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ORDER OF HOLONOMY AND GEOMETRIC OBJECTS OF MANIFOLDS WITH CONNECTION

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Our considerations are in the category C^{∞}. The standard notations of the theory of jets are used throughhout the paper, see [3].

1. Let $P(B,G)$ be a principal fibre bundle with base B and structure Lie group G and let $\Phi = PP^{-1}$ be the groupoid associated to P. Let H be a closed subgroup of G, let $F = G/H$ be the corresponding homogeneous space and let $E = E(B,F,G,P)$ denote the fibre bundle associated to P with standard fibre F; so that Φ is a groupoid of operators on E. Let p be the canonical projection $p: E \to B$; we shall write $E_x = p^{-1}(x)$, $x \in B$.

$\tilde{Q}^r(\Phi)$ or $\bar{Q}^r(\Phi)$ or $Q^r(\Phi)$ means the fibred manifold of all non-holonomic or semi-holonomic or holonomic elements of connection of order r on Φ respectively, see [4]. A non-holonomic or semi-holonomic or holonomic connection of order r (shortly: an r-connection) on Φ is a global section $C: B \to \tilde{Q}^r(\Phi)$ or $C: B \to \bar{Q}^r(\Phi)$ or $C: B \to Q^r(\Phi)$ respectively.

Let V be a manifold, let $Z \in \tilde{J}^{r}(V, E)$ and let $X \in \tilde{Q}^{r}(\Phi)$ such that $\alpha X = p(\beta Z) = x$. Then the <u>development</u> $X^{-1}(Z)$ <u>of</u> Z <u>into</u> E_x <u>by means of</u> X is defined by

$$X^{-1}(Z) = (X^{-1} p Z) \cdot Z \in \tilde{J}^{r}(V, E_x),$$

where $\cdot$ means the prolongation of the partial composition law $(\theta, x) \longmapsto \theta x$, $\theta \in \Phi$, $x \in E$, see [4]. (We remark that Ehresmann uses the term "the absolute differential of Z with respect to X" for $X^{-1}(Z)$.) Obviously, if $Z \in \bar{J}^{r}(V, E)$ or $J^{r}(V, E)$ and $X \in \bar{Q}^{r}(\Phi)$ or $Q^{r}(\Phi)$, then $X^{-1}(Z) \in \bar{J}^{r}(V, E_x)$ or $J^{r}(V, E_x)$ respectively. Furthermore, if $Z = j^{r}_{x}\sigma$, where σ is a local section in E, then we write $X^{-1}(\sigma)$ instead $X^{-1}(j^{r}_{x}\sigma)$ and $X^{-1}(\sigma)$ is called the <u>development</u> <u>of</u> σ <u>into</u> E_x <u>by means of</u> X.

Let C be an r-connection on Φ, then C' means the prolongation of C, which is an $(r+1)$-connection on Φ, see [4]. The k-th prolongation of C is defined by iteration $C^{(k)} = C^{(k-1)'}$. Every 1-connection C determines a sequence $C, C', \dots, C^{(k)}, \dots$ of semi-holonomic connections. The terms of such a sequence are called <u>simple connections</u>.

<u>Definition 1.</u> A space $\mathcal{S}$ with r-connection is a quintuple $\mathcal{S} = \mathcal{S}(B, \Phi, E, \sigma, C)$, where σ is a global section in E and C is an r-connection on Φ.

<u>Remarks</u>. For $r = 1$, our definition is equivalent to the definition of a space with connection by A. Švec [7]. The sequence $\mathcal{S}^{(r-1)}(B, \Phi, E, \sigma, C^{(r-1)})$, $r = 1, 2, \dots,$ of spaces with simple connections is canonically associa-

ted to every space $\mathcal{S}(B, \Phi, E, \sigma, C)$ with connection of the first order.

2. A (holonomic) contact element of dimension m and of order r (shortly: a <u>contact m^r-element</u>) on a manifold V at a point $x \in V$ is the set XL^r_m, where X is an m^r-velocity on V at x. Such a contact element is called regular, if $m < n = \dim V$ and if X is a regular velocity. The fibred manifold of all regular contact m^r-elements on V will be denoted by $K^r_m(V)$. Let U be another manifold and let $Z \in \in J^r(VU)$, then Z determines a contact m^r-element $k(Z)$ on U at βZ, $k(Z) = Z h L^r_m$, where h is a (holonomic) r-frame on V at αZ.

A manifold N together with a left action of a group G on N is called a G-<u>space</u>, see e.g. [1], p.31. A mapping φ of N into another G-space is called a G-<u>mapping</u>, if $\varphi(g x) = g \varphi(x)$ for every $x \in N$, $g \in G$. Let F be as above, then the action of G on F is canonically extended to an action on $K^r_m(F)$, so that $K^r_m(F)$ is a G-space.

<u>Definition 2</u>. A geometric m^r-object σ on F with values in a G-space S is a G-mapping of $K^r_m(F)$ into S. More generally, let W be an invariant subspace of $K^r_m(F)$, then a geometric m^r-object on F of type W with values in S is a G-mapping of W into S.

Let M be an m-dimensional submanifold of F, then M determines canonically a contact m^r- element

$k^r_x M$ at each point $x \in M$ and $\mathcal{O}(k^r_x M) \in S$ will be called the <u>value of</u> $\mathcal{O}$ <u>for</u> M <u>at</u> x. That's why we may also say that $\mathcal{O}$ is a <u>geometric object of order</u> r <u>for</u> m-<u>dimensional submanifolds of</u> F.

<u>Remarks</u>. We shall show in a next paper that our definition gives an invariant and deeper explanation of the so-called "method of prolongations and outflankings" by G.F. Laptěv, [6]. We shall also show that a modification of our ideas enables to define geometric objects for submanifolds of a space with fundamental Lie pseudogroup.

3. A semi-holonomic contact m^r-element on a manifold V is the set $Y\bar{L}^r_m$, where Y is a semi-holonomic m^r-velocity on V. Such a contact element is called regular, if $m < n = \dim V$ and Y is regular; the fibred manifold of all regular semi-holonomic contact m^r-elements on V will be denoted by $\bar{K}^r_m(V)$. Let U be another manifold and let $Z \in \bar{J}^r(V, U)$, then Z determines a semi-holonomic contact m^r-element $k(Z)$ on U, $k(Z) = Z\bar{h}\bar{L}^r_n$, where $\bar{h}$ is a semi-holonomic n-frame on V.

<u>Definition 3</u>. Let F be as above. A semi-holonomic geometric m^r-object on F with values in a G-space S is a G-mapping of $\bar{K}^r_m(F)$ into S.

<u>Remark</u>. Analogous definition relates to the non-holonomic case as well.

<u>Definition 4</u>. A space $\mathcal{S}(B, \Phi, E, \sigma, C)$ with r-connection will be called a manifold with r-connection,

if it holds

a) $m = \dim B < n = \dim F$,

b) $C^{-1}(x)(\sigma)$ is regular for every $x \in B$.

We shall also say that $m = \dim B$ is the dimension of $\mathcal{S}$.

<u>Remark</u>. A manifold with a 1-connection is locally equivalent to a submanifold of a space with Cartan connection, cf.[2].

Consider an m-dimensional manifold with a semi-holonomic r-connection and let σ be a semi-holonomic geometric m^r-object on E_x, $x \in B$. The development $C^{-1}(x)(\sigma)$ of σ into E_x determines a semi-holonomic contact m^r-element $k(C^{-1}(x)(\sigma))$ on E_x and $\sigma(k(C^{-1}(x)(\sigma)) \in S$ will be called the <u>value of σ for $\mathcal{S}$ at</u> $x \in B$, so that a semi-holonomic geometric m^r-object represents a <u>geometric object</u> for m -<u>dimensional manifolds with semi-holonomic r-connection</u>. Moreover, if $\mathcal{S}(B, \Phi, E, \sigma, C)$ is a manifold with 1-connection, then σ can be applied to the associated manifold $\mathcal{S}^{(r-1)}(B, \Phi, E, \sigma, C^{(r-1)})$ with semi-holonomic r -connection; that's why a semi-holonomic geometric m^r-object may also be considered as a <u>geometric object of order r for m -dimensional submanifolds of a space with Cartan connection.</u>

4. A semi-holonomic contact m^r-element $\bar{YL}^r_m$ will be said holonomic, if it contains a holonomic m^r-velocity.

Definition 5. A manifold $\mathcal{S}(B, \Phi, E, \sigma, C)$ with semi-holonomic r-connection is called holonomic at $x \in B$, if the contact element $k(C^{-1}(x)(\sigma))$ is holonomic.

Let σ be a semi-holonomic geometric m^r-object on E_x, then the restriction of σ to $K_m^r(E_x)$ is a holonomic geometric m^r-object on E_x, since $K_m^r(E_x)$ is an invariant subspace of $\overline{K}_m^r(E_x)$. This proves the following

Theorem. If a manifold $\mathcal{S}$ with semi-holonomic r-connection is holonomic at $x \in B$, then the value of every geometric object for $\mathcal{S}$ at x coincides with the value of a holonomic geometric m^r-object on E_x.

We can also restate this theorem in the following more intuitive way: if a manifold with semi-holonomic r-connection is holonomic at a point, then all its geometric objects at this point coincide with the geometric objects of order r of an m-dimensional submanifold of the corresponding homogeneous space.

5. A manifold $\mathcal{S}(B, \Phi, E, \sigma, C)$ with a 1-connection is called r-holonomic at $x \in B$, if the associated manifold $\mathcal{S}^{(r-1)}(B, \Phi, E, \sigma, C^{(r-1)})$ is holonomic at x. In this case, our theorem gives the conditions that every geometric object of order r of a submanifold of a space with Cartan connection coincides with a geometric object of a submanifold of the corresponding homogeneous space.

In [5], we consider a surface in a 3-dimensional space with projective connection from this point of view and we treat the conditions for r-holonomy geometrically in full details.

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