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MEAN VALUE THEOREMS IN THE THEORY OF LATTICE POINTS WITH WEIGHT II.

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§ 1. Introduction. Let r be a natural number, $r \geqq 2$ and let

$$Q(u) = Q(u_j) = \sum_{j,\ell=1}^{r} a_{j\ell} u_j u_\ell$$

be a positive definite quadratic form with a symmetric matrix of coefficients and determinant D, let $\bar{Q}$ denote the form conjugated with Q. Let further M_j, b_j and α_j be real numbers, $M_j > 0$ ($j = 1, 2, \ldots, r$). Let $0 < \lambda_1 < \lambda_2 < \ldots$ be the sequence of all positive values of the form $Q(m_j M_j + b_j)$ with integer m_1, $m_2, \ldots, m_r$, $\lambda_0 = 0$ and for integer m, $m \geqq 0$, let

$$a_m = \sum e^{2\pi i \sum_{j=1}^{r} \alpha_j u_j},$$

where summation runs over all systems of real numbers $u_1, u_2, \ldots, u_r$ such that $Q(u_j) = \lambda_m$ and $u_j \equiv b_j \pmod{M_j}$, $j = 1, 2, \ldots, r$.

For $x \geqq 0$, $\rho \geqq 0$ put

$$A_\rho(x) = \frac{1}{\Gamma(\rho+1)} \sum_{\lambda_m \leq x} a_m (x - \lambda_m)^\rho, \quad V_\rho(x) = \frac{M e^{2\pi i \sum_{j=1}^{r} \alpha_j b_j} \delta}{\Gamma(\frac{r}{2} + \rho + 1)} x^{\frac{r}{2} + \rho},$$

where $M = \frac{\pi^{k/2}}{\sqrt{D}\prod_{j=1}^{k} M_j}$, $\sigma = 1$ if $\alpha_1 M_1, \alpha_2 M_2, \ldots, \alpha_k M_k$ are integers, $\sigma = 0$ otherwise. Let further

(1) $$P_\varphi(x) = A_\varphi(x) - V_\varphi(x)$$

and

(2) $$M_\varphi(x) = \int_0^x |P_\varphi(t)|^2 dt \ .$$

For $\rho = 0$ we obtain the known "lattice rest" $P_0(x) = P(x)$, studied in a number of papers (cf.e.g. [3],the bibliography in [4],[5] etc.). From the definition of $P_\varphi(x)$ we can easily see that

$$\int_0^x P_\varphi(t)\,dt = P_{\varphi+1}(x) \ .$$

The significance of the study of the function $P_\varphi(x)$ has followed from classical papers by Landau [3]. In the recent paper [2] Jarník pointed the following circumstance:

Let the form Q have integer coefficients, $M_j = 1$, $\alpha_j = b_j = 0$ $(j = 1, 2, \ldots, k)$. Then for $0 \leqq \varphi < \frac{k}{2} - 2$ it is

(3) $$P_\varphi(x) = O(x^{\frac{k}{2}-1}), \quad P_\varphi(x) = \Omega(x^{\frac{k}{2}-1});$$

for $\rho > \frac{k}{2} - \frac{1}{2}$ we have then

(4) $$P_\varphi(x) = O(x^{\frac{k-1}{4}+\varphi/2}), \quad P_\varphi(x) = \Omega(x^{\frac{k-1}{4}+\varphi/2}),$$

while the estimates known for $\frac{\kappa}{2}-2 \leqq \wp \leqq \frac{\kappa}{2}-\frac{1}{2}$ are not definite: between O- and Ω-estimates there is still a gap (cf.[2]). Let us mention only that in the case of $\wp = 0$ this may happen for $\kappa = 2,3,4$ - i.e. we arrive to classical problems of the theory of lattice points.

As it has been proved in [2] both Ω-estimates (3) and (4) are valid (under assumptions brought there) for all $\wp \geq 0$. For $\wp \leqq \frac{\kappa}{2} - \frac{3}{2}$ better Ω-estimate is given by (3), for $\wp \geqq \frac{\kappa}{2} - \frac{3}{2}$ by the estimate (4). For this reason we can imagine that for $\wp = \frac{\kappa}{2} - \frac{3}{2}$ the estimates (3) "turn into" estimates (4).

Certain confirmation of this conjecture is given by the study of the function (2) which is the main object of the present paper. In [6], considered as the first part of this paper, the function (2) is studied for $\wp = 0$. Further results are brought in [8]. So we are giving here a generalization of the results from [6] and partly from [8].

The general method used, i.e. the representation of the function $M_{\wp}(x)$ by a two-fold curvilinear integral as well as exploitation of the transformation of the theta-function is due to Jarník (cf. e.g. [1]).

§ 2. Notations and auxiliary theorems.

If not said explicitly otherwise, we shall preserve throughout the whole paper the following conventions and notations (in addition to those introduced in § 1).

The letter c means (eventually also various) positive constants dependent only on $\wp$, Q, b_j, M_j and α_j , $j = 1, 2, \ldots, r$. A positive constant depending, moreover, for example on ε is denoted by $c(\varepsilon)$ etc . If $|A| \leqq cB$, we write $A \ll B$; if $A \ll B$ and $B \ll A$ simultaneously, we write shortly $A \asymp B$. The symbols O, o, Ω have the usual meaning, that is, they are related to a limit step for $x \to +\infty$ and the constants involved are of the "type" c . Moreover, constants of the "type" $c(\varepsilon)$ are admitted if a positive parameter ε occurs in O relations (and similarly in Ω). We eclude from our considerations the case if $A(x) = 0$ for all x .

$\wp$ denotes a nonnegative number, x a sufficiently large positive number, i.e. $x > c$, m, h, p (indexed as it happens, etc.) denote always some integers, k, n (again possibly indexed) denote natural numbers. If h and k occur simultaneously, it is always $(h,k) = 1$ (the same for h_1, k_1 etc.). By an integral we always mean the (absolutely convergent) Lebesgue integral. For a real let

$$\int_{(a)} f(s)\,ds = i\int_{-\infty}^{\infty} f(a+it)\,dt$$

and (for $x > 0, -\infty \leqq a \leqq b \leqq +\infty$, $I = [a, b)$)

$$\int_I f(s)\,dt = \int_a^b f\left(\frac{1}{x} + it\right)dt \quad ,$$

provided, of course, the integrals on the right hand sides exist.

For a real number t let $\langle t \rangle$ denote the distance of t to the nearest integer, i.e.

$$\langle t \rangle = \min_{p} |t - p| .$$

Let

$$R_k = \min_{m_1, m_2, \dots, m_k} \overline{Q}\left(\frac{m_j}{M_j} - \alpha_j k\right), \quad P_k = \max_{j=1,2,\dots,k} \langle \alpha_j M_j k \rangle .$$

It can be shown easily (cf.[4],Remark 2,p.431) that

$$(5) \qquad R_k \asymp P_k^2 \quad .$$

Let us put further $M_{\varphi}(t) = M_{\varphi,1}(t)$ and define by induction for every n, $t \geq 0$

$$(6) \qquad M_{\varphi,n+1}(t) = \int_0^t M_{\varphi,n}(y)\,dy \quad .$$

(Realize that in [6] the notation differs: $M_2(x)$ from there is according to this definition $M_{0,2}(x)$, etc.)

For s complex, $\mathrm{Re}\, s > 0$. Let

$$(7) \qquad \Theta(s) = \Theta(s;\alpha_j) = \sum_{m=0}^{\infty} a_m e^{-\lambda_m s}$$

and

$$(8) \qquad F(s) = F(s;\alpha_j) = \Theta(s) - \frac{M e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{s^{k/2}}\, \delta \quad ,$$

$$(9) \quad G(s) = \overline{F(\bar{s})} = F(s;-\alpha_j) = \Theta(s;-\alpha_j) - \frac{M e^{-2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{s^{k/2}}\, \delta .$$

(For s complex, $\mathrm{Re}\, s > 0$ and τ positive real, we denote by s^{τ} the branch of the function s^{τ} positive for positive values of s .) The functions (7)-(9) are, as known, holomorphic functions in the half plane

$Re\, s > 0$ and bounded in every domain of the form $Re\, s \geq \varepsilon > 0$.

Let us close this paragraph by several auxiliary assertions.

Lemma 1. Let $a > 0,\ b > 0$. Then

$$(10)\quad M_{\varphi,n}(x) = -\frac{1}{4\pi^2}\int_{(a)}\Big(\int_{(b)}\frac{F(s)\,G(s')\,e^{x(s+s')}}{s^{\varphi+1}s'^{\varphi+1}(s+s')^n}\,ds\Big)ds'$$

Proof. Let first $\varphi > 0$. By direct computation we get for $t > 0$

$$P_\varphi(t) = \frac{1}{2\pi i}\int_{(a)}\frac{F(s)e^{ts}}{s^{\varphi+1}}\,ds$$

and thus

$$|P_\varphi(t)|^2 = -\frac{1}{4\pi^2}\int_{(a)}\Big(\int_{(b)}\frac{F(s)\,G(s')}{s^{\varphi+1}s'^{\varphi+1}}\,e^{t(s+s')^m}ds\Big)ds' .$$

By absolute (and uniform for $0 < t \leq T < +\infty$) convergence of the integral we have for $y > 0$

$$M_\varphi(y) = M_{\varphi,1}(y) = -\frac{1}{4\pi^2}\int_{(a)}\Big(\int_{(b)}\frac{e^{y(s+s')}-1}{s^{\varphi+1}s'^{\varphi+1}(s+s')}F(s)\,G(s')\,ds\Big)ds'.$$

Since the function $F(s)\,G(s')$ is a holomorphic function and bounded in the domain $Re\, s \geq b,\ Re\, s' \geq a$ we can see easily, using the theorem of Cauchy that the value of the integral

$$(11)\qquad \int_{(a)}\Big(\int_{(b)}\frac{F(s)\,G(s')}{s^{\varphi+1}s'^{\varphi+1}(s+s')}\,ds\Big)ds'$$

does not depend on a and b (cf. e.g. [8],Lemma 2,p. 159), therefore the integral (11) can be estimated by the expression

$$c\int_0^\infty\int_0^{t'} \frac{dt\,dt'}{(R+t)(R+t')(2R+t'-t)}$$

for an arbitrary $R > \max(a,b)$ and thus (taking limit for $R \to +\infty$) the integral (11) is equal to zero.

By this our assertion is proved for $\rho > 0,\ n = 1$. For $\rho > 0$ and n arbitrary, we proceed by induction (on each step using the integral in (11) being zero). Validity of (10) also for $\rho = 0$ is now easily obtained having in view that both sides (with x and n fixed) are continuous functions of the variable ρ at the interval $[0, +\infty)$.

Remark 1. This lemma is usually (cf.[1],[6] - [8]) brought in the form containing a member of the form $O(x^{n-1})$. Extending (in an obvious way) the definition of both expressions in (10) for ρ complex, $\mathrm{Re}\,\rho \geq 0$, we obtain two functions of the complex variable ρ, both of them being holomorphic in the half-plane $\mathrm{Re}\,\rho > 0$, continuous for $\mathrm{Re}\,\rho \geq 0$ (continuity with regard to the set $\mathrm{Re}\,\rho \geq 0$ is meant here) and thus the equality (10) is valid also for ρ complex, $\mathrm{Re}\,\rho \geq 0$.

Lemma 2. Let a form Q have integer coefficients and let the numbers M_j be natural, b_j integer, $j = 1, 2, \dots, k$. Then for s complex, $\mathrm{Re}\,s > 0$, it is

$$(12)\quad \Theta(s) = \frac{M}{k^{r}\left(s - \frac{2\pi i h}{k}\right)^{r/2}} \sum_{m_1, m_2, \dots, m_r} S_{h,k,(m)}\, e^{-\frac{\pi^2 \overline{Q}\left(\frac{m_j}{M_j} - \alpha_j k\right)}{k^2\left(s - \frac{2\pi i h}{k}\right)}} ,$$

where

$$S_{h,k,(m)} = S_{h,k,(m_1, m_2, \dots, m_r)} =$$

$$= \sum_{a_1, a_2, \dots, a_r = 1}^{k} e^{-\frac{2\pi i h}{k} Q(a_j M_j + b_j) + \frac{2\pi i}{k} \sum_{j=1}^{r} \frac{m_j}{M_j} (a_j M_j + b_j)} .$$

If $S_{h,k,(m)} \neq 0$, then it is

$$(13)\qquad S_{h,k,(m)} \asymp k^{\frac{r}{2}} .$$

<u>Proof</u>. Cf.[5],Lemma 1,2,8.

For the rest of this paragraph let us assume that the assumptions of Lemma 2 are fulfilled. In [4],pp.430-431, it is proved that there exists a constant $c_1 = c$ such that for $R_k < c_1$ there exists exactly one system $m_1, m_2, \dots, m_r$ with

$$(14)\qquad R_k = \overline{Q}\left(\frac{m_j}{M_j} - \alpha_j k\right) .$$

If further $p_1, p_2, \dots, p_r$ is a system different from the system $m_1, m_2, \dots, m_r$ satisfying (14), then it is $\overline{Q}\left(\frac{p_j}{M_j} - \alpha_j k\right) \geq c_1$.

Now, we can define: if (14) is satisfied by just one system $m_1, m_2, \dots, m_r$, put

$$(15)\qquad S_{h,k} = S_{h,k,(m_1, m_2, \dots, m_r)} .$$

Otherwise we choose one of the systems satisfying (14) (e.g. according to the lexicographic order, to be definite) and define the value $S_{h,k}$ again by (15). (In this case it is necessarily $R_k \geq c_1$).

We shall further call the pair h, k singular, if (14) is satisfied by two distinct systems $m_1, m_2, \ldots, m_r$ (then it is $R_k \geq c_1$), or, if the system of the numbers $m_1, m_2, \ldots, m_r$ satisfying (14) is unique and the value (15) of the sum $S_{h,k}$ is zero. We shall further say the number k to be singular if any pair h, k is singular (all the time it is $(h,k)=1$). Finally, we shall the case we are dealing with, singular, if there exists a constant $c_2 = c$ such that all natural k, for which $R_k < c_2$ are singular [1]. (Remark that we should say more properly that k is singular with regard to $Q, \alpha_j, M_j, b_j, j = 1, 2, \ldots, r$ etc. But since we consider both the form Q and the numbers α_j, b_j and M_j fixed, there is no danger of confusion.) It is clear now what is meant by a non-singular pair, etc.

Lemma 3. [2] Let $s = \frac{1}{x} + it$.

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1) In [8] the singular case has been defined only for rational $\alpha_1, \alpha_2, \ldots, \alpha_r$ by an equivalent requirement: if $R_k = 0$ then $S_{h,k} = 0$ for all h. In this paper there is also the value $S_{h,k}$ defined in a slightly different way. Existence of the singular case is shown in [5], pp. 393-395.

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2) For rational $\alpha_1, \alpha_2, \ldots, \alpha_r$ see [8], Lemma 4, and also Lemma 2 in [6], Lemma 3 in [7].

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a) Let $k \leq \sqrt{x}$, $h \neq 0$, $|t - \frac{2\pi h}{k}| \ll \frac{1}{k\sqrt{x}}$. If $\sigma = 0$, then

$$(16) \qquad F(s) \ll \frac{x^{\frac{k}{2}} e^{-\frac{cR_k x}{k^2(1+x^2|t-\frac{2\pi h}{k}|^2)}}}{k^{k/2}(1+x^2|t-\frac{2\pi h}{k}|^2)^{\frac{k}{4}}}$$

and for a singular pair h, k we have even

$$F(s) \ll \frac{x^{\frac{k}{2}}}{k^{k/2}} \frac{e^{-\frac{cx}{k^2(1+x^2|t-\frac{2\pi h}{k}|^2)}}}{(1+x^2|t-\frac{2\pi h}{k}|^2)^{\frac{k}{4}}} .$$

For $\sigma = 1$ it is

$$F(s) \ll \frac{x^{\frac{k}{2}}}{k^{k/2}(1+x^2|t-\frac{2\pi h}{k}|^2)^{\frac{k}{4}}} .$$

b) For $t \ll x^{-\frac{1}{2}}$ it is

$$(17) \qquad \frac{F(s)}{s^{\varrho+1}} \ll x^{\frac{k}{4}+\frac{\varrho}{2}+\frac{1}{2}} .$$

Analogous statements hold good for the function $G(s)$.

<u>Proof</u>. a) If $\sigma = 0$ we get both estimates immediately by Lemma 2 (cf. e.g. [4],pp.432-434, the relation (36)), since by (8) it is $F(s) = \Theta(s)$. Take now $\sigma = 1$ (i.e., all the numbers $\alpha_1 M_1, \alpha_2 M_2, \ldots, \alpha_r M_r$ are integers and thus $R_k = 0$ for all k). As above, we shall see that the required estimate holds for the function $\Theta(s)$ and the rest is done by the estimate

$$\frac{1}{s^{\frac{k}{2}}} \ll x^{\frac{k}{4}} \ll \frac{x^{\frac{k}{4}}}{k^{\frac{k}{2}}(1+x^2|t-\frac{2\pi h}{k}|^2)^{\frac{k}{4}}} .$$

b) From the relation (12) for $h = 0$, $k = 1$ we obtain immediately for $t \ll x^{-\frac{1}{2}}$

$$\frac{F(s)}{s^{\rho+1}} \ll \frac{x^{\frac{k}{2}+\rho+1}\, e^{-\frac{cx}{1+x^2t^2}}}{(1+x^2t^2)^{\frac{k}{4}+\frac{\rho}{2}+\frac{1}{2}}} \ll x^{\frac{k}{4}+\frac{\rho}{2}+\frac{1}{2}} \quad ,$$

since $\xi^c e^{-c\xi} \ll 1$ for $\xi \in [0, +\infty)$.

Remark 2. The first part of the preceding lemma can obviously be formulated as follows: If $s = \frac{1}{x} + it$, $|t - \frac{2\pi h}{k}| \ll \frac{1}{k\sqrt{x}}$, $h \neq 0$, then (16) holds good and we can write there 1 instead R_k for a singular pair h, k.

Let us bring some further estimates we are going to use in the sequel without any reference. Let $s = \frac{1}{x} + it$, $s' = \frac{1}{x} + it'$. Then

$$\frac{1}{s+s'} \ll \frac{x}{1 + x\,|t+t'|} \quad ,$$

$$e^{x(s+s')} \ll 1$$

and for $|t - \frac{2\pi h}{k}| \ll \frac{1}{k\sqrt{x}}$, $h \neq 0$ it is

$$|s| \asymp |t| \asymp \frac{|h|}{k} \quad .$$

§ 3. The main theorem. In this paragraph we are going to prove the following

Main theorem. Let a quadratic form Q have integer coefficients and let the numbers b_j, M_j, $j = 1, 2, \dots, r$ be integers. Then it is

$$(18)\quad M_\varphi(x) \ll x^{\frac{k}{2}-\frac{1}{2}} \sum_{k \leq \sqrt{x}}' k^{2\varphi+1} \min{}^{\frac{k}{2}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right),$$

where we put $\min(A, \frac{1}{0}) = A$ and Σ' means that for singular k we put 1 instead of R_k.

Let us assume throughout this paragraph that the assumptions of the main theorem are satisfied. Clearly, the relation (18) is a special case (for $n = 1$) of the relation

$$(19)\quad M_{\varphi,n}(x) \ll x^{\frac{k}{2}+n-\frac{3}{2}} \sum_{k \leq \sqrt{x}}' k^{2\varphi+1} \min{}^{\frac{k}{2}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right)$$

(under the same conventions as for the main theorem).

Now,we can show easily that it holds the following

Lemma 4. Let there be some $n = c$ such that (19) holds. Then this relation holds for all n.

Proof. Denote the right side in (19) by $F_n(x)$. Both the function $F_n(x)$ and the function $M_{\varphi,n}(x)$ are non-negative and non-decreasing. Therefore, if (19) holds for a certain $n = c$, then also

$$M_{\varphi,n+1}(x) = \int_0^x M_{\varphi,n}(y)\,dy \leqq x M_{\varphi,n}(x) \ll x F_n(x) = F_{n+1}(x).$$

If (19) holds for some $n = c > 1$, then it is

$$M_{\varphi,n-1}(x) \leqq \frac{1}{3x} \int_x^{4x} M_{\varphi,n-1}(y)\,dy =$$

$$= \frac{1}{3x}\left(M_{\varphi,n}(4x) - M_{\varphi,n}(x)\right) \ll \frac{1}{x} F_n(4x).$$

But now we have

$$F_n(4x) \ll x^{\frac{k}{2}+\frac{3}{2}+n} \sum_{k \leqq \sqrt{x}}' k^{2\rho+1} \min{}^{\frac{k}{2}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right) \ll$$

$$\ll F_n(x) + x^{\frac{k}{2}-\frac{3}{2}+n} \sum_{\sqrt{x} < k \leqq 2\sqrt{x}} k^{2\rho+1} \frac{x^{\frac{k}{2}-\frac{1}{2}}}{k^{k-1}} \ll F_n(x) + x^{\frac{k}{2}+\rho+n-\frac{1}{2}} \ll F_n(x),$$

because of $(R_k \ll 1,\ k \leqq \sqrt{x})$

$$(20) \qquad F_n(x) \gg x^{\frac{k}{2}-\frac{3}{2}+n} \sum_{k \leqq \sqrt{x}} k^{2\rho+1} \asymp x^{\frac{k}{2}+\rho+n-\frac{1}{2}} \quad .$$

Putting together we obtain

$$M_{\rho, n-1}(x) \ll \frac{1}{x} F_n(x) = F_{n-1}(x) .$$

From what has just been proved, the assertion of Lemma follows by an easy induction.

In order to prove the main theorem it remains to prove

Lemma 5. There exists $n = c$ such that (19) holds.

Proof proceeds in two steps like the proof of the main theorem in [6] (pp.720-724), so we can do it more briefly here. For the rest of this paragraph let n be great enough, $n \ll 1$, $s = \frac{1}{x} + it$, $s' = \frac{1}{x} + it'$. Denote

$$H(t, t'; \alpha_j) = \frac{F(s)\, G(s')}{s^{\rho+1} s'^{\rho+1} (s+s')^n} e^{x(s+s')} .$$

Clearly it is

$$H(-t, -t'; \alpha_j) = \overline{H(t, t'; -\alpha_j)}, \quad H(t, t'; \alpha_j) = H(t', t; -\alpha_j) .$$

Hence by Lemma 3 (for $a = b = \frac{1}{x}$) we obtain

$$(21) \qquad M_{\rho, n}(x) \ll T_1 + T_2 + T_3 ,$$

where

$$T_1 = \int_{-2w}^{2w}\int_{-2w}^{2w} \dots dt'\,dt \ ,$$

$$T_2 = \int_{-w}^{w}\int_{2w}^{\infty} \dots dt'\,dt + \int_{-w}^{w}\int_{-\infty}^{-2w} \dots dt'\,dt \ ,$$

$$T_3 = \int_{w}^{\infty}\int_{w}^{\infty} \dots dt'\,dt + \int_{w}^{\infty}\int_{-\infty}^{-2w} \dots dt'\,dt$$

(all the integrands are $|H(t,t';\alpha_j)|$, $w = \frac{2\pi}{[\sqrt{x}]+1} \asymp x^{-\frac{1}{2}}$).
For T_1 we obtain by (17)

$$(22) \quad T_1 \ll x^{\frac{k}{2}+\rho+n} \int_0^w \Big(\int_0^t \frac{x\,dt'}{(1+x(t-t'))^n} \Big) dt \ll x^{\frac{k}{2}+\rho+n-\frac{1}{2}} .$$

For the estimates of T_2 and T_3 we shall need the following easily provable relations (cf. [6],p.721-722)

$$(23) \quad \int_0^{\frac{c}{k\sqrt{x}}} \frac{e^{-\frac{cTx}{k^2(1+x^2u^2)}}}{(1+x^2u^2)^\gamma}\,du \ll \begin{cases} \dfrac{k^{2\gamma-1}}{x^{\gamma+\frac{1}{2}}} \min{}^{\gamma-\frac{1}{2}}\Big(\dfrac{x}{k^2}, \dfrac{1}{T}\Big) & \text{for } \gamma = c > \frac{1}{2} \ , \\ \dfrac{\lg x}{x} & \text{for } \gamma = \frac{1}{2} \ , \end{cases}$$

where $T \geq 0$ and where we put $\min(A, \frac{1}{0}) = A$.

Now, consider the Farey's fractions corresponding to $\sqrt{x}$ i.e. the fractions of the form h/k , where $k \leq \sqrt{x}$ (cf. [3],pp.249-250): For each of these fractions h/k there exist uniquely determined neighbouring Farey's fractions h'/k', h''/k'', $k', k'' \leq \sqrt{x}$, i.e. $h'/k' < h/k < h''/k''$ and between h'/k' and h''/k'' there is exactly one Farey's fraction corresponding to $\sqrt{x}$ – that is h/k. Then, if we denote by $\mathcal{L}_{h,k}$ the interval

$$\left[\, 2\pi \frac{h+h'}{k+k'} \,,\ 2\pi \frac{h+h''}{k+k''} \right)$$

it is

$$\mathcal{L}_{h,k} = \left[2\pi \frac{h}{k} - \frac{\vartheta_1}{k\sqrt{x}} \,,\ 2\pi \frac{h}{k} + \frac{\vartheta_2}{k\sqrt{x}} \right) \quad ,$$

where $\pi \leqq \vartheta_1 , \vartheta_2 \leqq 2\pi$. Thus for $t \in \mathcal{L}_{h,k}$ it holds

$$\left| t - \frac{2\pi h}{k} \right| \ll \frac{1}{k\sqrt{x}} \quad .$$

All the intervals $\mathcal{L}_{h,k}$ are mutually disjoint, and, their union being the whole real axis,

$$\mathcal{L}_{0,1} = \langle -w, w \rangle \, .$$

Let us estimate first T_2. If $t \in \mathcal{L}_{h,k}, h \neq 0, t' \ll w$, then it is $|s|, |s+s'| \gg \frac{|h|}{k}$, therefore, by Lemma 3 we obtain (keeping the convention about R_k according to Remark 2)

$$T_2 \ll \int_0^w \Big(x^{\frac{\kappa}{4}+\frac{\varrho}{2}+\frac{1}{2}} \sum_{k \leqq \sqrt{x}}{}' \sum_{h=1}^{\infty} \left(\frac{k}{h}\right)^{\varrho+m+1} \frac{x^{\frac{\kappa}{2}}}{k^{\kappa/2}} \int_0^{\frac{c}{k\sqrt{x}}} \frac{e^{-\frac{cR_k x}{k^2(1+x^2u^2)}}}{(1+x^2u^2)^{\frac{\kappa}{4}}} du \Big) dt'$$

and thus by (23) ($\lg x$ can be omitted for $\kappa > 2$)

$$(24)\quad T_2 \ll x^{\frac{3\kappa}{4}+\frac{\varrho}{2}} \lg x \sum_{k \leqq \sqrt{x}} k^{\varrho+m-\frac{\kappa}{2}+1} \frac{1}{x} \ll x^{\frac{\kappa}{2}+\varrho+\frac{m}{2}} \lg x \ll x^{\frac{\kappa}{2}+\varrho+m-\frac{1}{2}} .$$

(Note that for both estimates we could assume $m > 1$, $m > \frac{\kappa}{2} - 2 - \varrho$.)

Applying the inequality $|ab| \leqq \frac{1}{2}(|a|^2 + |b|^2)$

to both integrals in T_3 we obtain with regard to the relation $|G(s)| = |F(\bar{s})|$ the estimate

$$T_3 \ll \int_w^\infty \int_w^\infty \frac{|F(s)|^2 + |F(\bar{s})|^2}{(tt')^{\varphi+1}(\frac{1}{x}+|t-t'|)^n}\, dt'dt \tag{25}$$

We shall prove first that for $t \geq w$ it holds

$$\int_w^\infty \frac{dt'}{t'^{\varphi+1}(\frac{1}{x}+|t-t'|)^n} \ll \frac{x^{n-1}}{t^{\varphi+1}} \quad . \tag{26}$$

First we have $(xt \geq w \gg \sqrt{x})$

$$\int_{t-\frac{1}{x}}^\infty \frac{dt'}{t'^{\varphi+1}(\frac{1}{x}+|t-t'|)^n} \ll \frac{1}{t^{\varphi+1}} \int_{-\infty}^\infty \frac{du}{(\frac{1}{x}+|u|)^n} \ll \frac{x^{n-1}}{t^{\varphi+1}} \quad ,$$

so it is enough to estimate for $t-\frac{1}{x} > w$ the integral

$$I = \int_w^{t-\frac{1}{x}} \frac{dt'}{t'^{\varphi+1}(\frac{1}{x}+t-t')^n} \ll x^{n-\varphi-2} \int_w^{t-\frac{1}{x}} \frac{dt'}{t'^{\varphi+1}(t-t')^{\varphi+2}}$$

(here it suffices to assume $n > \varphi + 2$). Now, we can find easily that the function $t'(t-t')$ has, for $t' \in [w, t-\frac{1}{x}]$, the minimum either for $t' = w$ or $t' = t-\frac{1}{x}$ and it is, therefore, greater than $c\, t/x$. From this it follows

$$I \ll x^{n-\varphi-2}\left(\frac{x}{t}\right)^\varphi \int_w^{t-\frac{1}{x}} \frac{dt'}{t'(t-t')^2} \quad .$$

The remaining integral can now be estimated by the expression $c\,\frac{x}{t}$ as easily checked by a direct computation (cf.[7], p.617 - in the formula (31) of [6] the sign in the last row is erroneous). This proves the relation (26).

From (25) and (26 we obtain

$$T_3 \ll x^{n-1} \int_{w}^{\infty} \frac{|F(s)|^2 + |F(\bar{s})|^2}{t^{2(\rho+1)}} \, dt \, .$$

The integration path will again be decomposed into intervals $\mathcal{L}_{h,k}$ and Lemma 3 will be used in each of them (if $t \in \mathcal{L}_{h,k}$ then $|t - \frac{2\pi h}{k}| \ll \frac{1}{k\sqrt{x}}$). Using (23) we obtain

$$T_3 \ll x^{\kappa+m-1} \sum_{k \leq \sqrt{x}}' \sum_{h=1}^{\infty} \frac{k^{2(\rho+1)}}{h^{2(\rho+1)}} \frac{1}{k^{\kappa}} \int_0^{c/k\sqrt{x}} \frac{e^{-\frac{cR_k x}{k^2(1+x^2u^2)}}}{(1+x^2u^2)^{\kappa}} \, du \ll$$

$$\ll x^{\frac{\kappa}{2}+m-\frac{3}{2}} \sum_{k \leq \sqrt{x}}' k^{2\rho+1} \min^{\frac{\kappa-1}{2}} \left(\frac{x}{k^2} \, , \, \frac{1}{R_k} \right) .$$

According to (20) the right side can be estimated from below by the expression $c\, x^{\frac{\kappa}{2}+m+\rho-\frac{1}{2}}$ which, having in view (21),(22) and (24), finishes the proof of the lemma.

§ 4. <u>General Ω -estimate.</u> In this paragraph we do not impose any conditions neither on the form Q nor on the numbers $M_j > 0$, b_j and α_j, $j = 1, 2, \dots, \kappa$ (except $A(x) \not\equiv 0$).

<u>Theorem 1.</u>

$$M_{\rho}(x) \gg x^{\frac{\kappa}{2}+\rho+\frac{1}{2}} \, .$$

From this theorem it follows immediately

Theorem 2.

$$P_\varphi(x) = \Omega(x^{\frac{k-1}{4}+\varphi/2}) .$$

Since we have

$$\int_0^x |P_\varphi(t)|\,dt \leq \sqrt{x\, M_\varphi(x)}$$

it suffices to prove the next stronger assertion:

Lemma 5. If $\operatorname{Re} A(x) \not\equiv 0$ then it is

$$\int_0^x \max(0, \operatorname{Re} P_\varphi(t))\,dt >> x^{\frac{k}{2}+\frac{\varphi}{2}+\frac{3}{4}} ,$$

$$\int_0^x \max(0, -\operatorname{Re} P_\varphi(t)\,dt >> x^{\frac{k}{4}+\frac{\varphi}{2}+\frac{3}{4}} .$$

Analogous statements are true for imaginary part, too.

Proof will be carried out on the basis of Landau's identity in a manner brought in some special cases by Jarník in [1],[2]. As known (cf. e.g. [3],pp.226, 8 , p.178), it is for $\varphi > \frac{k}{2}$

$$(27)\quad P_\varphi(x) = \frac{M e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\pi^{\varphi+\frac{k}{2}}}\, x^{\frac{k}{4}+\frac{\varphi}{2}} \sum_{n=1}^{\infty} b'_n \,\frac{J_{\frac{k}{2}+\varphi}(2\pi\sqrt{\lambda'_n x})}{{\lambda'_n}^{\frac{k}{4}+\frac{\varphi}{2}}} ,$$

where $J_{\frac{k}{2}+\varphi}(z)$ is Bessel function of the I^{st} kind, $0 < \lambda'_1 < \lambda'_2 < \ldots$ is a sequence of all positive numbers of the form $\overline{Q}(\frac{m_j}{M_j} - \alpha_j)$ and

$$b'_n = \sum e^{2\pi i \sum_{j=1}^{k} b_j (\frac{m_j}{M_j} - \alpha_j)}$$

(summing up over all systems $m_1, m_2, \ldots, m_k$ with $\overline{Q}(\frac{m_j}{M_j} - \alpha_j) = \lambda'_n$). Let us remark that this formula follows immediately from the expression $(a > 0, \rho > 0)$

$$P_\rho(x) = \frac{1}{2\pi i} \int_{(a)} \frac{e^{xs}\,\Theta(s)}{s^{\rho+1}}\, ds$$

using the transformation from Lemma 2 for $h = 0$, $k = 1$ (it is valid - in this case - even without any assumptions about Q, M_j and b_j) and having the order of integration and summation interchanged (for $\rho > \frac{k}{2}$).

Let now $w = e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}$, $\operatorname{Re} A(x) \not\equiv 0$ and let m be the least index such that $\operatorname{Re} a_m \neq 0$. From this we can easily determine $\operatorname{Re} A_\rho(x)$, $\operatorname{Re} P_\rho(x)$ in the interval $[\lambda_m, \lambda_{m+1})$ and from the form of $\operatorname{Re} P_\rho(x)$ it follows immediately that $\operatorname{Re} P_\rho(x) \not\equiv 0$. According to (17) there exists, therefore, the least index - let us denote it n - for which it is $\operatorname{Re} \omega b_n' \neq 0$ (hence $\operatorname{Re} \omega b_m' = 0$ for $m = 1, 2, \ldots, n-1$). Taking into consideration that, as known, it holds

$$\sum_{\lambda_m' \le x} |b_m'| \ll x^{\frac{k}{2}},$$

hence the series

$$\sum_{m=1}^{\infty} \frac{|b_m'|}{\lambda_m'^{\,t}}$$

converges for $t > \frac{k}{2}$, we obtain from (27) with aid of the known relation

$$J_{\frac{k}{2}+\rho}(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \pi\left(\rho/2 + k/4 + \frac{1}{2}\right)\right) + O\left(\frac{1}{z^{3/2}}\right)$$

(for $z \to +\infty$), $d_\rho = \rho/2 + k/4 + 1/2$

$$P_\rho(x) = \frac{M\omega}{\pi^{\frac{k}{2}+\rho+1}}\, x^{\frac{k-1}{4}+\frac{\rho}{2}} \sum_{m=1}^{\infty} \frac{b_m'}{\lambda_m'^{\frac{k+1}{4}+\frac{\rho}{2}}} \cos(2\pi\sqrt{\lambda_m' x} - \pi d_\rho) + O(x^{\frac{k-3}{4}+\frac{\rho}{2}})$$

for $\rho > \frac{\kappa}{2}$. Choose now $\rho_0 = c > \frac{\kappa}{2}$ so that for $\rho \geq \rho_0$ it is

$$\frac{|\mathrm{Re}\,\omega b'_n|}{\lambda_n'^{\rho/2+\frac{\kappa+1}{4}}} > 2\sum_{m=n+1}^{\infty}\frac{|\mathrm{Re}\,\omega b'_m|}{\lambda_m'^{\rho/2+\frac{\kappa+1}{4}}} .$$

For a natural m, $\rho \geq \rho_0$ let $(j = 1, 2)$

$$x_{j,\rho}(m) = \frac{1}{\lambda'_n}\left(m + \frac{d_\rho + j - 1}{2}\right)^2 .$$

Therefore

$$\cos\pi\left(2\sqrt{\lambda'_n x_{j,\rho}(m)} - d_\rho\right) = (-1)^{j+1}$$

and, according to the choice of ρ_0 for $m > c$ also

(28) $$(-1)^{j+1}\mathrm{Re}\,P_\rho(x_{j,\rho}(m)) >> x_{j,\rho}^{\frac{\kappa-1}{4}+\frac{\rho}{2}}(m) >> m^{\frac{\kappa-1}{2}+\rho} .$$

Thus we have proved that for $\rho \geq \rho_0$, $m > c$, $j = 1, 2$ there exist numbers $x_{j,\rho}(m)$ such that (28) holds and

(29) $$x_{j,\rho}(m) = \frac{m^2}{\lambda'_n} + O(m)$$

(for $m \to +\infty$). At the same time

(30) $$x_{1,\rho}(m) < x_{2,\rho}(m) < x_{1,\rho}(m+1) .$$

For the proof it suffices to show (for $\rho \geq 1$) that there exist numbers $x_{j,\rho-1}(m)$ (for $m > c, j = 1, 2$) satisfying analogous conditions. Because of $(m > c)$

$$\int_{x_{2,\rho}(m)}^{x_{1,\rho}(m+1)} \mathrm{Re}\,P_{\rho-1}(t)\,dt = \mathrm{Re}\,P_\rho(x_{1,\rho}(m+1)) - \mathrm{Re}\,P_\rho(x_{2,\rho}(m)) >> m^{\frac{\kappa+1}{2}-\rho}$$

and

$$-\int_{x_{1,\rho}(m)}^{x_{2,\rho}(m)} \mathrm{Re}\,P_{\rho-1}(t)\,dt = -\mathrm{Re}\,P_\rho(x_{2,\rho}(m)) + \mathrm{Re}\,P_\rho(x_{1,\rho}(m)) >> m^{\frac{\kappa-1}{2}+\rho}$$

and since the length of the integration path is $O(m)$ $(m \to +\infty)$ there exist (for $m > c$) the numbers

$$x_{1,\rho-1}(m) \quad \text{and} \quad x_{2,\rho-1}(m)$$

such that

$$x_{2,\rho}(m) < x_{1,\rho-1}(m) < x_{1,\rho}(m+1) ,$$

$$x_{1,\rho}(m) < x_{2,\rho-1}(m) < x_{2,\rho}(m)$$

and

$$(-1)^{j+1} \operatorname{Re} P_{\rho-1}(x_{j,\rho-1}(m)) >> m^{\rho-1+\frac{k-1}{2}}$$

$(j = 1, 2)$. The established values meet all our requirements.

For each $\rho \geqq 0$ and for all $m > c$ there exist numbers $x_{j,\rho}(m)$, $j = 1, 2$ such that (28),(29) and (30) hold [3]. But, according to (28) it is

$$\int_0^x \max(0, \operatorname{Re} P_\rho(t))dt \geqq \sum_{x_{1,\rho+1}(m) \leqq x} \int_{x_{2,\rho+1}(m)}^{x_{1,\rho+1}(m+1)} \operatorname{Re} P_\rho(t)dt >>$$

$$>> \sum_{1 \ll m \ll \sqrt{x}} m^{\rho+1+\frac{k-1}{2}} >> x^{\frac{k+3}{4}+\rho/2}$$

and analogously for the other integral.

<u>Remark 3.</u> Theorem 1 for $\rho = 0$ can be found in [7],Theorem 2 for an arbitrary $\rho \geqq 0$ and for $\alpha_j = b_j = 0$, $M_j = 1$, $j = 1, 2, .., k$ can be found in [2]. From the proof of Lemma 5 it follows, by the way, a stronger assertion: If $\operatorname{Re} A(x) \not\equiv 0$, $\rho \geqq 0$, then for all

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3) Note that this proves directly Theorem 2.

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natural $m > c$ there exist numbers $x_{j,\rho}(m)$, $j = 1, 2$ such that (28) to (30) hold. Analogously for the imaginary part.

§ 5. Consequences of the Main Theorem. In this last paragraph let the form Q have integer coefficients, let the numbers M_j be natural, b_j integers $(j = 1, 2, \dots, k)$. From the Main Theorem it follows immediately

Theorem 3.

$$x^{\frac{k}{2}+\rho+\frac{1}{2}} << M_\rho(x) << \begin{cases} x^{k-1} & \text{for } 0 \leq \rho < \frac{k}{2} - \frac{3}{2}, \\ x^{k-1} \lg x & \text{for } \rho = \frac{k}{2} - \frac{3}{2} \geq 0, \\ x^{\frac{k}{2}+\rho+\frac{1}{2}} & \text{for } \rho > \frac{k}{2} - \frac{3}{2}, \rho \geq 0. \end{cases}$$

Proof. The first half is Theorem 1; for the proof of the second half remind that by the Main Theorem it is

$$M_\rho(x) << x^{\frac{k}{2}-\frac{1}{2}} \sum_{k \leq \sqrt{x}} k^{2\rho+1} \left(\frac{x}{k^2}\right)^{\frac{k}{2}-\frac{1}{2}} = x^{k-1} \sum_{k \leq \sqrt{x}} k^{2\rho-k+2},$$

so we have only in each case to estimate the remaining sum in an obvious way.

Theorem 4. In the singular case it is for $\rho \geq 0$

$$M_\rho(x) \asymp x^{\frac{k}{2}+\rho+\frac{1}{2}}.$$

Proof. With regard to Theorem 1 we are only to prove the upper estimate. But, by the Main Theorem, we have in the singular case (1 written instead of R_k)

$$M_\rho(x) << x^{\frac{k}{2}-\frac{1}{2}} \sum_{k \leq \sqrt{x}} k^{2\rho+1} << x^{\frac{k}{2}+\rho+\frac{1}{2}}.$$

Remark 4. Theorem 3 shows that the lower estimate by Theorem 1 cannot be further improved for $\rho > \frac{\kappa}{2} - \frac{3}{2}$, $\rho \geq 0$; by Theorem 4 it follows that it is generally unimprovable for all $\rho \geq 0$. In [9] it is shown that, if the numbers $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ are rational and the case is not singular, then there exists a constant $K = c$ such that

$$M_\rho(x) = Kx^{\kappa-1} + o(x^{\kappa+1})$$

for $0 \leq \rho < \frac{\kappa}{2} - \frac{3}{2}$,

$$M_\rho(x) = Kx^{\kappa-1} \lg x + c(x^{\kappa-1} \lg x)$$

for $\rho = \frac{\kappa}{2} - \frac{3}{2} \geq 0$. It follows that generally the upper estimates by Theorem 3 cannot be improved for $0 \leq \rho \leq \frac{\kappa}{2} - \frac{3}{2}$ as well.

For derivation of further theorems we shall use the results from [10]. For $t \geq 0$, $\beta \geq 0$ let

$$F(x) = F_{t,\beta}(x) = \sum_{k \leq \sqrt{x}} k^t \min^\beta \left(\frac{\sqrt{x}}{k}, \frac{1}{P_k} \right).$$

According to (5) and (18) we have then

$$M_\rho(x) \ll x^{\frac{\kappa-1}{2}} F_{2\rho+1,\kappa-1}(x) \quad . \tag{31}$$

Let us gather the results on the function $F(x)$ in the following

Lemma 6. a) If $t < \beta - 1$ and if at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ is irrational, then

$$F(x) = o(x^{t/2}) \; ,$$

b) Let $t < \beta - 1$, $\beta - t \leq \kappa - 1$. Then for almost all the systems $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ (in the sense of the κ-dimensional Lebesgue measure) it is

$$F(x) \ll x^{\frac{t+1}{2}} \lg^{\tau} x ,$$

where $\tau = 3\kappa - 1$ for $\beta < \kappa - 1 + t$, $\tau = 3\kappa + 2$ for $\beta = \kappa - 1 + t$.

c) Let $t < \beta - 1$, $\gamma > 0$ and let the inequality

(32) $$P_k \gg k^{-\gamma}$$

be fulfilled for all k. Then

$$F(x) \ll x^{\frac{\beta\gamma + t + 1}{2(\gamma+1)}} .$$

d) Let $\alpha_1 = \alpha_2 = \dots = \alpha_\kappa = \alpha$, $t < \beta - 1$, $\gamma > 0$ and let for all k be

(33) $$\langle \alpha k \rangle \gg k^{-\gamma} .$$

Then

$$F(x) \ll \begin{cases} x^{\frac{\beta\gamma+t}{2(\gamma+1)}} & \text{for } t < \beta - 2 , \\ x^{\frac{\beta\gamma+t}{2(\gamma+1)}} + x^{\frac{t+1}{2}} \lg x & \text{for } t = \beta - 2 , \\ x^{\frac{\beta\gamma+t}{2(\gamma+1)}} + x^{\frac{t+1}{2}} & \text{for } t > \beta - 2 . \end{cases}$$

Proof. See [10], Theorems 3 - 6, 7.

On the basis of this lemma we obtain from (31) the following theorems:

Theorem 5. Let $0 \leq \rho < \frac{\kappa}{2} - \frac{3}{2}$. If at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ is irrational, then

$$M_\rho(x) = o(x^{\kappa-1}) .$$

For almost all the systems $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ (in the sense of r -dimensional Lebesgue measure) it is

$$M_\varphi(x) \ll x^{\frac{r}{2}+\varphi+\frac{1}{2}} \lg^{3r-1} x \quad .$$

<u>Theorem 6.</u> Let $0 \leq \varphi < \frac{r}{2} - \frac{3}{2}$, $\gamma > 0$ and let (32) be true for all k . Then it is

$$M_\varphi(x) \ll x^{\frac{r-1}{2}\frac{2\gamma+1}{\gamma+1} + \frac{\varphi+1}{\gamma+1}} \quad .$$

<u>Theorem 7.</u> Let $0 \leq \varphi < \frac{r}{2} - \frac{3}{2}$, $\gamma > 0$, $\alpha_1 = \alpha_2 = \dots = \alpha_r = \alpha$ and let (33) be true for all k . Then it is

$$M_\varphi(x) \ll \begin{cases} x^{\frac{r-1}{2}\frac{2\gamma+1}{\gamma+1} + \frac{2\varphi+1}{2(\gamma+1)}} & \text{for } 0 \leq \varphi < \frac{r}{2} - 2 \ , \\ x^{\frac{r-1}{2}\frac{2\gamma+1}{\gamma+1} + \frac{2\varphi+1}{2(\gamma+1)}} + x^{\frac{r}{2}+\varphi+\frac{1}{2}} \lg x & \text{for } \varphi = \frac{r}{2} - 2 \geq 0 \ , \\ x^{\frac{r-1}{2}\frac{2\gamma+1}{\gamma+1} + \frac{2\varphi+1}{2(\gamma+1)}} + x^{\frac{r}{2}+\varphi+\frac{1}{2}} & \text{for } \varphi > \frac{r}{2} - 2, \ \varphi \geq 0 \ . \end{cases}$$

<u>Remark 5.</u> As it is proved in [9], the σ -estimate by Theorem 5 cannot be generally improved.

In [12], Theorem 3, there is proved the following - estimate: Let γ be the supremum of all the numbers $\beta > 0$ such that there exists a sequence of nonsingular pairs $h = h_m, k = k_m$ such that $\lim_{m \to \infty} k_m = +\infty$, $h_m \ll 1$, and

$$R_k \ll k^{-2\beta} \tag{34}$$

(or equivalently $P_k \ll k^{-\beta}$ by (5)). Then for $0 \leq \varphi \leq \frac{r}{2} - 1$ it is for every $\varepsilon > 0$

$$M_\rho(x) = \Omega(x^{\frac{\kappa-1}{2}\frac{2\gamma+1}{\gamma+1} + \frac{2\rho+1}{2(\gamma+1)} - \varepsilon})$$

(if $\gamma = +\infty$, then the exponent is $\kappa - 1 - \varepsilon$). As stated in [12] (it follows, by the way, from Lemma 3,p.392 in [5]), if $b_1 = b_2 = \ldots = b_\kappa = 0$, then γ can be defined as the supremum of those β for which (34) is fulfilled by an infinite number of k.

Hence the result of Theorem 7 is, in general, the final one, while in Theorem 6 there is a certain gap left. The assumption of the just formulated assertion generally cannot be omitted. In [5],pp.393-399 there is, for a given number $\beta > 0$ constructed a form Q and systems of numbers M_j, α_j, b_j, $j = 1, 2, \ldots, \kappa$ such that the inequality (34) is fulfilled for an infinite number of k and - in our terminology - the singular case occurs. By Theorem 4 it is $M_\rho(x) \asymp x^{\frac{\kappa}{2} + \rho + \frac{1}{2}}$ (the construction can be also easily modified so that the inequality (34) will be now, for every $\beta > 0$ fulfilled for infinitely many k).

From this and from Theorems 6 and 7 it follows the final result, formulated as

<u>Theorem 8.</u> Let $0 \leq \rho < \frac{\kappa}{2} - \frac{3}{2}$, $\alpha_1 = \alpha_2 = \ldots = \alpha_\kappa = \alpha$, $b_1 = b_2 = \ldots = b_\kappa = 0$ and let γ be the infimum of all numbers β, for which the inequality (33) is satisfied for all k. Then

$$\limsup_{x \to +\infty} \frac{\lg M_\rho(x)}{\lg x} = \max\left(\frac{\kappa-1}{2}\,\frac{2\gamma+1}{\gamma+1} + \frac{2\rho+1}{2(\gamma+1)}, \frac{\kappa}{2} + \rho + \frac{1}{2}\right).$$

Remark 6. In [11] it is shown that for $\kappa > 4$ it holds

$$(35)\qquad P_\varphi(x) \ll x^{\frac{\kappa}{4}-\frac{1}{2}} \sum_{k \leq \sqrt{x}}' k^\varphi \min{}^{\frac{\kappa}{4}-\frac{1}{2}}\left(\frac{x}{k^2}, \frac{1}{R_k}\right),$$

where for $\varphi = 0$ k^φ is replaced by $\lg 2k$. From this (on the basis of Lemma 6) there are, alike as above, derived O-estimates of the function $P_\varphi(x)$. But from the relation (35) we cannot obtain such a number of final results as in the present paper from the relation (18). In the singular case we have for $\varphi > 0$ only $O(x^{\frac{\kappa}{4}+\frac{\varphi}{2}})$, which is for $1/4$ worse than Ω-estimate by Theorem 2. Analogously we obtain from (35) the estimate $O(x^{\frac{\kappa}{2}-1})$ only for $0 \leq \varphi < \frac{\kappa}{2}-2$, etc. For comparison let us say that, under assumptions of Theorem 7, the relation $(0 \leq \varphi < \frac{\kappa}{2}-2)$ (cf. [11]) can be proved.

$$\overline{\lim_{x\to+\infty}} \frac{\lg |P_\varphi(x)|}{\lg x} = \left(\frac{\kappa}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1} + \frac{\varphi}{2(\gamma+1)}$$

only for $\gamma > \dfrac{1}{\frac{\kappa}{2}-\varphi-2}$. For $\gamma \leq \dfrac{1}{\frac{\kappa}{2}-\varphi-2}$ we obtain only the inequality

$$\frac{\kappa}{4}-\frac{1}{4}+\frac{\varphi}{2} \leq \overline{\lim_{x\to+\infty}} \frac{\lg|P_\varphi(x)|}{\lg x} \leq \frac{\kappa}{4}+\varphi/2 .$$

References

[1] V. JARNÍK: Über die Mittelwertsätze der Gitterpunktlehre I,II. Math.Zeitschrift 33(1931), 62-97; III, Math.Zeitschrift 36(1933),

581 -617; V, Časopis pro pěstování matematiky 69(1940),148-174.

[2] V. JARNÍK: Bemerkungen zu Landauschen Methoden in der Gitterpunktlehre, Abhandlungen aus Zahlentheorie und Analysis zur Erinnerung an E. Landau. Berlin,1968.

[3] E. LANDAU: Ausgewählte Abhandlungen zur Gitterpunktlehre. Berlin, 1962.

[4] B. NOVÁK: Verallgemeinerung eines Peterssonschen Satzes und Gitterpunkte mit Gewichten, Acta Arithmetica XIII(1968),423-454.

[5] B. NOVÁK: On lattice points with weight in high dimensional ellipsoids. Acta Arithmetica XIV(1968),371-397.

[6] B. NOVÁK: Mean value theorems in the theory of lattice points with weight,Comm.Math.Univ. Carolinae 8(1967),711-733.

[7] B. NOVÁK: Über Gitterpunkte mit Gewichten in mehrdi--mensionalen Ellipsoiden: Mittelwertsätze. Czech.Math.Journal 92(1967),609-623.

[8] B. NOVÁK: Mittelwertsätze der Gitterpunktlehre. Czech. Math.Journal 94(1969),154-180.

[9] B. NOVÁK: Mittelwertsätze der Gitterpunktlehre,II. Czech.Math.Journal - to appear.

[10] B. NOVÁK: Über eine Summe in der Zahlentheorie. Czech. Math.Journal - to appear.

[11] B. NOVÁK: Über Gitterpunkte in mehrdimensionalen Ellipsoiden. Czech.Math.Journal - to appear.

[12] B. NOVÁK: Über eine Methode der Ω-Abschätzungen. Czech.Math.Journal - to appear.

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