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0. Introduction. In this note some topological extensions are studied. The notion of the p-topological extension is introduced and it is shown that every topological extension fulfilling the Myškis condition (Γ) is in fact a p-topological extension ($(T, \mathcal{O}_T)$ is a topological extension of the space $(G, \mathcal{O}_G)$ if G is a dense subset of the space T and if $\mathcal{O}_T/G = \mathcal{O}_G$). In part 2, the notion of the S^{φ}-topological extension is introduced which is a special case of the p-topological extension. Part 3 deals with the notion of the C-topological extension, which is a generalization of the Caratheodory method for compactification of a simply connected bounded plane domains and which applies also to general Moore spaces. Finally, in part 4, the equivalence of the C-topological extension with the S^{φ}-topological extension for plane domains is demonstrated.

1. p-topological extension. Let $(G, \mathcal{O})$ be a topological space with the system $\mathcal{O}$ of open sets; let Z be a set and $p : \mathcal{O} \longrightarrow \exp(G \cup Z)$ a mapping such that the following axioms are fulfilled:

$(O_p) : p(G) = G \cup Z$,

(1_p): $p(A\cap B) = p(A) \cap p(B)$ for $A, B \in \mathcal{O}$.

Then the system $\{p(H)\ ;\ H \in \mathcal{O}\}$ forms the base of a certain topology on $G \cup Z$; this topology will be denoted by the symbol $\mathcal{O}_p$. The original topology of the space $(G, \mathcal{O})$ will agree with the topology induced on G if $p(H) \cap G \in \mathcal{O}$ for every $H \in \mathcal{O}$. This is certainly the case if

(2_p): $H \in \mathcal{O} \Longrightarrow p(H) \cap G = H$.

<u>Lemma 1</u>. Let the mapping p fulfil the axioms (0_p), (1_p), (2_p). Then the set G is dense in the space $(G \cup Z, \mathcal{O}_p)$ iff the following axiom (3_p) is fulfilled:

(3_p): $p(A) = \emptyset \Longleftrightarrow A = \emptyset$.

Let $(G, \mathcal{O})$, Z and $p : \mathcal{O} \to \exp(G \cup Z)$ have the meaning described above and suppose that the axioms (0_p) – (3_p) are fulfilled. Then the topological space $(G \cup Z, \mathcal{O}_p)$ is a topological extension of the space $(G, \mathcal{O})$; we call this extension the <u>p-topological extension</u> (precisely the (p, Z)-topological extension).

<u>Lemma 2.</u> 1) $H_1, H_2 \in \mathcal{O}$, $H_1 \subset H_2 \Longrightarrow p(H_1) \subset p(H_2)$ provided p fulfils (1_p),

2) $H \in \mathcal{O}_p \Longrightarrow H \subset p(H \cap G)$ if (2_p) is fulfilled.

<u>Definition</u>. Let $(R, \mathcal{S})$ be a topological extension of the space $(G, \mathcal{O})$ (in the sense of the introduction). We say that $(R, \mathcal{S})$ and $(G, \mathcal{O})$ fulfil the condition (Γ) (see Myškis [4]), if

$x \in R$, $U \in \mathcal{U}^{\mathcal{S}}(x) \Longrightarrow [$ there is a

$\mathcal{U}_1 \in \mathcal{U}^{\mathcal{S}}(x)$, $\mathcal{U}_1 \subset \mathcal{U}$ such that $y \in R - (G \cup \mathcal{U}_1)$, $V \in \mathcal{U}^{\mathcal{S}}(y) \Rightarrow V \cap (G - G \cap \mathcal{U}_1) \neq \emptyset]$.

<u>Lemma 3.</u> $(G \cup Z, \mathcal{O}_p)$ and $(G, \mathcal{O})$ fulfil the condition (Γ).

<u>Proof</u>. Let $x \in G \cup Z$, let $\mathcal{U} \in \mathcal{U}(x)$ be open in the topology $\mathcal{O}_p$. There is a $\mathcal{L} \subset \mathcal{O}$ with $\mathcal{U} = \bigcup_{A \in \mathcal{L}} p(A)$; let $x \in p(A)$, $A \in \mathcal{L}$. If we put $\mathcal{U}_1 = p(A)$, then (Γ) is easily verified.

<u>Theorem 4</u>. Let $(R, \mathcal{S})$ be a topological extension of $(G, \mathcal{O})$ and put $Z = R - G$. Define the mapping p by

$$p(H) = H \cup \{x \in Z;\ \text{there is a } \mathcal{U} \in \mathcal{U}^{\mathcal{S}}(x) \text{ with } G \cap \mathcal{U} \subset H\},\quad H \in \mathcal{O}.$$

Then p fulfils the axioms $(0_p) - (3_p)$ and $\mathcal{O}_p \subset \mathcal{S}$; in addition,

$$\mathcal{O}_p = \mathcal{S} \Longleftrightarrow (R, \mathcal{S}), (G, \mathcal{O}) \text{ fulfil the condition } (\Gamma).$$

<u>Proof</u>. One easily verifies that p fulfils (0_p)-(3_p) and $\mathcal{O}_p \subset \mathcal{S}$. Let now $H \in \mathcal{S}$ and assume (Γ). Then $H \cap G \in \mathcal{O}$ and $H \subset p(H \cap G) \in \mathcal{O}_p$. Let us fix $x \in H$; then there is a $\mathcal{U}_1 \in \mathcal{U}^{\mathcal{S}}(x)$, $\mathcal{U}_1 \subset H$ with

$$y \in R - (G \cup \mathcal{U}_1),\ V \in \mathcal{U}^{\mathcal{S}}(y) \Rightarrow V \cap (G - G \cap \mathcal{U}_1) \neq \emptyset.$$

It is easy to show that $x \in p(\mathcal{U}_1 \cap G) \subset H$, hence $H \in \mathcal{O}_p$. The rest follows from lemma 3.

2. <u>S^{φ}-topological extensions.</u> Let again $(G, \mathcal{O})$ be a topological space, let $\mathcal{L} \subset \mathcal{O}$ be a system of open sets, $\emptyset \notin \mathcal{L}$. Suppose that φ is a relation on $\mathcal{L} \times \mathcal{L}$

fulfilling the following axiom

$(1_\wp)$: $X, Y \in \mathcal{L}, \ X \wp Y \Rightarrow X \subset Y$.

An <u>ideal element</u> of $(G, \mathcal{O})$ is every nonempty system of open sets $\mathcal{S} \subset \mathcal{L}$ fulfilling the following conditions

(1_S): $\bigcap_{S \in \mathcal{S}} S = \emptyset$,

(2_S): $S_1, S_2 \in \mathcal{S} \Longrightarrow$ there exists an $S \in \mathcal{S}$ with $S \subset S_1 \cap S_2$,

(3_S): $S \in \mathcal{S}, \ Q \in \mathcal{L}, \ S \wp Q \Longrightarrow Q \in \mathcal{S}$,

(4_S): $S \in \mathcal{S} \Longrightarrow$ there exists a $T \in \mathcal{S}$ with $T \wp S$,

(5_S): $A, B \in \mathcal{L}, \ A \wp B, \ A \cap S \neq \emptyset$ for every $S \in \mathcal{S} \Longrightarrow B \in \mathcal{S}$.

Let $S^\wp(G)$ denote the set of all ideal elements of $(G, \mathcal{O})$.

<u>Lemma 5</u>. 1) If $\mathcal{S} \in S^\wp(G)$ then each finite subsystem of $\mathcal{S}$ has a non-void intersection.

2) For $\mathcal{S}_1, \mathcal{S}_2 \in S^\wp(G)$

$$[\mathcal{S}_1 \neq \mathcal{S}_2 \Longleftrightarrow \text{there exist } S_i \in \mathcal{S}_i \ (i = 1, 2) \text{ with } S_1 \cap S_2 = \emptyset].$$

3) $\mathcal{S}, \mathcal{S}' \in S^\wp(G), \ \mathcal{S} \subset \mathcal{S}' \Longrightarrow \mathcal{S} = \mathcal{S}'$.

For every $H \in \mathcal{O}$ we put

$p(H) = H \cup \{\mathcal{S} \in S^\wp(G); \text{there is an } S \in \mathcal{S} \text{ with } S \subset H\}$.

It is easy to see that the mapping $p: H \longrightarrow p(H)$ fulfils the axioms $(0_p) - (3_p)$, so that we may form the $(p, S^\wp(G))$-topological extension of the space $(G, \mathcal{O})$ according to the preceeding paragraph; this extension will be called the S^φ -topological extension (precisely the $(S^\varphi(G); \mathcal{L})$-topological extension) and the topology of this extension will be denoted by $\mathcal{O}^\varphi$. For every $x \in G \cup S^\varphi(G)$, $\mathcal{U}(x) = \{p(H); H \in \mathcal{O}, x \in p(H)\}$ forms the local open base at x .

Lemma 6. $\mathcal{S}_1, \mathcal{S}_2 \in S^\varphi(G)$, $\mathcal{S}_1 \neq \mathcal{S}_2 \Longrightarrow$ there exist $U_i \in \mathcal{U}(\mathcal{S}_i)$ $(i = 1, 2)$ with $U_1 \cap U_2 = \emptyset$.

Proof: According to lemma 5 there are $S_i \in \mathcal{S}_i$ with $S_1 \cap S_2 = \emptyset$. We put $U_i = p(S_i)$, $i = 1, 2$.

In what follows we suppose that the relation φ fulfils the following strengthening $(\overline{1_\varphi})$ of the axiom (1_φ) :

$$(\overline{1_\varphi}): \quad X, Y \in \mathcal{L}, \quad X \varphi Y \Longrightarrow uX \subset Y$$

(where uX denotes the closure of X in the space $(G, \mathcal{O})$).

Lemma 7. $\mathcal{S} \in S^\varphi(G)$, $x \in G \Longrightarrow$ there exist $U_1 \in \mathcal{U}(\mathcal{S})$, $U_2 \in \mathcal{U}(x)$ with $U_1 \cap U_2 = \emptyset$.

Proof: Suppose that $A \cap H \neq \emptyset$ for every $A \in \mathcal{S}$ and for every $H \in \mathcal{U}(x) \cap \mathcal{O}$. Then $x \in \bigcap_{A \in \mathcal{S}} uA$ .According to (4_S) and $(\overline{1_\varphi})$, given $A \in \mathcal{S}$ there is a $B_A \in \mathcal{S}$ with $uB_A \subset A$. Thus $x \in \bigcap_{A \in \mathcal{S}} A$, in contradiction with (1_S).

Theorem 8. 1) The one-point sets in $S^\varphi(G)$ are closed in the space $(G \cup S^\varphi(G), \mathcal{O}^\varphi)$.
2) If $(G, \mathcal{O})$ is a T_0 $(T_1, T_2$ resp.) space, then $(G \cup S^\varphi(G), \mathcal{O}^\varphi)$ is a T_0 $(T_1, T_2$ resp.) space.

Further properties of the S^φ -topological extension are studied in [7]; J.C. Taylor demonstrated, besides other things, that the $S^{\wp}$-topological extension is even a compactification provided the relation $\wp$ fulfils the following axioms

$(1_{\wp})$: $A \wp B \Longrightarrow uA \subset B$,

$(4_{\wp})$: $A_i \wp B_i$, $i = 1, 2 \Longrightarrow (A_1 \cap A_2) \wp (B_1 \cap B_2)$,

$(5_{\wp})$: $A \wp B \Longrightarrow (G - uB) \wp (G - uA)$,

$(7_{\wp})$: $A \wp B \Longrightarrow$ there is a set C, $A \wp C \wp B$.

3. C -topological extensions. Let $(T, \mathcal{O})$ be a topological space, let $G \subset T$ be a domain (a nonempty connected open set). We say that an arc $\widehat{AB}$ in T is a cross-cut of G if $\widehat{AB} \subset G \cup \{A, B\}$, $A, B \notin G$. Let us denote by $Q(G)$ the set of all cross-cuts of G. For $q \in Q(G)$ put further $\mathring{q} = q \cap G$; obviously $\mathring{q}$ is a connected set. $G \subset T$ is called a Q-domain, if for every cross-cut $q \in Q(G)$ there exist the separate domains $G_1, G_2 \subset G$ with the property $G - q = G_1 \cup G_2$, $q \subset H(G_1) \cap H(G_2)$ (the symbol $H(M)$ denotes the boundary of $M \subset T$ in the space $(T, \mathcal{O})$). Every bounded simply connected domain in the euclidean plane or, more generally, every nonempty domain bounded by a continuum in the Moore space fulfilling axioms 1 - 5 (see Moore, [6], theorem 34) is an example of a Q-domain.

In the remainder of this paragraph G denotes a Q-domain in some topological space $(T, \mathcal{O})$.

Lemma 9. a) Let $q \in Q(G)$ and suppose that the domains G_1, G_2, G_1', G_2' in G fulfil the conditions $G_1 \cap G_2 = \emptyset = G_1' \cap G_2'$, $q \subset H(G_1) \cap H(G_2) \cap H(G_1') \cap H(G_2')$. Then G_1, G_2 are separated and either $G_1 = G_1'$ and $G_2 = G_2'$ or $G_1 = G_2'$ and $G_2 = G_1'$.

b) Let $q_1, q_2 \in Q(G)$, $\mathring{q}_1 \cap \mathring{q}_2 = \emptyset$, $G - q_1 = G_1 \cup G_2$, where G_1, G_2 are separated domains, $q_1 \subset H(G_1) \cap H(G_2)$. Then either $\mathring{q}_2 \subset G_1$ or $\mathring{q}_2 \subset G_2$.

Let $q_1, q_2 \in Q(G)$, $\mathring{q}_1 \cap \mathring{q}_2 = \emptyset$. According to previous lemma the arc q_1 separates G into two disjoint domains; the domain that has nonempty intersection with the arc q_2 will be denoted by $G(q_1, q_2)$. Let now $q_1, q_2, q_3 \in Q(G)$, $\mathring{q}_i \cap \mathring{q}_j = \emptyset$ for $i \neq j$. We say that the cross-cut q_2 separates the cross-cuts q_1, q_3, if $G(q_2, q_1) \cap G(q_2, q_3) = \emptyset$.

Lemma 10. a) $q_1, q_2 \in Q(G)$, $\mathring{q}_1 \cap \mathring{q}_2 = \emptyset \Longrightarrow \mathring{q}_2 \subset G(q_1, q_2)$,

b) $q_1, q_2 \in Q(G), \mathring{q}_1 \cap \mathring{q}_2 = \emptyset \Longrightarrow G - G(q_2, q_1) \subset G(q_1, q_2)$,

c) q_2 separates $q_1, q_3 \Longleftrightarrow q_2$ separates $q_3, q_1 \Longleftrightarrow G(q_2, q_3) \subset G(q_1, q_2) \Longleftrightarrow G(q_2, q_1) \subset G(q_3, q_2)$.

Proof: a) This follows immediately from lemma 9.

b) We may write $G - q_2 = G(q_2, q_1) \cup G'$, where $G(q_2, q_1)$, G' are separated domains, $q_2 \subset H(G(q_2, q_1)) \cap H(G')$. On account of the relation $G' \subset G' \cup \mathring{q}_2 \subset G' \cup H(G')$ we conclude that the set $G' \cup \mathring{q}_2$ is connected. Write again $G - q_1 =$

$= G(q_1, q_2) \cup G''$, where $G(q_1, q_2)$, G'' are separated domains, $q_1 \subset H(G(q_1, q_2)) \cap H(G'')$. We have

$$G' \cup \mathring{q}_2 \subset G(q_1, q_2) \cup G'', \; (G' \cup \mathring{q}_2) \cap G(q_1, q_2) \supset \mathring{q}_2 ,$$

whence $G' \cup \mathring{q}_2 \subset G(q_1, q_2)$.

c) This assertion follows from the preceding part.

Definition. The sequence $\{q_n ; q_n \in Q(G)\}_{n=1}^{\infty}$ is called a C -chain of the domain Q , if

1) $q_n \cap q_{n+1} = \emptyset$ for every $n = 1, 2, \dots$,

2) q_n separates q_{n-1} , q_{n+1} for every $n = 2, 3, \dots$,

according to lemma 10 we may replace the condition 2) by

2*) $G(q_n, q_{n+1}) \subset G(q_{n-1}, q_n)$ for every $n \geq 2$.

If $\{q_n\}$, $\{q'_n\}$ are the C-chains of the domain G , we define the following relations $\prec$, $\sim$:

I) $\{q_n\} \prec \{q'_n\} \overset{def}{\Longleftrightarrow} \forall n \; \exists k \; (G(q_k, q_{k+1}) \subset G(q'_n, q'_{n+1}))$,

II) $\{q_n\} \sim \{q'_n\} \overset{def}{\Longleftrightarrow} \{q_n\} \prec \{q'_n\}$ and $\{q'_n\} \prec \{q_n\}$.

It is easy to see that the relation $\sim$ just defined is an equivalence relation.

Every equivalent class of the C -chains is called the end of the domain G. If E_1, E_2 are the ends of G , we define

$$E_1 \prec E_2 \overset{def}{\Longleftrightarrow} \forall \{q^1_n\} \in E_1 , \forall \{q^2_n\} \in E_2 \; (\{q^1_n\} \prec \{q^2_n\}) .$$

The primend of the Q -domain G is the end E of G with the property:

$$E' \prec E , \; E' \text{ is the end} \Longrightarrow E' = E .$$

Let $C(G)$ denote the set of all primends of the domain G . For $A \subset G$ we put

$\rho(A) = A \cup \{E \in C(G);\ \forall \{q_n\} \in E \; \exists n_0 \; (G(q_{n_0}, q_{n_0+1}) \subset A)\}$.

It is easy to see that the mapping $\rho : H \to \rho(H)$ fulfils the axioms $(0_{\mathfrak{n}})-(3_{\mathfrak{n}})$ (where $\mathcal{O}$ is the system of all open subsets of a set G, $\mathcal{Z} = \mathcal{C}(G)$); we may form again the $\mathfrak{n}$-topological extension of the Q-domain G with the topology $\mathcal{O}$; we call this extension the $\mathcal{C}$-topological extension (precisely the $\mathcal{C}(T, G)$-topological extension).

For every Q-domain G of the topological space $(T, \mathcal{O})$ we define the system $\mathcal{L}(G)$ in the following way:

$A \in \mathcal{L}(G) \overset{def}{\Longleftrightarrow} A \subset G$ is a domain and there is a $q \in Q(G)$ such that $G - q = A \cup (G - \{q \cup A\})$, where the domains A, $G - (q \cup A)$ are separated, $q \subset H(A) \cap H(G - (q \cup A))$.

Lemma 11. a) $A \in \mathcal{L}(G)$ iff there is precisely one cross-cut $q \in Q(G)$ with the property just introduced (we denote this cross-cut by the symbol q_A),
b) $A, B \in \mathcal{L}(G)$, $A \cap B \neq \emptyset \neq B - A$, $\mathring{q}_A \cap \mathring{q}_B = \emptyset \Longrightarrow \mathring{q}_A \subset B$.

For $A, B \in \mathcal{L}(G)$ we define

$$A \wp B \overset{def}{\Longleftrightarrow} u A \cap G \subset B, \quad q_A \cap q_B = \emptyset .$$

It is easy to see that the relation $\wp$ on $\mathcal{L}(G)$ fulfils the axiom $(\overline{1}_{\wp})$ from the part 2, so that we may form the $S^{\wp}$-topological extension of the domain G , too. The relation between the $\mathcal{C}$-topological extension and the $S^{\wp}$-topological extension of a bounded simply connected plane domain will be examined in the next paragraph.

At this moment we remark only that already in the simplest cases (where G is not a bounded simply connected plane domain) the C-topological extension need not be a compactification, for example if $T = \{[x, y] \in R^2; y > 0\} \cup \{[x, y] \in R^2; y = 0, x = \frac{1}{n}, n = 2, 3, \dots\}$, $\mathcal{O}$ = the euclidean topology, $G = (0,1) \times (0,1)$.

4. <u>The equivalence in the euclidean plane.</u> In the following part G denotes a nonempty bounded simply connected domain in the euclidean plane R^2. According to the previous paragraph we may form the C-topological extension of the domain G, we may define the system $\mathcal{L}(G)$ and the relation $\wp$ on $\mathcal{L}(G)$ and hence we may form the $S^{\wp}$-topological extension of the domain G.

The relationship between C and $S^{\wp}$-extensions is explained by the following

<u>Theorem 12</u>. The $S^{\wp}$-topological extension of G and the C-topological extension of G are homeomorphic and the corresponding homeomorphism can be so chosen that it reduces to the identity map on G.

<u>Proof</u>: First of all we construct a one-to-one mapping F from $G \cup C(G)$ to $G \cup S^{\wp}(G)$. For $E \in C(G)$ we define $F(E)$ as follows:

$A \in F(E) \overset{def}{\Longleftrightarrow}$ there is a C-chain $\{q_n\} \in E$ and a natural number k such that $A = G(q_k, q_{k+1})$. We shall show that $F(E) \in S^{\wp}(G)$. We must verify the axioms (1_S) - (5_S) from the part 2. The axioms (1_S) - (4_S) are obviously fulfilled. We are going to verify the axiom (5_S); let $A, B \in \mathcal{L}(G)$, $A \wp B$, $A \cap X \neq \emptyset$

for every $X \in F(E)$. According to [1] there exist concentric circles $K(s, r_n)$ with the centre s and the radii r_n and a C-chain $\{k_n\} \in E$ such that

$$k_n \subset K(s, r_n), \ \lim r_n = 0 .$$

We put $K_n = G(k_n, k_{n+1})$. Clearly $A \cap K_n \neq \emptyset$ for every n. There are three following possibilities:

I) $A \subset K_n$ for all n; consequently, $A \subset \bigcap_{n=1}^{\infty} K_n = \emptyset$ - in contradiction with $A \in \mathcal{L}(G)$.

II) There exists an N such that $K_N \subset A$; then there are again two possibilities:

a) There is an $n \geq N$ such that $(k_n - \mathring{k}_n) \cap (q_A - \mathring{q}_A) = \emptyset$. This implies $K_n \wp A$, whence $A \in F(E)$ and, consequently, $B \in F(E)$.

b) For no $n \geq N$ is $(k_n - \mathring{k}_n) \cap (q_A - \mathring{q}_A) = \emptyset$. If X, Y are the end-points of the cross-cut q_A, it follows in this case that either $r_n = |s - X|$ or $r_n = |s - Y|$ for every $n \geq N$. But this is impossible on account of $\lim r_n = 0$.

III) There is an N such that $A - K_n \neq \emptyset \neq K_n - A$ for all $n \geq N$; we distinguish two cases again:

a) $\mathring{k}_n \cap \mathring{q}_A = \emptyset$ for infinitely many $n \geq N$; for those n we have $\mathring{q}_A \subset K_n$ (lemma 11) and $\mathring{q}_A \subset \bigcap_{n=1}^{\infty} K_n = \emptyset$.

b) There is an $N_1 \geq N$ such that $\mathring{k}_n \cap \mathring{q}_A \neq \emptyset$ for all $n \geq N_1$. We choose an arbitrary $P_n \in \mathring{k}_n \cap \mathring{q}_A$ for every $n \geq N_1$. The set q_A being compact we may choose a subsequence $\{P_{n_k}\}$ and a point $P \in q_A$ such that $P_{n_k} \to P$. Hence $P = s \in H(G)$ and at least one end point of the arc q_A coincides with s. In the case III b) there are three possibilities again:

I*) $B \subset K_n$ for all n is easily seen to be impossible.

II*) There exists an $N_2 \geq N_1$ such that $K_{N_2} \subset B$ and

a*) $(k_n - \mathring{k}_n) \cap (q_B - \mathring{q}_B) = \emptyset$ for some $n \geq N_2$; it is easy to see that in this case $B \in F(E)$.

b*) $(k_n - \mathring{k}_n) \cap (q_B - \mathring{q}_B) \neq \emptyset$ for all $n \geq N_2$; an argument similar to that used in II b) shows that this is impossible.

III*) There exists an $N_2 \geq N_1$ such that $B - K_n \neq \emptyset \neq$ $\neq K_n - B$ for all $n \geq N_2$ and

a*) $\mathring{k}_n \cap \mathring{q}_B = \emptyset$ for infinitely many $n \geq N_2$; as in III a) one can show that this is impossible.

b) There exists an $N_3 \geq N_2$ such that $\mathring{k}_n \cap \mathring{q}_B \neq \emptyset$ for all $n \geq N_3$; as in III b) we have $s \in q_B - \mathring{q}_B$ and we see that the arcs q_A, q_B are not disjoint (in contradiction with $A \wp B$).

All possibilities have been exhausted and in every case $B \in F(E)$.

It is easy to see that $F(E_1) \neq F(E_2)$ whenever $E_1 \neq E_2$. We want now to show that $F(C(G)) = S^{\wp}(G)$. Let $\mathcal{S} \in S^{\wp}(G)$ and suppose that $F(E) = \mathcal{S}$ for no $E \in C(G)$.

For every $H \subset G$ we put

$p_s(H) = H \cup \{\mathcal{S} \in S^{\wp}(G);$ there is an $A \in \mathcal{S}$ with $A \subset H\}$,

$p_c(H) = H \cup \{E \in C(G)$; for every C-chain $\{q_n\} \in E$

there exists an m_0 such that $G(q_{m_0}, q_{m_0+1}) \subset H\}$.

According to lemma 5, for every $E \in C(G)$ there are $A_E \in F(E)$, $S_E \in \mathcal{S}$ such that $A_E \cap S_E = \emptyset$. Obviously $E \in \kappa_c(A_E)$, whence $\bigcup_{E \in C(G)} \kappa_c(A_E) \supset C(G)$. According to lemma 7, for every $X \in G$ there are the sets $U_X \in \mathfrak{U}(X)$, $B_X \in \mathcal{S}$ such that $U_X \cap \kappa_s(B_X) = \emptyset$ and, consequently, $(U_X \cap G) \cap B_X = \emptyset$. Obviously $\bigcup_{X \in G} (U_X \cap G) = G$. The sets $\kappa_c(A_E)$, $U_X \cap G$ are open in $G \cup C(G)$ and

$$\bigcup_{E \in C(G)} \kappa_c(A_E) \cup \bigcup_{X \in G} (U_X \cap G) = G \cup C(G) .$$

The C-topological extension of the plane domain G is a compactification (see Caratheodory [1]); there are $E_1, \ldots, E_m \in C(G)$, $X_1, \ldots, X_k \in G$ such that

$$\bigcup_{i=1}^{m} \kappa_c(A_{E_i}) \cup \bigcup_{i=1}^{k} (U_{X_i} \cap G) = G \cup C(G) .$$

Hence it follows

$$\bigcap_{i=1}^{k} B_{X_i} \cap \bigcap_{i=1}^{m} S_{E_i} = \emptyset ,$$

in contradiction with lemma 5. Further we define F as the identity map on G. Then F is a one-to-one correspondence between $G \cup C(G)$ and $G \cup S\varphi(G)$. It is easy to verify the following implications:

$$H \subset G, \ X \in \kappa_c(H) \Longrightarrow F(X) \in \kappa_s(H) ,$$

$$H \subset G, \ X \in \kappa_s(H) \Longrightarrow F^{-1}(X) \in \kappa_c(H) .$$

We see that F is a homeomorphism.

R e f e r e n c e s

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