty} A_n$. It is obvious that $A \cap P_n = A_n$ for each $n = 1, 2, \dots$.

Now, we shall prove that the set A is finitely open in X. Hence, let P be an arbitrary finite-dimensional subspace of X. The case $P = \{0\}$ being trivial, suppose that $P \neq \{0\}$ and choose a finite basis $v_1, \dots, v_d$ of P. There is, of course, a natural number N such that for each of the points $v_1, \dots$ $\dots, v_d$ the following is valid: all coordinates beginning from the $(N+1)$-th place are zeros. Therefore $P \subseteq P_N$, so that, obviously, it is sufficient to prove (for every $n = 1, 2, \dots$) the openness of the set $A \cap P_n$ in the subspace P_n. But it is true according to our construction.

Further, let us prove that the set $A \subset X$ has no interior point. For every natural number n, there exists a point $x_n \in (K_{\frac{1}{n}} \cap P_{n+1}) - A$. It is, for instance, the point whose $(n+1)$-th coordinate equals to $\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)$ and the other ones are zeros. As $\frac{1}{n+1} < \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < \frac{1}{n}$, this point lies in $K_{\frac{1}{n}} \cap P_{n+1}$ but does not lie in A_{n+1} and since the intersection $A \cap P_{n+1}$ is exactly the set A_{n+1}, this point cannot lie in A. Thus, we have obtained a sequence of points $\{x_n\}_{n=1}^{\infty}$ contained in $X - A$ with $x_n \to 0$ in X. It follows that the point $0 \in A$ is not an interior one of the set A.

If x is an arbitrary point of the set A, then we have $x \in A_N$ for some N and for every $n \geq N$ we can find again (similarly as in the case $x = 0$ before a while) a point $y_n \in [(x + K_{\frac{1}{n}}) \cap P_{n+1}] - A$ (it will be, for instance, the point $y_n = x + x_n$, where x_n is that from the previous case). Thus, we proved that the set A has no interior point with respect to X.

Moreover, we shall show that our set A is nowhere dense in X. If M is a point set in a space P, we shall denote by $\overline{M}^{P}$ its closure in P. First of all, it is clear that $\bigcup_{n=1}^{\infty} \overline{A}_n^{P_n} \subset \overline{A}^{X}$. Let us take a point $x \notin \bigcup_{n=1}^{\infty} \overline{A}_n^{P_n}$. Since $X = \bigcup_{n=1}^{\infty} P_n$, there is an index N such that $x \in P_N$. In accordance with the above supposition $x \notin \overline{A}_N^{P_N}$ and, hence, there is a suitable $\varepsilon > 0$ for which $[(x + K_\varepsilon) \cap P_N] \cap A_N = \emptyset$ or (because $A_N \subset P_N$) $(x + K_\varepsilon) \cap A_N = \emptyset$.

With regard to inclusions $A_1 \subset \ldots \subset A_N$, it holds also that $(x + K_\varepsilon) \cap A_n = \emptyset$ for each $n = 1, \ldots, N$. If we assume, for a $k \geq N$, that $(x + K_\varepsilon) \cap A_k = \emptyset$, then it follows from the construction of A_{k+1} that

also $(x + K_\varepsilon) \cap A_{k+1} = \emptyset$. Indeed, if the contrary was true, then there would be a point $z \in (x + K_\varepsilon) \cap$ $\cap A_{k+1}$ so that we could write it in the form (unambiguously) $z = m + \lambda r_{k+1}$, where $m \in A_k$, $|\lambda| < \frac{1}{k+1}$. Hence, the points z, m may differ in the $(k+1)$-th coordinate only, so that from the inequality $\| x - z \| < \varepsilon$ it follows (if we recollect the definition of our norm and that $x \in P_N \subseteq P_k$, $m \in P_k$) the inequality $\| x - m \| < \varepsilon$. Consequently, a point $m \in A_k$ lies in the ball $x + K_\varepsilon$, which is impossible. By induction we have $(x + K_\varepsilon) \cap A_n =$ $= \emptyset$ for every $n = 1, 2, \dots$, so that also $(x + K_\varepsilon) \cap$ $\cap A = \emptyset$.
It means, however, that $x \notin \bar{A}^X$.

Thus, we proved the equality $\bar{A}^X = \bigcup_{n=1}^{\infty} \bar{A}_n^{P_n}$.
Now, if we realize what $\bar{A}_n^{P_n}$ are, we can show easily that this set $\bar{A}^X$ has no interior point with respect to X (similarly as it was done for the set A).

Thus, we have really constructed a (non-void) set $A \subset X$ which is finitely open in X although it is nowhere dense in X. Obviously, we could consider our space X also as a complex one.

3. Remarks. Let X be a linear normed space. A point set $A \subset X$ is said to be countably open (in X) if and only if its intersection $A \cap L$ with an arbit-

rary span L generated by a countable set of points is open in the linear normed subspace $L \subset X$.

Proposition. If a set $A \subset X$ is countably open in a linear normed space X, then it is open in X.

Proof. If we assume the contrary, there exists an $x \in A$ which is not an interior point of A. Hence, there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points from the set $X - A$ with $x_n \to x$ in X. If we put $L = \operatorname{lin}\{x, x_1, x_2, \dots\}$, then the points $x_1, x_2, \dots$ lie in $L - (A \cap L)$, x lies in $A \cap L$ and, obviously, $x_n \to x$ in the subspace L. It follows that the set $A \cap L$ is not open in the subspace L. This contradiction proves our assertion. Naturally, the converse is also true.

Obviously, there are spaces which have no countable set of generators; such is, e.g., every Banach space with an infinite dimension. It justifies the proposition mentioned above.

It is worth mentioning that the family $\mathcal{T}$ of all finitely open sets in the space X can define a new topology on the point set X. Indeed, an easy verification shows that $X \in \mathcal{T}$ and that the union (the intersection) of an arbitrary (a finite) subfamily of $\mathcal{T}$ belongs to $\mathcal{T}$, too. Thus, we can consider $\mathcal{T}$ as the collection of all open sets for this new topology on X and this new topology will be finer than that defined by the given norm. Consequently, our set A is open in the finer topology and nowhere dense in the coarser one, so that we have also a contribution concerning the comparison of topologies.

R e f e r e n c e

[1] E. HILLE, R.S. PHILLIPS: Functional Analysis and Semigroups, Providence 1957.

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