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COMPARABILITY AND CONDITIONAL MAXIMALITY OF MEASURES SUPPORTED BY FINITE SETS OF REAL NUMBERS

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The notion of ordering of measures, generally introduced by G. Choquet, is in the present paper investigated in a relation to stochastic matrices of the type (m, n). The purpose of this paper is to obtain effective necessary and sufficient conditions for the comparability of measures. These results are applied to finding of conditionally maximal measures. Illustration of the present theory is given through some numerical examples.

Contents

1. Notations. Let R_n be the euclidean space of dimension n, $n = 1, 2, \dots$. We shall denote by $x \cdot y$ scalar product of the elements x and y and by $co\{x\}$ convex

hull of the set $\{b_1, b_2, \dots, b_n\}$ for any $b = (b_k)_{k=1}^n \in R_n$ and $y = (y_k)_{k=1}^m \in R_m$. The set of all convex functions on the space R_1 will be denoted by $\check{C}$.

Define

$$\mathcal{P}_n = \{\beta \in R_n \,;\, \beta = (\beta_k)_{k=1}^n, \beta_k \geq 0, \sum_{k=1}^n \beta_k = 1\} \quad \text{and}$$

$$\mathcal{P}_n^+ = \{\beta \in \mathcal{P}_n \,;\, \beta_k > 0 \quad \text{for} \quad k = 1, 2, \dots, n\} \quad .$$

Let $Q = (q_{jk})_{j,k}$ be a matrix of the type (m, n). We denote by $Q^* = (q_{jk})_{k,j}$ the adjoint matrix of the type (n, m). If $q_{jk} \geq 0$ and $\sum_{k=1}^n q_{jk} = 1$ for $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, n$ then the matrix Q is called <u>stochastic</u>.

If moreover $b \in R_n$ and $\alpha \in \mathcal{P}_m$ then we denote by $a = Qb$ and $\beta = Q^* a$ such elements $a \in R_m$, $\beta \in \mathcal{P}_n$ that

$$a_j = \sum_{k=1}^n q_{jk} b_k \,, \qquad \beta_k = \sum_{j=1}^m q_{jk} \alpha_j \quad ,$$

δ_t will denote a translated Dirac measure on R_1 for $t \in R_1$.

2. <u>Ordering of a set of measures and stochastic matrices.</u>

(2.1) Let $\mathcal{P}$ be the set of all probability-measures, supported by finite sets of real numbers, i.e.

$$\mathcal{P} = \{\mu = \sum_{k=1}^n \beta_k \delta_{b_k} \,;\, \beta = (\beta_k)_{k=1}^n \in \mathcal{P}_n^+, b = (b_k)_{k=1}^n \in R_n, n = 1, 2, \dots\},$$

where $\mu(f) = \sum_{k=1}^n \beta_k f(b_k)$ for each function f on R_1.

(2.2) Let $\prec$ be a relation such that

$$\lambda \prec \mu \text{ iff } \lambda, \mu \in \mathcal{P} \text{ and } \lambda(f) \leq \mu(f) \text{ for all } f \in \check{C}.$$

Then this relation is transitive and reflexive, i.e.

$\prec$ is a quasi-ordering on the set $\mathcal{P}$.

The following theorem (2.3) follows from the theorem 2 in [2] or from a theorem in [6],chap.13, but the present proof is more simple.

(2.3) **Theorem**. Let λ, μ be two elements of the set $\mathcal{P}$ of the form

$$\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j}, \quad \mu = \sum_{k=1}^{n} \beta_k \delta_{b_k}, \quad \alpha = (\alpha_j)_{j=1}^{m} \in \mathcal{P}_m^+, \quad \beta \in \mathcal{P}_n^+ .$$

Then $\lambda \prec \mu$ if and only if there is a stochastic matrix Q of the type (m, n) such that

$$Qb = a \quad \text{and} \quad Q^*\alpha = \beta .$$

Proof. 1° Let $Q = (q_{jk})$ be a stochastic matrix of the type (m,n), $Qb = a$, $Q^*\alpha = \beta$. Then

$$\lambda(f) = \sum_{j=1}^{m} \alpha_j f(a_j) = \sum_{j=1}^{m} \alpha_j f(\sum_{k=1}^{n} q_{jk} b_k) \leq \sum_{j=1}^{m} \alpha_j \sum_{k=1}^{n} q_{jk} f(b_k) =$$

$= \sum_{k=1}^{n} (\sum_{j=1}^{m} q_{jk} \alpha_j) f(b_k) = \sum_{k=1}^{n} \beta_k f(b_k) = \mu(f)$ for all $f \in \check{C}$, i.e. $\lambda \prec \mu$.

2° Let $\lambda \prec \mu$. Then $\{a_j\}_{j=1}^{m} \subset co\{b\}$. If the opposite statement holds, take a convex function f such that $f = 0$ on the set $co\{b\}$, f positive on $R_1 \dot{-} co\{b\}$. Then $\mu(f) = 0 < \lambda(f)$ which is a contradiction. Hence each set $S^j = \{p = (p_k)_{k=1}^{n} \in \mathcal{P}_n ; \sum_{k=1}^{n} p_k b_k = a_j\}$, $j = 1, 2, \ldots, m$, is nonempty and convex.

Define $e^{(j)} = (0, 0, \ldots, 1_j, 0, \ldots, 0) \in R_m$, $S = \bigcup_{j=1}^{m} ((e^{(j)}) \times S^j)$.

Hence

$$\emptyset \neq S \subset R_{m+n}, \text{ convex hull } co(S) \neq \emptyset .$$

If $(\alpha, \beta) \in co(S)$ then by [1] there are $p^j = (q_{jk})_{k=1}^{n} \in S^j$ such that $\sum_{j=1}^{m} \alpha_j p^j = \beta$, i.e. $q_{rs} \geq 0$, $\sum_{k=1}^{n} q_{rk} = 1$,

$\sum_{k=1}^{n} q_{rk} b_k = a_r$, $\sum_{j=1}^{m} q_{js} \alpha_j = \beta_s$ for $r = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, n$. Hence the matrix $Q = (q_{jk})$ is stochastic, $Qb = a$ and $Q^*\alpha = \beta$.

If $(\alpha, \beta) \notin co(S)$ then by Caratheodory's theorem [5] and Hahn-Banach's theorem [5] there is a hyperplane that strictly separates those sets. Since $\sum_{j=1}^{m} \alpha_j + \sum_{k=1}^{n} \beta_k = 1 + \sum_{k=1}^{n} p_k = 2$ for all $p = (p_k)_{k=1}^{n} \in \bigcup_{j=1}^{m} S^j$, there is a linear functional F on R_{m+n} such that

$F(\alpha, \beta) > 0$ and $F \leq 0$ on the set S ,i.e. there is an element $x \in R_m$ and an element $y \in R_n$ such that

$\alpha \cdot x - \beta \cdot y > 0$ and $x_j - p \cdot y \leq 0$ for all $p \in S^j$, $j = 1, 2, \dots, m$.

Define $\check{y}(a_j) = \inf\{p \cdot y ; p \in S^j\}$. Then $x_j \leq \check{y}(a_j)$. Let $\bar{y}$ be the greatest convex function on the set $co\{b\}$ such that $\bar{y}(b_k) \leq y_k$ for $k = 1, 2, \dots, n$ .If $j \in \{1, 2, \dots, m\}$ then $\check{y}(a_j) \leq \bar{y}(a_j)$, since the point $(a_j, \bar{y}(a_j))$ is an element of the line segment, joining points (b_{k_1}, y_{k_1}) and (b_{k_2}, y_{k_2}) such that $b_{k_1} \leq a_j \leq b_{k_2}$. Hence there exists $p = (0, 0, \dots 0, p_{k_1}, 0, \dots 0, p_{k_2}, 0, \dots, 0) \in S^j$, so that $\check{y}(a_j) \leq p_{k_1} y_{k_1} + p_{k_2} y_{k_2} = p_{k_1} \bar{y}(b_{k_1}) + p_{k_2} \bar{y}(b_{k_2}) = \bar{y}(a_j)$. The convex function $\bar{y}$ has a convex extension in $\check{C}$. Using the condition $\lambda \prec \mu$, we obtain following inequalities

$$\alpha \cdot x \leq \sum_{j=1}^{m} \alpha_j \check{y}(a_j) \leq \sum_{j=1}^{m} \alpha_j \bar{y}(a_j) \leq \sum_{k=1}^{n} \beta_k \bar{y}(b_k) \leq \beta \cdot y$$

i.e. a contradiction.

Now we shall prove that the quasi-ordering on $\mathcal{P}$ is moreover an ordering. This result is obtained by theorem (2.3) without the classical Stone-Weierstrass's theorem and it can be generalized for any linear space R .

(2.4) A convex function $g \in \check{C}$ is called strictly convex iff the following condition holds:

if $b_1, b_2 \in R_1$, $b_1 \neq b_2$, $t \in \langle 0,1\rangle$,

$g(tb_1+(1-t)b_2) = tg(b_1)+(1-t)g(b_2)$ then $t=0$ or $t=1$.

(2.5) <u>Lemma</u>. Let B be a finite subset of the set R_1. Let p be a nonnegative function on the set B such that $\sum\{p(b);\ b\in B\}=1$. Let g be a strictly convex function in $\check{C}$ and let

$$g(\sum\{p(b)b;\ b\in B\}) = \sum\{p(b)g(b);\ b\in B\}.$$

Then there is an element $b_0 \in B$ such that

$$p(b_0)=1 \text{ and } p(b)=0 \text{ for all } b\in B,\ b\neq b_0.$$

<u>Proof</u>. If $b_1\in B$, $p(b_1)\in(0,1)$ then there is an element $b_2\in B$, $b_1\neq b_2$, that also $p(b_2)\in(0,1)$. Put $p(b_1)+p(b_2)=\varepsilon$. We obtain the following inequalities:

$$g(\sum\{p(b)b;\ b\in B\}) \leq \varepsilon g(\varepsilon^{-1}(p(b_1)b_1+p(b_2)b_2)+$$
$$+(1-\varepsilon)g((1-\varepsilon)^{-1}\sum\{p(b)b;\ b\in B,\ b_1\neq b\neq b_2\}) \leq$$
$$\leq p(b_1)g(b_1)+p(b_2)g(b_2)+$$
$$+(1-\varepsilon)g((1-\varepsilon)^{-1}\sum\{p(b)b;\ b\in B,\ b_1\neq b\neq b_2\}) \leq$$
$$\leq \sum\{p(b)g(b);\ b\in B\} = g(\sum\{p(b)b;\ b\in B\}).$$

Hence

$$g(\varepsilon^{-1}p(b_1)b_1+\varepsilon^{-1}p(b_2)b_2) = \varepsilon^{-1}p(b_1)g(b_1)+\varepsilon^{-1}p(b_2)g(b_2).$$

Since the function g is strictly convex, it follows that $p(b_1)=0$ or $p(b_2)=0$, which is a contradiction. Hence $p(b)=0$ or $p(b)=1$ for all $b\in B$. Since $\sum\{p(b);b\in B\}=1$, there is an element $b_0\in B$ such that $p(b_0)=1$, $p(b)=0$ for all $b\in B$, $b\neq b_0$.

(2.6) <u>Remark.</u> If μ is a measure, $\mu\in\mathcal{P}$, then there is one and only one expression of μ such that

$$\beta = (\beta_k)_{k=1}^{n} \in \mathcal{P}_n^+,\ b_1 > b_2 > \dots > b_n,\ \mu = \sum_{k=1}^{n} \beta_k b_k .$$

(2.7) __Theorem__. If $\lambda \in \mathcal{P}$, $\mu \in \mathcal{P}$, $\lambda \prec \mu$, $\mu \prec \lambda$ then $\lambda = \mu$.

__Proof__. Let g be a function of $\check{C}$, which is strictly convex. Then using the expression

$$\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j},\quad \mu = \sum_{k=1}^{n} \beta_k \delta_{b_k},$$

$$\alpha \in \mathcal{P}_m^+,\ \beta \in \mathcal{P}_n^+,\ a_1 > a_2 > \dots > a_m,\ b_1 > b_2 > \dots > b_n$$

and choosing by theorem (2.3) a stochastic matrix $Q = (q_{jk})$ of the type (m, n) such that $Qb = a$ and $Q^*\alpha = \beta$, we obtain

$$\lambda(g) = \sum_{j=1}^{m} \alpha_j g(a_j) = \sum_{j=1}^{m} \alpha_j g\Big(\sum_{k=1}^{n} q_{jk} b_k\Big) \leq$$

$$\leq \sum_{j=1}^{m} \alpha_j \sum_{k=1}^{n} q_{jk}\, g(b_k) = \sum_{k=1}^{n} \beta_k g(b_k) = \mu(g) \leq \lambda(g) .$$

Since $\alpha_j > 0$ and $\sum_{k=1}^{n} q_{jk}\, g(b_k) - g\big(\sum_{k=1}^{n} q_{jk} b_k\big) \geq 0$, it follows from the equality $\sum_{j=1}^{m} \alpha_j \big(\sum_{k=1}^{n} q_{jk}\, g(b_k) - g\big(\sum_{k=1}^{n} q_{jk} b_k\big)\big) = 0$ that

$$g\Big(\sum_{k=1}^{n} q_{jk} b_k\Big) = \sum_{k=1}^{n} q_{jk}\, g(b_k) \text{ for } j = 1, 2, \dots, m .$$

Lemma (2.5) implies the existence of numbers $k(j) \in \{1, 2, \dots, n\}$ such that

$$q_{jk(j)} = 1,\ q_{jk} = 0 \quad \text{for all } k \neq k(j),\ j = 1, 2, \dots, m .$$

From the fact that b and $a = Qb$ are both strictly decreasing finite sequences follows that

$k(1) < k(2) < \dots < k(m)$ and $m \leq n$. If $m < n$, then there is an integer $k \in \{1, 2, \dots, n\}$ such that $q_{1k} = q_{2k} = \dots = q_{mk} = 0$. Then $0 < \beta_k = \sum_{j=1}^{m} q_{jk} \alpha_k = 0$ which is a contradiction. Hence $m = n$, $k(1) = 1$, $k(2) = 2, \dots, k(m) = m$,

$\mathcal{Q}$ is the unit matrix

$$a = b, \ \alpha = \beta, \ \lambda = \mu .$$

(2.8) Corollary. The quasi-ordering $\prec$ is moreover an ordering on the set $\mathcal{P}$.

3. Conditions for comparability of measures

(3.1) Lemma. Suppose b_k, r_k, u_k are real numbers for $k = 1, 2, \dots, m$, $b_1 \geq b_2 \geq \dots \geq b_m$, $r_k \geq 0$, $\sum_{k=1}^{m} r_k = 1$ and $u_k \geq r_k$ for all k. Let s be such integer of the set $\{1, 2, \dots, m\}$ that $\sum_{k=1}^{s-1} u_k \leq 1 \leq \sum_{k=1}^{s} u_k$ (put $\sum_{k=1}^{0} = 0$).

Then

$$\sum_{k=1}^{m} r_k b_k \leq \sum_{k=1}^{s-1} u_k b_k + (1 - \sum_{k=1}^{s-1} u_k) b_s .$$

Proof. Put $v_k = \dfrac{u_k}{\sum_{i=1}^{k-1} u_i}$, $v_k = 0$ if $u_k = 0$. Clearly,

$$v_k \geq u_k \geq 0, \sum_{k=1}^{s-1} v_k = 1, \ \sum_{k=s}^{m} r_k b_k \leq (1 - \sum_{k=1}^{s-1} r_k) b_s ,$$
$$0 \leq \left(\frac{u_k}{v_k} - \frac{r_k}{v_k}\right)(b_k - b_s) \quad \text{for} \quad k \leq s - 1 .$$

We obtain the following inequalities

$$\sum_{k=1}^{m} r_k b_k \leq \sum_{k=1}^{s-1} r_k b_k + (1 - \sum_{k=1}^{s-1} r_k) b_s = \sum_{k=1}^{s-1} (r_k b_k + (v_k - r_k) b_s) =$$
$$= \sum_{k=1}^{s-1} \left(v_k \left(\frac{r_k}{v_k} b_k + (1 - \frac{r_k}{v_k}) b_s\right)\right) \leq \sum_{k=1}^{s-1} \left(v_k \left(\frac{u_k}{v_k} b_k + (1 - \frac{u_k}{v_k}) b_s\right)\right) =$$
$$= \sum_{k=1}^{s-1} u_k b_k + (1 - \sum_{k=1}^{s-1} u_k) b_s .$$

(3.2) Theorem. Suppose $\lambda, \mu \in \mathcal{P}$, $\lambda \prec \mu$,
$\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j}$, $\mu = \sum_{k=1}^{m} \beta_k \delta_{b_k}$, $\alpha = (\alpha_j)_{j=1}^{m} \in \mathcal{P}_m^+$,
$\beta = (\beta_k)_{k=1}^{m} \in \mathcal{P}_m^+$, $b_1 \geq b_2 \geq \dots \geq b_m$.

Then the following conditions (for comparability) are fulfilled:

(3.3) $$\sum_{j=1}^{m} \alpha_j a_j = \sum_{k=1}^{n} \beta_k b_k$$

and if $r \in \{1, 2, \ldots, m-1\}$, $s(r) \in \{1, 2, \ldots, n\}$,

$\sum_{k=1}^{s(r)-1} \beta_k \leq \sum_{j=1}^{r} \alpha_j < \sum_{k=1}^{s(r)} \beta_k$ then

(3.4) $$\sum_{j=1}^{r} \alpha_j a_j \leq \sum_{k=1}^{s(r)-1} \beta_k b_k + \left(\sum_{j=1}^{r} \alpha_j - \sum_{k=1}^{s(r)-1} \beta_k\right) b_{s(r)} .$$

Proof. Theorem (2.3) implies there exists a stochastic matrix $Q = (q_{jk})$ of the type (m, n) such that $Qb = a$ and $Q^* \alpha = \beta$. Using the lemma (3.1) we obtain the following inequalities:

$$\sum_{j=1}^{m} \alpha_j a_j = \sum_{j=1}^{m} \alpha_j \sum_{k=1}^{n} q_{jk} b_k = \sum_{k=1}^{n} \sum_{j=1}^{m} q_{jk} \alpha_j b_k = \sum_{k=1}^{n} \beta_k b_k ,$$

$$u_k = \frac{1}{\sum_{i=1}^{r} \alpha_i} \sum_{j=1}^{m} q_{jk} \alpha_j \geq \frac{1}{\sum_{i=1}^{r} \alpha_i} \sum_{j=1}^{r} q_{jk} \alpha_j = v_k \geq 0 ,$$

$$\sum_{j=1}^{r} \alpha_j a_j = \sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_j q_{jk} b_k = \sum_{j=1}^{r} \alpha_j \sum_{k=1}^{n} v_k b_k \leq \sum_{k=1}^{s(r)-1} \beta_k b_k + \left(\sum_{j=1}^{r} \alpha_j - \sum_{k=1}^{s(r)-1} \beta_k\right) b_{s(r)} .$$

(3.5) Theorem. Suppose $\lambda, \mu \in \mathcal{P}$:

$$\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j}, \quad \mu = \sum_{k=1}^{n} \beta_k \delta_{b_k}, \quad \alpha \in \mathcal{P}^+_m, \quad \beta \in \mathcal{P}^+_n ,$$

$a_1 \geq a_2 \geq \ldots \geq a_m$. Let the conditions for comparability (3.3) and (3.4) hold.

Then $$\lambda \prec \mu .$$

Proof. Put $K = \{f \in \mathcal{C};\ \sum_{j=1}^{m} \alpha_j f(a_j) \leq \sum_{k=1}^{n} \beta_k f(b_k)\}$. Clearly, K is a convex wedge. From (3.3) follows that the wedge K contains all affine functions of the set $\mathcal{C}$. We shall prove that K contains also all convex functions g_x for $x \in R_1$, where $g_x(\xi) = \xi - x$ for $\xi > x$, $g_x(\xi) = 0$ for $\xi \leq x$.

If $k \in \{1,2,\dots,m-1\}$, $x \in R_1$, $a_1 > x \geq a_{k+1}$ then using (3.3) and (3.4) we obtain the following inequalities

$$\sum_{j=1}^{m} g_x(a_j) = \sum_{j=1}^{k} \alpha_j (a_j - x) = \sum_{j=1}^{k} \alpha_j a_j - x \sum_{j=1}^{k} \alpha_j \leq \sum_{k=1}^{s(k)-1} \beta_k b_k +$$
$$+ \Big(\sum_{j=1}^{k} \alpha_j - \sum_{k=1}^{s(k)-1} \beta_k\Big) b_{s(k)} - x \sum_{j=1}^{k} \alpha_j = \sum_{k=1}^{s(k)-1} \beta_k (b_k - x) + \Big(\sum_{j=1}^{k} \alpha_j -$$
$$- \sum_{k=1}^{s(k)-1} \beta_k\Big)(b_{s(k)} - x) \leq \sum_{k=1}^{s(k)-1} \beta_k (b_k - x) \leq \sum_{k=1}^{n} \beta_k g_x(b_k) \text{ for } b_{s(k)} \leq x$$

or $$\sum_{j=1}^{m} g_x(a_j) \leq \sum_{k=1}^{s(k)} \beta_k (b_k - x) + \Big(\sum_{k=1}^{s(k)} \beta_k - \sum_{j=1}^{k} \alpha_j\Big)(x - b_{s(k)}) \leq$$
$$\leq \sum_{k=1}^{s(k)} \beta_k (b_k - x) \leq \sum_{k=1}^{n} \beta_k g_x(b_k) \quad \text{for } x < b_{s(k)} .$$

If $x \geq a_1$ then $\sum_{j=1}^{m} g_x(a_j) = 0 \leq \sum_{k=1}^{n} \beta_k g_x(b_k)$.

If $a_m > x$ then $\sum_{j=1}^{m} \alpha_j g_x(a_j) = \sum_{j=1}^{m} \alpha_j (a_j - x) =$

$$= \sum_{j=1}^{m} \alpha_j a_j - x = \sum_{k=1}^{n} \beta_k b_k - x = \sum_{k=1}^{m} \beta_k (b_k - x) \leq \sum_{k=1}^{m} \beta_k g(b_k) .$$

Now, let f be any element of the set $\check{C}$.
The set of all a_j and b_k, $j \in \{1,2,\dots,m\}$, $k \in \{1,2,\dots,n\}$, can be ordered. Suppose $d_1 < d_2 < \dots < d_q$ are all its elements. Define a function h: $h(d_i) = f(d_j)$ for $i = 1,2,\dots,q$ and h as an affine function between the numbers d_i, d_{i+1} This function h is convex, moreover h is a linear non-negative combination of an affine function and the functions

g_{d_i}, $i = 1,2,\dots,q$. Hence $h \in K$ and $f \in K$, i.e.

$$K = \check{C}, \quad \lambda \prec \mu .$$

(3.6) Corollary. Suppose λ, $\mu \in \mathcal{P}$, $\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j}$, $\mu = \sum_{k=1}^{n} \beta_k \delta_{b_k}$, $\alpha \in \mathcal{P}_m^+$, $\beta \in \mathcal{P}_n^+$, $a_1 \geq a_2 \geq \dots \geq a_m$, $b_1 \geq b_2 \dots \geq b_n$.
Then the following statements are equivalent:

1° $\lambda \prec \mu$. 2° the conditions (3.3) and (3.4) are fulfilled.

4. Applications to conditional maximality of measures.

In this section it will be shown how the theorems (3.2) and (3.5) about the comparability of measures can be applied to finding of measures σ which are "conditionally maximal". In particular:

(4.1) Theorem. Suppose $\mu \in \mathcal{P}$, $\mu = \sum_{k=1}^{n} \beta_k \delta_{b_k}$, $b_1 \geq b_2 \geq \ldots \geq b_n$, $\beta = (\beta_k)_{k=1}^{n} \in \mathcal{P}_n^+$, $\alpha = (\alpha_j)_{j=1}^{m} \in \mathcal{P}_m^+$.

Then there exists one and only one measure $\sigma \in \mathcal{P}$, fulfilling the following conditions:

1° $\sigma = \sum_{j=1}^{m} \alpha_j \delta_{c_j}$, $c_1 \geq c_2 \geq \ldots \geq c_m$,

2° $\sigma \prec \mu$,

3° if $\lambda \in \mathcal{P}$, $\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j}$, $a_1 \geq a_2 \geq \ldots \geq a_m$ and $\lambda \prec \mu$ then $\lambda \prec \sigma$.

Proof. Define the following elements of the set $\mathcal{P}_m$:

$v^r = \frac{1}{A_r}(\alpha_1, \alpha_2, \ldots, \alpha_r, 0, 0, \ldots, 0)$, where $r = 1, 2, \ldots, m$ and $A_r = \sum_{j=1}^{r} \alpha_j$, and the following elements of the set $\mathcal{P}_n$:

$w^r = \frac{1}{A_r}(\beta_1, \beta_2, \ldots, \beta_{s(r)-1}, A_r - \sum_{k=1}^{s(r)-1} \beta_k, 0, 0, \ldots, 0)$, where $r = 1, 2, \ldots, m-1$ and $s(r) \in \{1, 2, \ldots, n\}$, $\sum_{k=1}^{s(r)-1} \beta_k \leq$ $\leq A_r < \sum_{k=1}^{s(r)} \beta_k$, and $w^m = (\beta_1, \beta_2, \ldots, \beta_n) = \beta$.

Then the conditions of comparability (3.3) and (3.4) can be written in a simple form

(3.3) $$v^m . a = w^m . b ,$$

(3.4) $$v^\kappa . a \leq w^\kappa . b \quad \text{for } \kappa = 1, 2, \dots, m-1 .$$

The element $c = (c_j)_{j=1}^m \in R_m$, destinating the measure σ in (4.1), can be chosen as a solution of the following system of m independent linear equations

$$v^\kappa . c = w^\kappa . b \quad \text{for } \kappa = 1, 2, \dots, m , \quad \text{i.e.}$$

$$\sum_{j=1}^m v_j^\kappa c_j = \sum_{k=1}^n w_k^\kappa b_k .$$

Define a matrix $V = (v_j^\kappa)_{\kappa, j}$ of the type (m, m) and a matrix $W = (w_k^\kappa)_{\kappa, k}$ of the type (m, n).

Then there exists the inverse matrix V^{-1} and a matrix $E_{\alpha,\beta} = (e_{jk})_{j,k} = V^{-1}W$ of the type (m, n).

Clearly, $E_{\alpha,\beta} b = c$, $E^*_{\alpha,\beta} V^* = W^*$, $E^*_{\alpha,\beta} v^\kappa = w^\kappa$ for $\kappa = 1, 2, \dots, m$.

Now, we shall find all elements e_{jk} of the matrix $E_{\alpha,\beta}$. Put $e^{(\kappa)} = (0, 0, \dots, 1_\kappa, 0, \dots, 0) \in \mathcal{P}_m$, $A_0 = 0$, $v^0 = 0$, $w^0 = 0$, $s(m) = n$. Then

$$e^{(\kappa)} = \frac{1}{\alpha_\kappa}(A_\kappa v^\kappa - A_{\kappa-1} v^{\kappa-1}), \quad (e_{\kappa k})_{k=1}^n = E^*_{\alpha,\beta} e^{(\kappa)} .$$

It follows that

$$(e_{\kappa k})_{k=1}^n = \frac{1}{\alpha_\kappa}(A_\kappa w^\kappa - A_{\kappa-1} w^{\kappa-1}) =$$

$$= \frac{1}{\alpha_\kappa}(\beta_1, \dots, \dots, \beta_{s(\kappa)-1}, A_\kappa - \sum_{k=1}^{s(\kappa)-1} \beta_k, 0, \dots, 0) -$$

$$- \frac{1}{\alpha_\kappa}(\beta_1, \dots, \beta_{s(\kappa-1)-1}, A_{\kappa-1} - \sum_{k=1}^{s(\kappa-1)-1} \beta_k, 0, \dots, 0) \text{ for } \kappa = 1, 2, \dots, m .$$

Hence $e_{\kappa k} = 0$ for $k < s(\kappa - 1)$.

1° If $s(\kappa-1) = s(\kappa)$ then

$$e_{\kappa s(\kappa)} = \frac{1}{\alpha_\kappa}(A_\kappa - A_{\kappa-1}) = 1, \quad e_{\kappa k} = 0 \quad \text{for } k > s(\kappa) .$$

2^o If $s(\kappa-1)<s(\kappa)$ then

$$e_{\kappa s(\kappa-1)}=\frac{1}{\alpha_\kappa}\Big(\beta_{s(\kappa-1)}-A_{\kappa-1}+\sum_{k=1}^{s(\kappa-1)-1}\beta_k\Big)=\frac{1}{\alpha_\kappa}\Big(\sum_{k=1}^{s(\kappa-1)}\beta_k-A_{\kappa-1}\Big)>0,$$

$e_{\kappa k}=\frac{1}{\alpha_\kappa}A_\kappa w_k^\kappa \geq 0$ for $k>s(\kappa-1)$, $e_{\kappa k}=0$ for $k>s(\kappa)$,

3^o $s(\kappa-1)>s(\kappa)$ is impossible; i.e. all elements e_{jk} of the matrix $E_{\alpha,\beta}$ are nonnegative,

$e_{\kappa k}=0$ for $k<s(\kappa-1)$, $\kappa>1$ and

$e_{\kappa k}=0$ for $k>s(\kappa)$, $\kappa<m$. Moreover, since $v^m=\alpha$, $E^*_{\alpha,\beta}v^m=w^m=\beta$ and

$$\sum_{k=1}^{m}e_{\kappa k}=E^*_{\alpha,\beta}e^{(\kappa)}\cdot e=\frac{1}{\alpha_\kappa}(A_\kappa w^\kappa\cdot e-A_{\kappa-1}w^{\kappa-1}\cdot e)=\frac{1}{\alpha_\kappa}(A_\kappa-A_{\kappa-1})=1$$

for $\kappa=1,2,\dots,m$, $e=(1,1,\dots,1)\in R_m$,

we obtain the following statement:

The matrix $E_{\alpha,\beta}$ is stochastic, $E_{\alpha,\beta}b=c$, $E^*_{\alpha,\beta}\alpha=\beta$.

It follows from the theorem (2.3) that $\sigma \prec \mu$.
On the other hand if $a_1\geq a_2\geq\dots\geq a_m$, $\lambda=\sum_{j=1}^{m}\alpha_j\delta_{a_j}$,

$\lambda\prec\mu$ then, by theorem (3.2) and definition of c,

$v^m\cdot a=w^m\cdot b=v^m\cdot c$ and $v^\kappa\cdot a\leq w^\kappa\cdot b=v^\kappa\cdot c$ for

$\kappa=1,2,\dots,m$.

Hence, the conditions for comparability of measures λ and σ hold. It follows from the theorem (3.5) that $\lambda\prec\sigma$.

It remains to show that $c_1\geq c_2\geq\dots\geq c_m$. Clearly, $s(1)\leq s(2)\leq\dots\leq s(m)$. Using the properties of e_{jk}, we obtain the following inequalities:

$$b_1\geq c_\kappa=\sum_{k=1}^{m}e_{\kappa k}b_k=\sum_{k=1}^{s(\kappa)}e_{\kappa k}b_k\geq b_{s(\kappa)}\sum_{k=1}^{s(\kappa)}e_{\kappa k}=b_{s(\kappa)},$$

$$b_{s(\varkappa)} = b_{s(\varkappa)} \sum_{k=s(\varkappa)}^{m} e_{\varkappa+1,k} \geq \sum_{k=s(\varkappa)}^{m} e_{\varkappa+1,k} b_k = \sum_{k=1}^{m} e_{\varkappa+1,k} b_k = c_{\varkappa+1} \geq b_m .$$

Hence $b_1 \geq c_\varkappa \geq b_{s(\varkappa)} \geq c_{\varkappa+1} \geq b_m$ for $\varkappa = 1,2,\dots,m-1$.

If σ' is a measure having also the properties 1°, 2° and 3° in (4.1), then $\sigma \prec \sigma'$, $\sigma' \prec \sigma$. Hence by the theorem (2.7) $\sigma' = \sigma$.

The proof contents moreover the following statement:

(4.2) Corollary. Let $\alpha \in \mathcal{P}_m^+$ and $\beta \in \mathcal{P}_n^+$ be given. Then there exists one and only one stochastic matrix $E_{\alpha,\beta}$ of the type (m,n) such that

1° $E^*_{\alpha,\beta}\,\alpha = \beta$.

2° If $b_1 \geq b_2 \geq \dots \geq b_n$, $\mu = \sum_{k=1}^{n} \beta_k \, \delta_{b_k}$,

$c = E_{\alpha,\beta}\, b$, $\sigma = \sum_{j=1}^{m} \alpha_j \, \delta_{c_j}$ then

σ is a greatest element of the set

(4.3) $\{\lambda \in \mathcal{P};\ \lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j},\ a_1 \geq a_2 \geq \dots \geq a_m,\ \lambda \prec \mu\}$.

This matrix $E_{\alpha,\beta}$ has the following property:

(4.4) If $s(\varkappa) \in \{1,2,\dots,n\}$, $\sum_{k=1}^{s(\varkappa)-1} \beta_k \leq \sum_{j=1}^{\varkappa} \alpha_j < \sum_{k=1}^{s(\varkappa)} \beta_1$

for $\varkappa = 1,2,\dots,m-1$, $s(m) = n$, $b_1 \geq b_2 \geq \dots \geq b_n$, $c = E_{\alpha,\beta}\, b$,

then $b_1 \geq c_1 \geq b_{s(1)} \geq c_2 \geq b_{s(2)} \geq \dots \geq c_\varkappa \geq b_{s(\varkappa)} \geq c_{\varkappa+1} \geq \dots \geq b_{s(m-1)} \geq c_m \geq b_n$.

(4.5) Note. If $\alpha = (\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m}) \in \mathcal{P}_m^+$, $\beta = (\frac{1}{n}, \frac{1}{n}, \dots$ $\dots, \frac{1}{n}) \in \mathcal{P}_n^+$ then the matrix $E_{\alpha,\beta}$ coincides with the doubly-stochastic unit matrix E of the type (m,n) introduced in [3] as an exposed element.

5 . Numerical examples.

(5.1) Suppose $\beta = (\frac{1}{2}t, 1-t, \frac{1}{2}t)$, $\alpha = (x, 1-x)$ where $t \in (0,1)$, $x \in (0, \frac{1}{2}\rangle$, $b = (1,0,-1)$, $\mu = \sum_{k=1}^{3} \beta_k \delta_{b_k}$.

We want to calculate real numbers c_1 and c_2 such that the measure $\sigma = x\delta_{c_1} + (1-x)\delta_{c_2}$ is a greatest element of the set (4.3)

1° If $x < \frac{1}{2}t$ then $s(1)=1, w^1=(1, 0, 0)$, $s(2)=3, w^2=(\frac{1}{2}t, 1-t, \frac{1}{2}t)$, $E_{\alpha,\beta} = \begin{bmatrix} 1, & 0, & 0 \\ \frac{t-2x}{2(1-x)}, & \frac{1-t}{1-x}, & \frac{t}{2(1-x)} \end{bmatrix}$

$c = (1, -\frac{x}{1-x})$.

2° If $\frac{1}{2}t \leq x < \frac{1}{2}t + (1-t)$ then $\frac{1}{2}t \leq x$, $\frac{1}{2}t < 1-x$,

$s(1) = 2, w^1 = (\frac{t}{2x}, 1-\frac{t}{2x}, 0)$,
$s(2) = 3, w^2 = (\frac{1}{2}t, 1-t, \frac{1}{2}t)$, $E_{\alpha,\beta} = \begin{bmatrix} \frac{t}{2x}, & 1-\frac{t}{2x}, & 0 \\ 0, & 1-\frac{t}{2(1-x)}, & \frac{t}{2(1-x)} \end{bmatrix}$

$c = (\frac{t}{2x}, -\frac{t}{2(1-x)})$.

3° If $\frac{1}{2}t + (1-t) \leq x$ then $1-x \leq \frac{1}{2}t$,

$s(1) = 3, w = (\frac{t}{2x}, \frac{1}{x}-\frac{t}{x}, 1-\frac{1}{x}+\frac{t}{2x})$,
$s(2) = 3, w = (\frac{1}{2}t, 1-t, \frac{1}{2}t)$, $E_{\alpha,\beta} = \begin{bmatrix} \frac{t}{2x}, & \frac{1-t}{x}, & \frac{t-2x-2}{2x} \\ 0, & 0, & 1 \end{bmatrix}$

$c = (\frac{1}{x} - 1, -1)$.

In particular,

if $x = \frac{1}{2}$ then $\alpha = (\frac{1}{2}, \frac{1}{2})$,

$E_{\alpha,\beta} = \begin{bmatrix} t, & 1-t, & 0 \\ 0, & 1-t, & 0 \end{bmatrix}$, $c = (t, -t)$, $\sigma = \frac{1}{2}(\delta_t + \delta_{-t})$;

if moreover $\mu = \frac{1}{3}(\delta_1 + \delta_0 + \delta_1)$ then $\sigma = \frac{1}{2}(\delta_{\frac{2}{3}} + \delta_{-\frac{2}{3}})$.

(5.2) In order to illustrate the property (4.4) of the conditionally maximal measure σ take the following example:

$\alpha=(\frac{1}{5},\frac{1}{5},\dots,\frac{1}{5})\in \mathcal{P}_5^+,\ \beta=(\frac{1}{3},\frac{1}{3},\frac{1}{3})\in \mathcal{P}_3^+,\quad b=(3,0,-3)$.

Then $p(1)=1,\ p(2)=2,\ p(3)=2,\ p(4)=3,\ p(5)=3,$

$E_{\alpha,\beta}\, b = c = (3,2,0,-2,-3)$.

Clearly

$$b_1 \geq c_1 \geq b_1 \geq c_2 \geq b_2 \geq c_3 \geq b_2 \geq c_4 \geq b_3 \geq c_5 \geq b_3 .$$

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