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FACTOR-SPLITTING ABELIAN GROUPS OF RANK TWO

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In this paper we shall give a structural description of factor-splitting torsion free abelian groups of rank two.

Throughout this paper by a group it is always meant an additively written abelian group. A torsion free group G is called factor-splitting if any its factor-group G/H splits (see [3]). We shall use the following notation: If g is an element of infinite order of a mixed group G then $h_p(g)$ denotes the p-height of g in the group G (see [2]). $\{H\}_*^G$ denotes the pure closure of a subgroup H in the torsion free group G. Instead $\{\{h\}\}_*^G$, $h \in G$ we shall write simply $\{h\}_*^G$. R_p will denote the group of rationals with denominators prime to p. Other notation will be essentially that as in [1].

It will be useful to formulate the following statement (see Theorem 2 from [2]):

Let G be a mixed group of torsion free rank one. Two following conditions are necessary and sufficient for G to be split:

(α) If T is the maximal torsion subgroup of G, then to any $g \in G \dot{-} T$ there exists an integer $m \neq 0$ such that mg has in G the same type as $g+T$ in G/T .

(β) To any $g \in G \dot{-} T$ there exists an integer $m \neq 0$ such that for any prime p with $h_p(g+T) = \infty$ there exist the elements $h_0^{(p)} = mg, h_1^{(p)}, h_2^{(p)}, \dots,$ such that $p h_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, 2, \dots$.

Now we are ready to prove the main result:

Theorem 1: A torsion-free group G of rank two is factor-splitting if and only if:

(1) For any two linearly independent elements $g, h \in G$ there is $(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p = G \otimes R_p$ for almost all primes p with $h_p(g) \neq h_p(h)$.

Proof: Proving the necessity let us suppose that there exist elements g, $h \in G$ which do not satisfy the condition (1). Without loss of generality we can assume that there exists an infinite set Π' of primes with $h_p(g) < h_p(h)$ and $(\{g\}_*^G + \{h\}_*^G) \otimes R_p \subsetneqq G \otimes R_p$. For any prime $p \in \Pi'$ there is $h_p(h) < \infty$ (in the other case it is easy to see that $(\{g\}_*^G + \{h\}_*^G) \otimes R_p = G \otimes R_p$). Let us denote $\ell_p = h_p(h) - h_p(g)$ and let h'_p be the solution of the equations $p^{\ell_p} x = h$.

In view of $(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p \subsetneqq G \otimes R_p$ there exist elements g_p and non-zero integers α_p with $p^{h_p(g)+1} g_p = g + \alpha_p h'_p$. Hence $h_p(g + \{h\}) = h_p(g)$

but $h_p(g+\{h\}_*^G) \geq h_p(g)+1$ such that $G/\{h\}$ does not satisfy the condition (α) and hence does not split.

Now we shall prove the sufficiency. In view of Lemma 2.6 from [4] it suffices to prove that for any $h \in G$ the factor-group $G/\{h\}$ splits. Let $g \in G \dot{-} \{h\}_*^G$ be an arbitrary element. Let

$\Pi_1 = \{p;\ h_p(g) = h_p(h)\}$,

$\Pi_2 = \{p;$ either $h_p(g) > h_p(h)$ or $h_p(g) < h_p(h)$ and

$(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p = G \otimes R_p\}$,

$\Pi_3 = \{p,\ h_p(g) < h_p(h)$ and $(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p \neq G \otimes R_p\}$.

Then Π_1, Π_2, Π_3 are disjoint subsets of the set Π of all primes whose union is Π. The set Π_3 is finite by hypothesis and it was mentioned in the proof of necessity that $h_p(h) < \infty$. Let us put

(2) $$m = \prod_{p \in \Pi_3} p^{h_p(h) - h_p(g)}$$

Now we are going to prove that

(3) $$h_p(mg + \{h\}) = h_p(mg + \{h\}_*^G)$$

holds for any prime p. For $p \in \Pi_1$ we can assume $h_p(g) < \infty$ (if $h_p(g) = h_p(h) = \infty$ then (3) holds evidently). Suppose that the equation $p^{h_p(g)+k}x = g + h'$ is solvable in G where $\rho h = \sigma h'$ for suitable relatively prime integers ρ, σ. Then $(\sigma, p) = 1$ (in the other case there is $h_p(h') < h_p(g)$ and the given equation has no solution). For suitable inte-

gers κ, s there is $\sigma\kappa + p^{h_p(g)+k} s = 1$ and it holds $p^{h_p(g)+k}(\sigma\kappa x + s g) = g + \varrho\kappa h$.
Hence

$$(4) \qquad h_p(g + \{h\}) = h_p(g + \{h\}_*^G).$$

In view of $(p, m) = 1$ the formula (3) is valid, too.

Similar calculations show that (3) holds also in the case $p \in \Pi_2$, $h_p(g) > h_p(h)$ and in the case $p \in \Pi_3$. Finally, let $p \in \Pi_2$, $h_p(g) < h_p(h)$ and $(\{g\}_*^G \dotplus \{h\}_*^G) \otimes R_p = G \otimes R_p$. For $h_p(h) = \infty$ it holds (4) and hence (3) evidently. Suppose that $h_p(h) < \infty$. Let the equation $p^k x = g + h'$, $h' \in \{h\}_*^G$ have a solution in G. In G there exists an element g' with $p^{h_p(g)} g' = g$. It is easy to see that any element of $\{g\}_*^G \otimes R_p$ is of the form $g' \otimes \alpha$, $\alpha \in R_p$. Now we have $p^k(x \otimes 1) = p^k x \otimes 1 = g \otimes 1 + h' \otimes 1$. By hypothesis there exists an element $g' \otimes \alpha$ in $\{g\}_*^G \otimes R_p$ for which $p^k(g' \otimes \alpha) = g \otimes 1 = g' \otimes p^{h_p(g)}$. Hence $g' \otimes (p^k \alpha - p^{h_p(g)}) = 0$ and then $p^k \alpha = p^{h_p(g)}$, which implies $k \le h_p(g)$. We have shown

$$(5) \qquad \begin{aligned} & h_p(g) < h_p(h),\ (\{g\}_*^G \dotplus \{h\}_*^G) \otimes R_p = G \otimes R_p \Longrightarrow \\ & \Longrightarrow h_p(g + \{h\}_*^G) = h_p(g). \end{aligned}$$

Now it is easy to derive the validity of (3) which shows that the condition (α) is satisfied.

Now we are proceeding to the condition (β). Suppose that $h_p(g + \{h\}_*^G) = \infty$. At first, let $p \in \Pi$ be such a prime that $h_p(g) \ge h_p(h)$. Then there exists a p-adic integer $\mathfrak{A} = (a^{(k)})$ with $p^k x_k = g + a^{(k)} h$ solvable in G for any $k = 1, 2, \dots$ (see [5]). Hence $p^k(p x_{k+1} - x_k) = (a^{(k+1)} - a^{(k)}) h$ such that $p(x_{k+1} + \{h\}) = x_k + \{h\}$. If m is defined by (2) then clearly the same holds for mg and $m x_k$.

In the case of $h_p(g) < h_p(h)$ and $(\{g\}_*^G + \{h\}_*^G) \otimes R_p = G \otimes R_p$ there is $h_p(g + \{h\}_*^G) = h_p(g) < \infty$ by (5) and hence there is nothing to prove. Finally, for $p \in \Pi_3$ there is $h_p(mg) = h_p(h)$ and it suffices to repeat the first part for mg and h. Hence the condition (β) is also satisfied which finishes the proof of our Theorem.

<u>Theorem 2</u>: Any homogeneous torsion free group of rank two is factor-splitting.

<u>Proof</u>: The condition (1) is clearly satisfied in this case.

The following Theorem shows that there is a great variety of factor-splitting torsion free groups of rank two. For any subset $\Pi' \subset \Pi$ we shall define the group $R_{\Pi'}$ as the group of all rationals with denominators relatively prime to any $p \in \Pi'$.

<u>Theorem 3</u>: Let Π_1, Π_2 be disjoint subsets of Π such that $\Pi \dot{-} (\Pi_1 \cup \Pi_2)$ is finite and let G be a torsion free group of rank two.

If $G \otimes R_{\Pi_1}$ is completely decomposable and

$G \otimes R_{\Pi_2}$ homogeneous then G is factor-splitting.

Proof: If Π' is any set of primes, then

(6) $h_p(g) = h_p(g \otimes 1)$, $p \in \Pi'$ and the second height is meant in $G \otimes R_{\Pi'}$.

Clearly, $h_p(g) \leq h_p(g \otimes 1)$. On the other hand let $p^\ell (\sum_{i=1}^{m} g_i \otimes \frac{k_i s}{s_i}) = g \otimes 1$, $(s_i, p) = 1$. If we put $s = s_1 \cdot s_2 \cdot \ldots \cdot s_m$ we have $(s, p) = 1$ and

$s \cdot p^\ell (\sum_{i=1}^{m} g_i \otimes \frac{k_i}{s_i}) = p^\ell (\sum_{i=1}^{m} \frac{k_i s}{s_i} g_i) \otimes 1 = sg \otimes 1$, therefore $p^\ell \sum_{i=1}^{m} \frac{k_i s}{s_i} g_i = sg$ and hence the equation $p^\ell x = g$ is solvable in G .

Now let g, h be any two linearly independent elements from G . Then in view of homogeneity of $G \otimes R_{\Pi_2}$ and (6) it holds $h_p(g) = h_p(h)$ for almost all primes $p \in \Pi_2$. Suppose that $p \in \Pi_1$, $h_p(g) \neq h_p(h)$ and $(\{g\}_*^G + \{h\}_*^G \otimes R_p \subsetneqq G \otimes R_p$. It may be easily shown that there exists an element

$u \otimes 1 \in G \otimes R_p \dot{-} (\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p$ with

$p(u \otimes 1) \in (\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_p$. Hence $p(u \otimes 1 \otimes 1)$ lies in $(\{g\}_*^G \dot{+} \{h\}_*^G) \otimes R_{\Pi_1} \otimes R_p$ and in view of (6) $u \otimes 1 \otimes 1$ does not lie in this group. But this can occur for a finite number of $p \in \Pi_1$ only in view of the complete decomposability of $G \otimes R_{\Pi_1}$, Theorem 3 from [4] and Theorem 1. Hence G satisfies the condition (1) and our proof is finished.

Let Π' be any set of primes. We call a torsion free group G homogeneous with respect to Π' if the types of any two non-zero elements from G restricted on Π' are equal. Now it is easy to see that Theorem 3 can be formulated in the following way:

<u>Theorem 3</u>$'$: Let G be a torsion free group of rank two and x_1, x_2 any its basis. Let us denote by Π_1 the set of those primes p for which the p-primary component of $G/(\{x_1\}_*^G \dotplus \{x_2\}_*^G)$ vanishes. If G is homogeneous with respect to $\Pi_2 \dot{-} \Pi'$ where Π' is finite and $\Pi_2 = \Pi \dot{-} \Pi_1$ then G is factor-splitting.

<u>Remark</u>: The special cases of Theorem 4 are the following: 1) If Π_1 is finite and G is divisible with respect to $\Pi_2 \dot{-} \Pi'$ then G is almost divisible (see [3],Theorem 5). If $\Pi_2 = \Pi \dot{-} \Pi_1$ is finite then G is primitive (see [3],Theorem 2).

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