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SOME REMARKS ON POLYNOMIAL OPERATORS

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Introduction. The polynomials on abstract spaces have been introduced by M. Fréchet in [5],[6]. But a systematical study on abstract polynomials and their properties we can find in [14]. Some questions about the existence of eigenvectors of homogeneous compact symmetric polynomials on the space L_2 were studied in [3] and a similar problem for positive polynomial operators was discussed in [4]. Further, the theory of analytical operators both in complex and real Banach spaces is based on the notion of the abstract polynomial [2],[8]. The first part of this note deals with the problem of continuity of the polynomial operator which is an inverse operator to a continuous operator on a Banach space. In the second part we consider some problems on the existence of eigenvectors for symmetric and positive polynomial operators on a Hilbert space.

1. Notations and definitions. Let X, Y be linear spaces. An operator $P(x_1, \dots, x_k)$ from $X \times X \times \dots \times X$ into Y is said to be the k-linear operator if it is linear in each variable $x_1, \dots, x_k$. The k-linear operator will be called the symmetric k-linear operator if it is invariant under arbitrary permutation of variables $x_1, \dots, x_k$.

We shall say that an operator $P(x)$ from X to Y is the homogeneous polynomial operator of the order $k \geqslant 1$ (briefly h.p.-operator) if there is a symmetric k-linear operator $P^*(x_1,\dots,x_k)$ such that $P(x) = P^*(x,\dots,x)$. It is easy to show that for some h.p.-operator $P(x)$ the symmetric k-linear operator $P^*(x_1,\dots,x_k)$ is defined unambiguously. This operator will be called the polar operator to the operator $P(x)$. We shall say that an operator $P(x) = P_0 + P_1(x) + \dots + P_m(x)$ from X to Y is the polynomial operator of the order $m \geqslant 1$ if $P_0 \in Y$ is a constant and $P_i(x)$, $i = 1,2,\dots,m$ are homogeneous polynomial operators of the order i from X to Y.

The following algebraic properties of the homogeneous polynomial operator $P(x)$ and its polar operator $P^*(x_1,\dots,x_k)$ are well-known:

(1.1) $$P^*(x_1,\dots,x_k) = \frac{1}{2^k k!} \sum \varepsilon_1 \dots \varepsilon_k P(\varepsilon_1 x_1 + \dots + \varepsilon_k x_k),$$

where the summation is related to all groups $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$ of numbers $+1$ and -1.

(1.2) $$P(x_1 + \dots + x_n) = \sum \frac{k!}{i_1! \dots i_n!} P^*(x_1^{i_1}, \dots, x_n^{i_n}),$$

where $P^*(\dots, x^j, \dots)$ denotes $P^*(\dots, \underbrace{x,\dots,x}_{j\text{-times}}, \dots)$ and the summation is related to all groups $\{i_1,\dots,i_n\}$ of numbers $0,1,2,\dots,n$ such that $i_1 + i_2 + \dots + i_n = k$.

(1.3) $$P(x) - P(y) = \sum_{i=1}^{k} P^*(x^{k-i}, y^{i-1}, x-y)$$

for any $x, y \in X$.

Let $P(x) = P_0 + P_1(x) + P_2(x) + \dots + P_m(x)$ be a polynomial operator of the order $m \geqslant 1$. Then for arbitrary mutual different real numbers $t_0, t_1, \dots, t_m$ there exist real numbers τ_{ij} ; $i, j = 0, 1, \dots, m$ such that

$$(1.4) \qquad P_i(x) = \sum_{j=0}^{m} \tau_{ij} P(t_j x)$$

for any $x \in X$ and $i = 0, 1, \dots, m$.

2. The continuity and boundedness of the polynomial operators

In this section we recall some based properties of polynomial operators on linear topological spaces and, especially, on Banach spaces.

Theorem 2.1. Let X be a linear topological space and let Y be a normed linear space. Then the k-linear operator $P(x_1, \dots, x_k)$ from $X \times \dots \times X$ to Y is continuous if and only if it is bounded on an open subset $U \times \dots \times U \subset X \times \dots \times X$.

For the proof of this theorem see, for example, [13] (the case $k = 2$).

Corollary. Let X be a locally convex linear topological space, Y be a normed linear space. Then the k-linear operator $P^*(x_1, \dots, x_k)$ from $X \times \dots \times X$ to Y is continuous if and only if there is a seminorm $\| \cdot \|_X$ and a positive real number M such that for any $x_i \in X$, $i = 1, 2, \dots, k$ it holds

$$\| P^*(x_1, \dots, x_k) \| \leqq M \|x_1\|_X \cdots \|x_k\|_X .$$

Proposition 2.3. Let X, Y be normed linear spaces and let $P(x)$ be a polynomial operator from X to Y of the order $m \geqslant 1$. Then the following assertions are equivalent.

a) $P(x)$ is continuous at a point $x_0 \in X$.

b) $P(x)$ is continuous at any point in X.

c) $P(x)$ is bounded on every ball in X.

d) $P(x)$ is uniformly continuous on every ball in X.

e) $P(x)$ satisfies the Lipschitz's condition on every ball in X.

The proof of this proposition follows easily from (1.1) - (1.4).

Proposition 2.4 ([14],p.182).Let X, Y be Banach spaces and let $P(x)$ be a polynomial operator of the order m from X to Y having the Baire's property (i.e., $P(x)$ is continuous on X except, maybe, a set of the first category). Then $P(x)$ is continuous on X.

Definition 2.5 ([2],[3],[4]). Let X, Y be normed linear spaces and let P be a continuous h.p.-operator of the order $k \geqslant 1$ from X to Y. Then we define the norms $\|P\|$ and $\|P^*\|$ of P and its polar operator P^* as follows

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\|, \quad \|P^*\| = \sup_{\|x_i\| \leq 1, i=1,2,\dots,k} \|P^*(x_1,\dots,x_k)\|.$$

Remark 2.6. Using (1.1) and Definition 2.5 we obtain the following inequalities

$$\| P \| \leqq \| P^* \| \leqq \frac{k^k}{k!} \| P \| .$$

The latter inequality cannot be generally improved as it is shown in [11].

Definition 2.7. Let X, Y be Banach spaces. Denote $\mathcal{P}_Y(X)^m$ the normed linear space of all continuous polynomial operators of the order m from X to Y with the norm $\| P \| = \| P_0 \| + \| P_1 \| + \dots + \| P_m \|$, where P_i are continuous h.p.-operators of the order $i = 0, 1, \dots, m$. Similarly, denote $\mathcal{L}_Y(X)^k$ and $\mathcal{H}_Y(X)^k$ the space of all continuous k-linear operators and the space of all continuous h.p.-operators from X to Y of the order $k \geqslant 1$.

Remark 2.8. It is obvious that $\mathcal{P}_Y(X)^m, \mathcal{L}_Y(X)^k, \mathcal{H}_Y(X)^k$ are Banach spaces with norms defined above. Indeed, if $\{P_n\} \in \mathcal{H}_Y(X)^k$ is a fundamental sequence, then $P(x) = \lim_{n \to \infty} P_n(x)$ is a h.p.-operator having the Baire's property and thus, due to Proposition 2.4, $P \in \mathcal{H}_Y(X)^k$ and $\| P_n - P \| \to 0$.

Finally we show that there is a theorem on polynomial operators which is an analogy to the well-known Banach-Steinhaus theorem.

Theorem 2.9. Let $M \subset \mathcal{P}_Y(X)^m$ be the set of continuous polynomial operators such that for any $P \in M$ the set $\{P(x) / x \in X\}$ is bounded in Y. Then the set M is bounded in the space $\mathcal{P}_Y(X)^m$.

Proof. According to (1.4) it is sufficient to prove this theorem provided that $P \in M$ are h.p.-operators of the order $k \geqslant 1$. Let P^* be the polar operator to P. Denote

$$B_n = \{x_i \in X;\ i = 1, 2, \dots, k \,/\, \|P^*(x_1, \dots, x_k)\| \leq n, P \in M\}.$$

Then $X = \bigcup_{n=1}^{\infty} B_n$ and there is a set B_p of the second category. The set of all differences $x - y$; $x, y \in B_p$ contains a neighbourhood of the point zero in X and thus, there is a positive real number $r > 0$ such that for $\|x_i\| \leq r$; $i = 1, 2, \dots, k$ we have $\|P^*(x_1, \dots, x_k)\| \leq$ $\leq p$. Hence, for $z = x - y$, $\|z\| \leq r$ we can write

$$\|P(z)\| = \Big\|\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} P^*(x^i, y^{k-i})\Big\| \leq p \cdot 2^k$$

Let $x \in X$, $\|x\| \leq 1$ and choose a positive real number σ such that $\sigma \cdot r > 1$. Then for any $P \in M$ we obtain

$$\|P(x)\| = \|\sigma^k P(\frac{x}{\sigma})\| \leq \sigma^k \cdot p \cdot 2^k .$$

Hence $\|P\| = \sup_{\|x\| \leq 1} \|P(x)\| \leq \sigma^k \cdot p \cdot 2^k$ and the theorem is proved.

<u>Proposition 2.10</u> ([14]). Let $\{P_n\}$ be a sequence of continuous polynomial operators from $\mathcal{P}_Y(X)^m$. Then the set

$$M = \{x \in X \,/\, \overline{\lim_{n \to \infty}} \|P_n(x)\| < +\infty\}$$

is either the set of the first category or M is equal to X.

It is obvious that so-called "principle of the condensation of singularities" reminds valid also for polynomial operators.

3. The continuity of inverse operators

In this section let X, Y denote Banach spaces and for $x_0 \in X$, $r > 0$ let $K_r(x_0) = \{x \in X / \|x - x_0\| < r\}$.

Proposition 3.1. Let F be an operator from X to Y having an inverse operator F^{-1}. Suppose F^{-1} is a h.p.-operator of the order $k \geq 1$. If there is a positive real number $\gamma > 0$ such that

$$\| F(x) \|^{k} \geq \gamma \| x \|$$

for any $x \in X$ then F^{-1} is the continuous h.p.-operator.

Proof. Let $y \in Y$, $x = F^{-1}(y)$. Then

$$\|x\| = \| F^{-1}(y) \| \leqq \frac{1}{\gamma} \| F(F^{-1}(y)) \|^{k} = \frac{1}{\gamma} \| y \|^{k}$$

and thus, according to Proposition 2.3, the operator F^{-1} is continuous.

Corollary 3.2. Let F be an operator mapping a Hilbert space H onto itself. Suppose F has an inverse h.p.-operator F^{-1} of the order $k \geq 1$. If there is a positive constant $\gamma > 0$ such that

$$(F(x), x) \geq \gamma \| x \|^{\frac{k+1}{k}}$$

for any $x \in X$ then F^{-1} is the continuous h.p.-operator.

Lemma 3.3. Let P be a h.p.-operator of the order $m \geq 1$ from X to Y with its polar operator P^*. For any natural number n let E_n be the set in X such that

$$\| P^*(x_1, \dots, x_k) \| \leqq n \|x_1\| \dots \|x_m\|$$

for each $x_i \in E_n$, $i = 1, 2, \dots, m$. Then, at least, one of the sets E_n is dense in X.

<u>Proof</u>. The sets E_n are nonempty because each of them contains the point zero. For arbitrary $x \in X$, $x \neq 0$ we can choose the smallest natural number n such that $n > \frac{\|P(x)\|}{\|x\|^m}$. Then $x \in E_n$ and thus $X = \bigcup_{n=1}^{\infty} E_n$. Being X Banach space, X is the set of the second category and there is a set E_{n_0} and a ball $K_r(x_0)$ such that $K_r(x_0) \cap E_{n_0}$ is dense in $K_r(x_0)$. Let $K_{r_1}(x_1)$ be a ball such that $\overline{K_{r_1}(x_1)} \subset \overline{K_r(x_0) \cap E_{n_0}}$, where $x_1 \in E_{n_0}$.

Then for $x \in X$, $\|x\| = r_1$ we have $x_1 + x \in \overline{K_{r_1}(x_1)}$ and there is a sequence $\{z_k\}$, $z_k \in K_{r_1}(x_1) \cap E_{n_0}$ which converges to $x_1 + x$. Hence, the sequence $x_k = z_k - x_1$ converges to x and we can assume that $\|x_k\| \geqslant r_1 2^{-\frac{1}{m}}$. Then, using (1.2), we obtain

$$\|P(x_k)\| = \|P(z_k - x_1)\| \leqq n_0 \sum_{i=0}^{m} \binom{m}{i} \|z_k\|^i \|x_1\|^{m-i} \leqq$$

$$\leqq n_0 \sum_{i=0}^{m} \binom{m}{i} \|x_1\|^{m-i} (r_1 + \|x_1\|)^i \leqq \frac{2n_0 (r_1 + 2\|x_1\|)^m}{r_1^m} \cdot \|x_k\|^m .$$

Let n denote the smallest natural number which is greater than $\frac{2n_0 (r_1 + 2\|x_1\|)^m}{r_1^m}$. Then $\|P(x_k)\| \leqq n \|x_k\|^m$ and thus $x_k \in E_n$. Let $x \in X$, $x \neq 0$ be an arbitrary point. Then for $\xi = r_1 \cdot \frac{x}{\|x\|}$ we have $\|\xi\| = r_1$ and there is a sequence $\xi_k \in E_n$ such that $\lim_{k \to \infty} \xi_k = \xi$. Hence, the sequence $\{x_k\} = \{\xi_k \cdot \frac{\|x\|}{r_1}\}$ converges to x and we obtain

$$\| P(x_k) \| = \| P(\xi_k \cdot \frac{\|x\|}{r_1}) \| = \frac{\|x\|^m}{r_1^m} \| P(\xi_k) \| \leq$$

$$\leq \frac{\|x\|^m}{r_1^m} \cdot n \| \xi_k \|^m = n \| x_k \|^m .$$

Consequently, $x_k \in E_n$ and E_n is dense in X.

Theorem 3.4. Let F be a continuous operator from X onto Y having an inverse operator F^{-1}. Suppose F^{-1} is a homogeneous polynomial operator of the order $m \geq 1$. Then F^{-1} is the continuous polynomial operator from Y to X.

Proof. Denote F^{-1*} the polar operator to F^{-1} and let

$$Y_k = \{ y_i \in Y, i = 1, 2, \ldots, m / \| F^{-1*}(y_1, \ldots, y_m) \| \leq k \| y_1 \| \ldots \| y_m \| \}$$

where $k = 1, 2, \ldots$. According to Lemma 3.3 there is a set Y_n which is dense in Y. Let $y \in Y$, $\| y \| = r > 0$. Then the set $\overline{K_r(0)} \cap Y_n$ is dense in $\overline{K_r(0)}$ and we can find elements $y_k \in Y_n$, $k = 1, 2, \ldots$, such that

$$\| y - (y_1 + \ldots + y_k) \| \leq \frac{r}{2^k} , \| y_k \| \leq \frac{r}{2^{k-1}} ,$$
$$y = \lim_{k \to \infty} \sum_{i=1}^{k} y_i .$$

Let us put

$$x_{i_1 \ldots i_k} = \frac{m!}{i_1! \ldots i_k!} F^{-1*}(y_1^{i_1}, \ldots, y_k^{i_k}) ,$$

where $\{ i_1, \ldots, i_k \}$ are the groups of integers $0, 1, \ldots, m$ such that $i_1 + \ldots + i_k = m$. Denote $x_k = \sum x_{i_1 \ldots i_k}$ adding over all such groups of integers. Then we obtain

$$\|x_{i_1\dots i_k}\| \leq \frac{m!}{i_1!\dots i_k!} n \|y_1\|^{i_1}\dots\|y_k\|^{i_k}$$

and thus, the sequence $\{x_k\}$ is a fundamental sequence in X. But X is a Banach space and there is a point $x \in X$ such that $\lim_{k\to\infty} x_k = x$. Using the continuity of the operator F and (1.2) we obtain

$$F(x) = F(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} F(F^{-1}(y_1+\dots+y_k)) = y.$$

Hence

$$\|F^{-1}(y)\| = \|x\| \leq n \lim_{k\to\infty} \sum \frac{m!}{i_1!\dots i_k!} \|y_1\|^{i_1}\dots\|y_k\|^{i_k} \leq n\cdot 2^m\cdot \|y\|^m.$$

Finally, the Proposition 2.3 completes the proof.

Remark 3.5. As an example of the continuous homogeneous polynomial operator of the order $k \geq 1$ can serve the operator $P(x)$ defined on the space $L_2([0,1])$ as follows

$$P(x) \equiv y(s) = \int_0^1\dots\int_0^1 K(s,t_1,\dots,t_k)x(t_1)\dots x(t_k)dt_1\dots dt_k ,$$

where $K(s,t_1,\dots,t_k)$ is a quadratically integrable function on $[0,1]\times\dots\times[0,1]$. Similarly, the operator

$$P(x) \equiv y(s) = \sum_{i=0}^{m} \int_0^1 K_i(s,t)x^i(t)dt ,$$

where $K_i(s,t)$, $i=0,1,\dots,m$ are continuous functions on $[0,1]\times[0,1]$, is the continuous polynomial operator of the order m on the space $C([0,1])$.

4. Eigenvectors of symmetric homogeneous polynomial operators

In this section let X', Y' denote dual spaces to

Banach spaces X, Y. The symbol $\langle y, f' \rangle$ denotes the valueeof a continuous linear functional $f' \in Y'$ at a point $y \in Y$ and (x, y) is the inner product in a Hilbert space.

Definition 4.1. Let P be a h.p.-operator of the order $k \geq 1$ from X to Y and let Q be a h.p.-operator of the order ℓ from X to Y'. We shall say that P is Q-symmetric if the following functional

$f(x_1, \dots, x_k, y_1, \dots, y_\ell) = \langle P^*(x_1, \dots, x_k), Q^*(y_1, \dots, y_\ell) \rangle$

is the symmetric $k+\ell$ -linear functional. We shall say that P is the symmetric h.p.-operator if $Y = X'$ and P is I-symmetric. (I is the identity operator.)

Lemma 4.2. Let for any continuous homogeneous polynomial functional $p(x)$ of the order $k \geq 1$ on X hold

$$\|p\| = \|p^*\| . \tag{4.2'}$$

Then for any continuous h.p.-operator $P \in \mathcal{H}_Y(X)^k$ it holds

$$\|P\| = \|P^*\| . \tag{4.2''}$$

Proof. It is obvious that $\|P\| \leqq \|P^*\|$. Now we prove the opposite inequality. For arbitrary positive real number ε we can choose $\bar{x}_1, \dots, \bar{x}_k \in X$, $\|\bar{x}_i\| = 1$, $i = 1, 2, \dots, k$ such that $\|P^*(\bar{x}_1, \dots, \bar{x}_k)\| > \|P^*\| - \varepsilon$. Let us denote $y = P^*(\bar{x}_1, \dots, \bar{x}_k)$. Using the well-known corollary of the Hahn-Banach theorem we can find $f' \in Y'$ such that $\|f'\| = 1$ and $\langle y, f' \rangle = \|y\|$.

Thus, for the continuous homogeneous polynomial functional $p(x) = f'(P(x)) = \langle P(x), f' \rangle$ we obtain $\|p\| = \|p^*\| \leqq \|P^*\|$. But

$$p^*(\bar{x}_1,\dots,\bar{x}_k) = \langle y, f' \rangle = \|y\| = \|P^*(\bar{x}_1,\dots,\bar{x}_k)\| > \|P^*\| - \varepsilon .$$

Hence $\|P^*\| \leqq \|p^*\| = \|p\| \leqq \|P\|$ and our lemma is proved.

We shall say that a normed linear space has the "p_k-property" if the assumption of Lemma 4.2 is satisfied.

Remark 4.3. There are spaces which have not the "p_k-property" for $k > 1$ as, for example, the space $L([0,1])$ (see [11]). But S. Banach has proved in [3] that the space L_2 has the "p_k-property" for any $k \geqslant 1$. An inspection of the proof shows that every abstract Hilbert space has also this property for any $k \geqslant 1$.

Remark 4.4. Every continuous homogeneous polynomial functional $p(x)$ defined on a subspace H_0 of a Hilbert space H can be extended to a continuous h.p.-functional $\bar{p}(x)$ on H such that $p(x) = \bar{p}(x)$ for any $x \in H_0$ and $\|p\| = \|\bar{p}\|$. It follows immediately from ([16],Theorem 7) and the Remark 4.3.

Theorem 4.5. Let X be a Hilbert space, Y be a Banach space. Suppose A is a continuous linear operator from X to Y' and A' is its adjoint operator. If P is A-symmetric continuous h.p.-operator of the order $k \geqslant 1$ from X to Y, then

$$\|A'P\| = \sup_{\|x\| \leqq 1} |\langle P(x), Ax \rangle| .$$

Proof. Denote $p(x) = \langle P(x), A(x) \rangle$. Then $|p(x)| \leqq \|A'P\| \cdot \|x\|^{k+1}$ and thus $\|p\| \leqq \|A'P\|$. For arbitrary positive real number $\varepsilon > 0$ we can choose $\bar{x}_1, \dots,$

$\bar{x}_k \in X$, $\|\bar{x}_i\| = 1$, $i = 1, 2, \ldots, k$ such that $\|A'P^*(\bar{x}_1, \ldots$ $\ldots, \bar{x}_k)\| > \|A'P\| - \varepsilon$. Let us denote

$$\bar{x}_{k+1} = \frac{A'P^*(\bar{x}_1, \ldots, \bar{x}_k)}{\|A'P^*(\bar{x}_1, \ldots, \bar{x}_k)\|} .$$

Then $\|\bar{x}_{k+1}\| = 1$ and we obtain

$$p^*(\bar{x}_1, \ldots, \bar{x}_{k+1}) = \langle P^*(\bar{x}_1, \ldots, \bar{x}_k), A\bar{x}_{k+1}) =$$
$$= (A'P^*(\bar{x}_1, \ldots, \bar{x}_k), \bar{x}_{k+1}) = \|A'P^*(\bar{x}_1, \ldots, \bar{x}_k)\| > \|A'P\| - \varepsilon .$$

Using Remark 4.3 we can conclude that $\|p\| = \|p^*\| \geq \|A'P\|$ and this implies $\|p\| = \|A'P\|$.

Corollary. Let X be a Hilbert space and let P be a continuous symmetric h.p.-operator mapping X into itself. Then

$$\|P\| = \sup_{\|x\| \leq 1} |(P(x), x)| .$$

Lemma 4.6. Let F be an operator from a Hilbert space X into X having the Fréchet derivative $F'(0)$ at the point 0 and let $F(0) = 0$, $F'(0)h = 0$ for any $h \in$ $\in X$. Then F has at most one point of bifurcation, namely the point zero.

Proof. If λ is a point of bifurcation of the operator F then there is a sequence $\{\lambda_n\}$ of eigenvalues and a sequence $\{x_n\}$ of eigenvectors such that $\lambda_n \to \lambda$ and $\lambda_n = \dfrac{(F(x_n), x_n)}{(x_n, x_n)}$. Further

$$F(x) = F(0) + F'(0)(x) + \omega(x) = \omega(x) ,$$

where $\omega(x)$ is an operator in a neighbourhood of 0 such that $\lim\limits_{\|x\| \to 0} \dfrac{\|\omega(x)\|}{\|x\|} = 0$. Then $\lim\limits_{\|x_n\| \to 0} \dfrac{(F(x_n), x_n)}{(x_n, x_n)} \leq$

$\leq \lim_{\|x_n\| \to 0} \frac{\|\omega(x_n)\|}{\|x_n\|} = 0$ and thus $\lambda = 0$.

Proposition 4.7. Let X be a Hilbert space, Y a Banach space. Let A' be the adjoint operator to a continuous linear operator A from X to Y'. Suppose P is a continuous A-symmetric h.p.-operator of the order $k \geq 1$ from X to Y. Then for any positive real number ε there is a point $x_\varepsilon \in X$, $\|x_\varepsilon\| = 1$ and a real number λ, $|\lambda| = \|A'P\|$ such that

$$\|A'P(x_\varepsilon) - \lambda x_\varepsilon\| < \varepsilon .$$

Proof. Let $\varepsilon > 0$. According to Theorem 4.5 there is $x_\varepsilon \in X$, $\|x_\varepsilon\| = 1$ such that

$$|(A'P(x_\varepsilon), x_\varepsilon)| > \|A'P\| - \frac{\varepsilon}{2\,\|A'P\|} .$$

Denote $\lambda = \operatorname{sign}(A'P(x_\varepsilon), x_\varepsilon)\,\|A'P\|$. Then we obtain

$$\|A'P(x_\varepsilon) - \lambda x_\varepsilon\|^2 = \|A'P(x_\varepsilon)\|^2 - 2|(A'P(x_\varepsilon), x_\varepsilon)| \cdot$$

$$\cdot \|A'P\| + \|A'P\|^2 < 2\|A'P\|^2 - 2(\|A'P\|^2 - \tfrac{\varepsilon}{2}) = \varepsilon .$$

Theorem 4.8. (The existence of eigenvectors.) Let P be a homogeneous A-symmetric polynomial operator of the order k from a Hilbert space X into a Banach space Y. Let, at least, one of the following conditions hold.

(i) P is a completely continuous operator and A is a continuous linear operator.

(ii) P is continuous and A is a completely continuous linear operator. Then for any $a > 0$ there is an eigenvector $x \in X$, $\|x\| = a$ of the operator $A'P$ with a real eigenvalue λ, $|\lambda| = \|A'P\| a^{k-1}$. The only one point of bifurcation of the operator $A'P$ is the point zero.

Proof. Let $a = 1$. According to Proposition 4.7 there is a sequence $\{x_n\} \in X$, $\|x_n\| = 1$ and a real number λ_0 such that $\|A'P(x_n) - \lambda_0 x_n\| \longrightarrow 0$. Using the condition (i) or (ii) we can assume that $\{A'P(x_n)\}$ is a convergent sequence. Then for arbitrary natural numbers m, n we have

$$\|x_n - x_m\| \leqq \|x_n - \frac{1}{\lambda_0} A'P(x_n)\| + \frac{1}{|\lambda_0|} \|A'P(x_n) -$$
$$- A'P(x_m)\| + \|x_m - \frac{1}{\lambda_0} A'P(x_m)\| .$$

Hence $\|x_n - x_m\| \longrightarrow 0$ as $m, n \longrightarrow \infty$ and there is a point $x_0 \in X$ such that $\|x_0\| = 1$, $x_n \longrightarrow x_0$. It is obvious that $A'P(x_0) - \lambda_0 x_0 = 0$. Let $a > 0$ be arbitrary positive real number. Then for $x = a x_0$ and $\lambda = \lambda_0 \cdot a^{k-1}$ we obtain

$$P(x) - \lambda x = a^k P(x_0) - \lambda_0 a^k x_0 = a^k (P(x_0) - \lambda_0 x_0) = 0 .$$

If $k \geqslant 2$ then the uniqueness of the bifurcation point zero follows immediately from Lemma 4.7. The case $k = 1$ is well-known from linear analysis.

Proposition 4.9. Let X be a Hilbert space, A be a linear selfadjoint positively defined (i.e., there is a constant $\gamma > 0$ such that $(Ax, x) \geqslant \gamma \|x\|$ for any $x \in X$) and P be a compact symmetric h.p.-operator mapping X into itself. Then for any $a > 0$ there is a point $x \in X$, $\|x\| = a$ and a real number λ

such that

$$P(x) - \lambda A x = 0 .$$

Proof. Denote X_A the space with the inner product $[x, y] = (Ax, y)$ for any $x, y \in X$. Then the operator $Q = A^{-1}P$ is a continuous symmetric h.p.-operator on X_A and, according to Proposition 4.7, there is a sequence $x_n \in X_A$ and a real number λ such that $[Q(x_n) - \lambda x_n, Q(x_n) - \lambda x_n] \to 0$ as $n \to \infty$. But $[Q(x_n) - \lambda x_n, Q(x_n) - \lambda x_n] = (A(Q(x_n) - \lambda x_n), Q(x_n) - \lambda x_n) \geq$

$\geq \gamma \| Q(x_n) - \lambda x_n \|$ and thus $\| A^{-1}P(x_n) - \lambda x_n \| \to 0$. Now we proceed as in the proof of Theorem 4.8.

Proposition 4.10. Let F be an operator mapping a Hilbert space X onto a Banach space Y. Suppose F has an inverse operator F^{-1} which is a h.p.-operator of the order $k \geq 1$ and let P be a compact h.p.-operator from X to Y. Assume further that $F^{-1}P$ is symmetric and let, at least, one of the following conditions hold.

(i) There is a positive constant γ such that
$$(F(x), x) \geq \gamma \| x \|^{\frac{k+1}{k}} ,$$

(ii) F is a continuous operator.

Then for any $a > 0$ there is a point $x \in X$, $\| x \| = a$ and a real number λ $|\lambda|^{k} = \| F^{-1} P \|$ such that

$$P(x) - \lambda F(x) = 0 .$$

Proof. Using Proposition 3.1 or Theorem 3.4 we obtain that $F^{-1}P$ is a symmetric completely continuous h.p.-operator. Then Theorem 4.8 gives the assertion.

<u>Proposition 4.11</u>. Let $P = P_1 + \dots + P_m$ be a completely continuous polynomial operator of the order $m \geq 1$ from a Hilbert space X into X satisfying

$$\sup_{\|x\| \leq 1} |(P(x), x)| = \sum_{i=1}^{m} \|P_i\| .$$

Then there is a point $x_0 \in X$, $\|x_0\| = 1$ and a real number λ, $|\lambda| = \sum_{i=1}^{m} \|P_i\|$ such that

$$P(x_0) - \lambda x_0 = 0 .$$

<u>Proof</u>. We can proceed as in the proof of Proposition 4.7 and the proof of Theorem 4.8.

<u>Remark 4.12</u>. There are polynomial operators (even on finite-dimensional spaces) which have no eigenvectors. A theorem on the existence of a continuous branch of positive eigenvectors is shown in [4]. An example of a polynomial operator which has discrete spectrum we can find in [15].

5. <u>Positive polynomial operators</u>

In this section let X, Y denote Hilbert spaces.

<u>Definition 5.1</u>. We shall say that a h.p.-operator P of the odd order $k \geq 1$ from X to X is positive on a set $M \subset X$ if

$$(P^*(x^{k-1}, h), h) \geq 0$$

for any $x \in M$ and any $h \in X$. We shall say that P is positively defined on M if there is a positive constant α such that

$$(P^*(x^{k-1}, h), h) \geq \alpha \|x\|^{k-1} \cdot \|h\|^2$$

for any $x \in M$ and any $h \in X$.

Lemma 5.2. Let P be a continuous positive symmetric h.p.-operator on a bounded closed convex set $M \subset X$. Then the functional

$$f(x) = (P(x), x)$$

is convex and weakly lower semi-continuous functional.

Proof. It is obvious that $f(x)$ has the Fréchet differential of the second order $D^2 f(x, h^2) = k(k+1)(P^*(x^{k-1}, h), h) \geqslant 0$. Then, according to [9], $f(x)$ is convex on M and the reflexivity of the Hilbert space X implies the rest of the assertion.

Proposition 5.3. Let P_i, $i = 1, 2, \dots, m$ be continuous symmetric h.p.-operators of the order i from X into X. Suppose P_i are positively defined on the ball $K = \{x \in X / \|x\| \leq R, R > 0\}$ with constants α_i, $i = 1, 2, \dots, m$. Then the equation

$$\sum_{i=1}^{m} P_i(x) = y$$

has the unique solution in K for any $y \in X$ such that $\|y\| < \sum_{i=1}^{m} \frac{\alpha_i R^i}{i+1}$.

Proof. Let us put

$$f(x) = \sum_{i=1}^{m} \frac{1}{i+1} (P_i(x), x) - (x, y).$$

Then, according to Definition 5.1, we obtain for $x \in K$

$$f(x) \geqslant R\left[\sum_{i=1}^{m} \frac{\alpha_i R^i}{i+1} - \|y\|\right] > 0, \text{ hence } f(x) > f(0).$$

Using Lemma 5.2 we conclude that $f(x)$ possesses its minimum at a point x_0, $\|x_0\| < R$. But $f(x)$ has the Fréchet derivative in K and thus

$$\operatorname{grad} f(x_0) = \sum_{i=1}^{m} P_i(x_0) - y = 0 .$$

The uniqueness is obvious.

<u>Proposition 5.3</u>. Let P be a symmetric positively defined h.p.-operator of the order $n \geq 1$ from X to X and let Q be a continuous symmetric positive h.p.-operator of the order $m \leq n$ from X to X. If there is a point $x_0 \in X, x_0 \neq 0$ such that

$$\frac{m}{n} \inf_{\substack{x \in X \\ x \neq 0}} \frac{(P(x), x)}{(Q(x), x)} = \frac{m}{n} \frac{(P(x_0), x_0)}{(Q(x_0), x_0)} = \lambda_0 .$$

Then

$$P(x_0) - \lambda_0 Q(x_0) = 0$$

and λ_0 is the smallest eigenvalue.

<u>Proof</u>. Denote

$$f(t) = \frac{m}{n} \frac{(P(x_0 + th), x_0 + th)}{(Q(x_0 + th), x_0 + th)} ,$$

where t is a real variable and $h \in X$ is an arbitrary point. Then $f(t)$ possesses its minimum at $t = 0$ and thus $f'(0) = 0$. But

$$f'(0) = \frac{1}{(Q(x_0), x_0)^2} \Big[2n (P(x_0), h) \cdot (Q(x_0), x_0) - $$

$$- (P(x_0), x_0) \cdot \frac{m}{n} (Q(x_0), h) \cdot 2m \Big] .$$

Hence

$(P(x_0) - \lambda_0 Q(x_0), h) = 0$ for any $h \in X$ and thus $P(x_0) - \lambda_0 Q(x_0) = 0$.

If there exists $x_1 \in X$, $x_1 \neq 0$ and a real number λ_1 such that $P(x_1) - \lambda_1 Q(x_1) = 0$, then

$$\lambda_1 = \frac{(P(x_1), x_1)}{(Q(x_1), x_1)} \geq \frac{m}{n} \inf_{\substack{x \in X \\ x \neq 0}} \frac{(P(x), x)}{(Q(x), x)} = \lambda_o \ .$$

This completes the proof.

Theorem 5.4. Let P, Q be continuous symmetric positively defined h.p.-operators of the order $m \geq 1$ from X to X. Assume that the set

$$E = \{x \in X \, / \, (P(x), x) \leq C, \ C > 0\}$$

is compact in X. Then there is a point $x_o \in X, \|x_o\| = 1$ and a positive real number λ_o such that

$$P(x_o) - \lambda_o Q(x_o) = 0$$

and

$\lambda_o = \inf\limits_{\substack{x \in X \\ x \neq 0}} \dfrac{(P(x), x)}{(Q(x), x)}$ is the smallest eigenvalue.

Proof. Let $f(x) = \dfrac{(P(x), x)}{(Q(x), x)}$ and $\lambda_o = \inf\limits_{0 \neq x \in X} f(x)$.

Then there is a sequence $x_n \in X$ such that

$$\lambda_o \leq \frac{(P(x_n), x_n)}{(Q(x_n), x_n)} < \lambda_o + \frac{1}{n} \ .$$

Define $\bar{x}_n = \dfrac{x_n}{\|x_n\|}$. Then $\|\bar{x}_n\| = 1$, $f(\bar{x}_n) = f(x_n)$ and thus

$$(P(\bar{x}_n), \bar{x}_n) < (Q(\bar{x}_n), \bar{x}_n)(\lambda_o + \tfrac{1}{n}) \leq \|Q\|(\lambda_o + \tfrac{1}{n}) \leq \|Q\|(\lambda_o + 1).$$

Using the compactness of E we can choose a subsequence $\bar{x}_{n_k} \to x_o \in X$ as $k \to \infty$. Then $f(\bar{x}_{n_k}) \to f(x_o) = \lambda_o$. Now we use Proposition 5.3 and the theorem is proved.

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