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CONCERNING THE BANACH-STONE THEOREM

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In the sequel we consider compact Hausdorff spaces X, Y and corresponding B-algebras $C(X)$, $C(Y)$ of all complex-valued continuous functions.

Recently (cf.[5]), it was estabslished that for any linear isometry u of the Banach algebra $C(Y)$ into the Banach algebra $C(X)$, X and Y being compact, there exists a closed subset $Q \subseteq X$ and a continuous mapping φ of Q onto Y with $\langle uf, x\rangle = \alpha(x)\cdot\langle f, \varphi(x)\rangle$ for all $x \in Q$ and for all $f \in C(Y)$, where $\alpha \in C(X)$ and $|\alpha(x)| = 1$ for any $x \in Q$.

With respect to this fact the following question arises: suppose that u is a linear isometry of $C(Y)$ into $C(X)$; to investigate under what conditions the mapping φ induced by u is a homeomorphism from Q onto Y. This question is completely solved in the statement (b) of the following Theorem *). Further we shall show a straightforward proof of the theorem of

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*) The presented results were communicated as Appendix in [6] firstly.

Holsztyński and, finally, we shall state some closely related results concerning the Šilov boundary.

Let u denote a linear isometry of $C(Y)$ into $C(X)$. Putting $E = \{uf ; f \in C(Y)\}$ we mean by t_u the corresponding adjoint mapping of the topological dual space E' onto $C'(X)$. The isometric image E of $C(Y)$ by u is a Banach space with the topology induced by the norm-topology in $C(Y)$. Denote by $E(X)$ the family of all extremal points of the unit ball in E'; similarly $E(Y)$ stands for the collection of all extremal points of the unit ball in $C'(Y)$. It is obvious that $t_u(E(X)) = E(Y)$. The canonical embedding $q : X \longrightarrow E'$ is defined by $\langle q(x), f \rangle = f(x)$. Similarly q_0 means the canonical embedding of Y into $C'(Y)$. In the sequel we put

(1) $$B = q(X) \cap E(X), \quad Q = q^{-1}(B) .$$

For an arbitrary Banach space F the set of all extremal points in the unit ball of F' need not be, in general, weakly compact. In what follows we shall prove the weak compactness of $E(X)$; consequently and with regard to the weak continuity of q the subset Q defined by (1) is closed, hence compact, in X.

<u>Theorem</u>. Let X and Y be two compact Hausdorff spaces and let u be a linear isometry from $C(Y)$ into $C(X)$. The subsets Q and B are defined by (1). Then it holds:

(a) There exists a continuous mapping φ from Q onto Y such that

$$\langle uf, x \rangle = \alpha(x) \cdot \langle f, \varphi(x) \rangle$$

for any $x \in Q$ and any $f \in C(Y)$, where $\alpha \in C(X)$, $\|\alpha\| = 1$ and $|\alpha(x)| = 1$ for all $x \in Q$.

(b) The mapping φ defined in (a) is a homeomorphism from Q onto Y if and only if the following conditions are satisfied:

1° The collection $E = u(C(Y))$ separates the points of Q.

2° For any $q(x_1) \in B$, $q(x_2) \in B$, $q(x_1) \neq q(x_2)$, and for any complex number β, $|\beta| = 1$, it holds $q(x_1) \neq \beta q(x_2)$.

Proof. (a). The proof of the statement (a) is a modification of the proof of the Banach-Stone theorem (cf.[3]). First we recall that (cf.[3])

$$E(Y) = \bigcup_{|\alpha|=1} \alpha Y .$$

Hence, the subset $E(Y)$ is weakly compact in $C'(Y)$. From the weak continuity of t_u and from $t_u(E(X)) = E(Y)$ we may conclude that $E(X)$ is weakly compact in E', thus $Q = q^{-1}(B)$ is compact in X. For any $q(x) \in B$ there exists a unique element $y \in Y$ such that

$$t_u(q(x)) = \alpha(x) \cdot q_0(y) , \tag{2}$$

where $|\alpha(x)| = 1$ for each $x \in Q$. We define now a mapping t of B onto Y by $t(q(x)) = y$, where

$q(x)$ and y satisfy the relation (2). It is easy to see that $t(B) = Y$.

For any $x \in Q$, $\langle u(e), x \rangle = \langle q(x), u(e) \rangle =$

$$= \langle \alpha(x) \cdot q_0(y), e \rangle = \alpha(x) \quad ,$$

where $e \in C(Y)$, $e(y) = 1$ for all $y \in Y$. This implies $u(e) = \alpha \in E$, hence t is continuous on B. The function $\varphi(x) = t(q(x))$, $x \in Q$, satisfies evidently the properties stated in (a). To verify the equality in (a), it suffices to note that $\langle uf, x \rangle =$ $= \langle {}^t u(q(x)), f \rangle = \langle \alpha(x) \cdot q_0(y), f \rangle = \alpha(x) \cdot f(\varphi(x))$ for any $x \in Q$, $y = \varphi(x)$ and $f \in C(Y)$.

(b) Suppose now that the properties 1^0 and 2^0 are satisfied. To establish that t, hence φ, is a homeomorphism, it suffices to prove that t is one-to-one. If for some $q(x_1) \in B$, $q(x_2) \in B$ we have $t(q(x_1)) = t(q(x_2)) = y$, then

$${}^t u(q(x_1)) = \alpha(x_1) \cdot q_0(y), \; {}^t u(q(x_2)) = \alpha(x_2) \cdot q_0(y).$$

But ${}^t u$ is a linear isometry, consequently, $q(x_1) =$ $= \beta q(x_2)$ for $\beta = \alpha(x_1) \cdot (\alpha(x_2))^{-1}$. According to the property 2^0 we conclude $q(x_1) = q(x_2)$.

On the other hand, if φ is a homeomorphism from Q onto Y, then obviously E separates the points of Q. Suppose that $q(x_1) = \beta \cdot q(x_2)$ for some $|\beta| = 1$. By the definition of t we obtain $t(q(x_1)) = t(q(x_2))$, hence $q(x_1) = q(x_2)$. This completes the proof of the statement (b).

Remark 1. The condition 2^o of the statement (b) may be formulated in the following way

$2^{o\prime}$ Let x_1 and x_2 be two points of Q. If for some $f \in E$ $f(x_1) \neq f(x_2)$, then for any complex $\beta, |\beta| = 1$, there exists a function $f_\beta \in E$ with $f_\beta(x_1) \neq \beta \cdot f_\beta(x_2)$.

Especially, if the vector space E separates the points of Q and if there exists a function in E with constant non-zero values on Q, then the mapping φ from Q onto Y defined in the precedent Theorem is a homeomorphism. Indeed, we we may suppose without loss of generality that $e \in E$, $e(x) = 1$ for any $x \in Q$. From $q(x_1) = \beta \cdot q(x_2)$, $x_1 \in Q$, $x_2 \in Q$, we obtain

$$1 = \langle q(x_1), e \rangle = \beta \cdot \langle q(x_2), e \rangle = \beta .$$

Remark 2. The statements of Theorem hold also in the case that $C(X)$ and $C(Y)$ represent the spaces of all continuous and real-valued functions on X and Y. Especially, if the image $u(e)$ of the unit element e of the algebra $C(Y)$ is a positive function on Q (e.g., if u is an isotonic linear isometry), then evidently $\alpha(x) = 1$ for all $x \in Q$. From $\alpha \in E$ it follows the property 2^o. The last case has been investigated from another point of view in [4].

Now we are ready to apply the previous results to the abstract Dirichlet's problem (in the sense of Bauer, cf.[1]).

First we recall that if X is a topological space and E is a family of bounded and continuous functions on X, then a compact subset $C \subseteq X$ will be termed a Šilov boundary of E whenever the following conditions are satisfied:

(i) For any $f \in E$

$$\max_{x \in C} |\langle f, x \rangle| = \sup_{x \in X} |\langle f, x \rangle| .$$

(ii) For any compact subset $C' \subseteq C$, $C' \neq C$, there exists $f \in E$ such that

$$\max_{x \in C'} |\langle f, x \rangle| < \max_{x \in C} |\langle f, x \rangle| .$$

Now we complete the precedent Theorem by

<u>Corollary</u>. Suppose that all assumptions of Theorem are fulfilled and that, moreover, φ is a homeomorphism. Then the subset Q defined by (1) is a Šilov boundary of E.

<u>Proof</u>. To prove the property (i), we may assume that the mapping φ defined by Theorem is continuous. For any such φ and any $f \in C(Y)$ we obtain

$$\sup_{x \in X} |\langle uf, x \rangle| = \sup_{y \in Y} |\langle f, y \rangle = $$
$$= \sup_{z \in Q} |\langle f, \varphi(z) \rangle| = \max_{z \in Q} |\langle uf, z \rangle| .$$

The last inequality implies (i).*)

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*) It should be noticed that a subset $C \subseteq X$ satisfying only the property (i) is called by some authors the boundary of the family E. In any case the subset Q defined by (1) is the boundary of the family E.

Suppose now that φ is a homeomorphism and that Q_0 is a proper and closed subset of Q. For some $x_0 \in Q \setminus Q_0$ the subset $V = \varphi(Q \setminus Q_0)$ is a neighborhood of $y_0 = \varphi(x_0)$ in Y. We choose a function $f \in C(Y)$, $0 \leq f \leq 1$, $f(y_0) = 1$ and $f(y) = 0$ for all $y \notin V$. Since $uf \in E$ and

$$|\langle uf, x_0 \rangle| = |\langle f, \varphi(x_0) \rangle| = 1, \ |\langle uf, x \rangle| = |\langle f, \varphi(x) \rangle| = 0$$

for all $x \in Q_0$, we obtain the property (ii).

<u>Remark 3</u>. In particular, if E separates the points of Q and if some constant non-zero function on Q is contained in E, then Q is the Šilov boundary of E.

This result is a complex modification of the Bauer's maximum principle (cf.[2]).

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