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A REMARK ON COMPACTNESS OF EMBEDDINGS

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In the theory of nuclear spaces the following situation is important: we are given two Banach spaces E_1 and E_3 and a Hilbert space E_2 with continuous embeddings

$$E_1 \xrightarrow{T_{12}} E_2 \xrightarrow{T_{23}} E_3$$

In [2], V.G. Ramm investigated the connection between the compactness of the mappings T_{12} and compactness of $T_{13} = T_{12} \cdot T_{23}$. In his work the fact that E_2 is Hilbert is used in an essential manner. It is the purpose of the present note to show that his result holds in a more general setting which simplifies both the statement and the proof of the proposition.

We use the following terminology. A mapping T of a normed space P into another normed space Q is said to be praecompact if the image of the closed unit ball of P is a praecompact subset of Q. A continuous injection is a one-to-one continuous embedding; we do not assume that it is onto.

<u>Proposition</u>. Let E_1, E_2, E_3 be three normed spaces, let $A \in L(E_1, E_2)$ and $T \in L(E_1, E_3)$. Suppose that

1° T is praecompact,

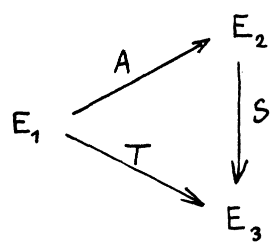
2^o for each $\varepsilon > 0$ there exists an $\omega(\varepsilon) > 0$ such that

$$|Ax| \leqq \varepsilon |x| + \omega(\varepsilon) |Tx| .$$

Then A is praecompact as well.

On the other hand, praecompactness of A implies conditions 1^o and 2^o provided the following additional assumption is made;

3^o there exists a continuous injection $S \in L(E_2, E_3)$ such that $T = S \circ A$.



Proof. Assume 1^o and 2^o. Denote by $\mathcal{U}$ the closed unit ball of E_1. Let $\varepsilon > 0$ be given. Since T is praecompact, there exists a finite set $F \subset \mathcal{U}$ such that, for each $x \in \mathcal{U}$

$$\inf_{z \in F} |Tx - Tz| \leqq \frac{\varepsilon}{2(\omega(\frac{\varepsilon}{4}) + 1)} .$$

By condition 2^o, we have for each $x \in \mathcal{U}$ and each $z \in F$

$$|Ax - Az| \leqq \frac{\varepsilon}{4} |x - z| + \omega(\tfrac{\varepsilon}{4}) |Tx - Tz| \leqq$$

$$\leqq \frac{\varepsilon}{2} + \omega(\tfrac{\varepsilon}{4}) |Tx - Tz| .$$

It follows that

$$\inf_{z\in F} |Ax - Az| \leq \frac{\varepsilon}{2} + \omega(\frac{\varepsilon}{4}) \inf_{z\in F} |Tx - Tz| \leq \varepsilon \quad .$$

To prove the second part, assume that A is praecompact and that condition 3° is satisfied. It follows immediately that T, a superposition of a continuous and a praecompact mapping, is praecompact. To prove 2°, note first that, S being an injection, the range of S' is $\sigma(E_2', E_2)$ dense in E_2'. Suppose now that $\varepsilon > 0$ is given and that no $\omega(\varepsilon)$ with the properties stipulated in 2° exists. It follows that there exists a sequence $x_n \in E$, such that

$$|Ax_n| > \varepsilon |x_n| + n |Tx_n| \quad .$$

We may clearly assume that $|x_n| = 1$ so that $|A| \geq$ $\geq |Ax_n| > \varepsilon$ and $Tx_n \to 0$. The operator A being praecompact, it is possible to extract a subsequence y_n of x_n such that Ay_n is a Cauchy sequence. Since all $|Ay_n| >$ $> \varepsilon$, there exists a z' in the range of S' such that $\langle Ay_n, z' \rangle$ tends to a limit different from zero. Now $z' =$ $= S'v'$ for some $v' \in E_3'$ so that

$$\langle Ay_n, z' \rangle = \langle Ay_n, S'v' \rangle = \langle SAy_n, v' \rangle = \langle Ty_n, v' \rangle \; .$$

Since $Tx_n \to 0$, this is a contradiction.

References

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[2] A.G. RAMM: Neobchodimyj i dostatočnyj priznak kompaktnosti vloženija, Vestnik leningradskogo universiteta,seria matematiki,mechaniki, astronomii 18(1963),150-151.

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