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ON THE CONTINUITY PROPERTIES OF NONLINEAR OPERATORS AND FUNCTIONALS

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Introduction. A number of the fixed-point theorems and approximative methods of solution of nonlinear equations involving strongly and weakly continuous operators were recently established by the methods of Functional Analysis. Some nonlinear problems require the study of the following questions: under which conditions the Fréchet derivative $F'(u)$ of a mapping $F: X \rightarrow Y$ is Lipschitzian, resp. strongly continuous in the norm-topology of the space of all linear continuous transformations of X into Y. Recall that there is a necessary and sufficient condition for the strong continuity of an operator F which acts in reflexive Banach space X with base (Vajnberg [1,Th.7.1]; Citlanadze [2], Rothe [3]). M.I. Kadec [4] has removed the existence of the base of X and he has shown that Theorem 7.1 [1] is valid for separable reflexive Banach spaces. V.I. Anosov [5] has established the validity of Th.7.1 [1] for functionals defined in non-reflexive Banach space X under some another restrictive condition on X. Some simple conditions for the strong and weak continuity of $F: X \rightarrow Y$ have been also given in [6]. For smooth operators there is known the following result of Vajnberg [1,Th.4.4]: Let W be bounded convex subset of a Banach space X, $F: W \to X$ completely continuous on W having compact (in the norm-topology of $(X \to X)$) Fréchet derivative $F'(u)$ on W. Then F is strongly continuous on W.

The strong continuity of the Fréchet derivative $F'(u)$ has been investigated by M.M. Vajnberg. His result is as follows [1,Th.4.5]: Let X be a reflexive Banach space, $F: B_R(0) \to X$ a completely continuous mapping of an open ball $B_R(0) = \{u \in X : \|u\| < R\}$ into X. If the Fréchet derivative $F'(u)$ is uniformly continuous on $B_R(0)$ and compact in $B_{R+\alpha}(0)$, $(\alpha > 0)$, then $F'(u)$ is strongly continuous in $B_R(0)$. (See also [1,Th.4.6]). For the results concerning the strong continuity of gradient maps see [1,§ 7],[4],[5],[7].

The purpose of this note is to establish some further conditions for the strong and weak continuity of nonlinear operators. Furthermore, we derive some conditions under which a) a convex functional f possesses a strongly continuous Fréchet derivative $f'(u)$ on $B_R(0)$; b) a mapping $F: X \to Y$ possesses Lipschitzian and strongly continuous Fréchet derivative $F'(u)$ on a convex open bounded subset $W \subset X$.

1. Let X, Y be linear normed spaces, X^*, Y^* their (adjoint) dual spaces. The pairing between the points of X^* (or Y^*) and the elements of X (or Y)

we denote by $\langle\ ,\ \rangle$. We use the symbols " $\longrightarrow$ ", " $\xrightarrow{w}$ " to denote the strong and weak convergence in X, Y. An operator $L: X \to Y$ is said to be compact (a linear operator L is called also completely continuous) if for each bounded subset $M \subset X$ the set $L(M)$ is compact in Y (a subset $N \subset Y$ is called compact in Y if from each sequence $\{y_n\} \in N$ we can select a subsequence $\{y_{n_k}\}$ such that $\{y_{n_k}\}$ converges to some point $y_0 \in Y$). A mapping $F: M \to \to Y$, $M \subset X$, is said to be completely continuous on a bounded set M, if F is compact and continuous on M. Recall that $F: X \to Y$ is called weakly (strongly) continuous at $u_0 \in X$ if $u_n \xrightarrow{w} u_0 \Longrightarrow \Longrightarrow F(u_n) \xrightarrow{w} F(u_0)$ $(F(u_n) \to F(u_0))$. For the Gâteaux, Fréchet differentials and derivatives we shall use the terminology and notations given in the Vajnberg's book [1,chapt.I], for the weak Fréchet derivatives see for instance [8]. For other notions which will occur in this paper, we refer the reader to [1,chapt.I],[6].

Through this note $D_R(u_0)$, resp. $B_R(u_0)$ will denote the closed resp. open ball centered about the point u_0 and with the radius R, $(X \to X)$ the space of all linear continuous operators of X into Y.

We start with the following

Theorem 1. Let X, Y be linear normed spaces, $F: X \to Y$ a mapping such that $F(tu) = t\,F(u)$ for each $t \in (0, 1)$ and $u \in X$. Suppose F possesses a linear Gâteaux differential $DF(0, h)$ at 0.

Then F is an additive mapping on X.

<u>Proof</u>. First of all we note that $F(0) = 0$. Let u, v be arbitrary (but fixed) elements of X. Since F has a linear Gâteaux differential $DF(0, h)$ at 0, we have for $t > 0$ that

$$
\begin{aligned}
F(t(u+v)) &= DF(0, t(u+v)) + \omega(0, t(u+v)), \\
\text{(1)} \qquad F(tu) &= DF(0, tu) + \omega(0, tu), \\
F(tv) &= DF(0, tv) + \omega(0, tv).
\end{aligned}
$$

Let t be so that $0 < t < 1$. By our hypothesis we have that

$$
t(F(u+v) - F(u) - F(v)) = \omega(0, t(u+v)) - \\
- \omega(0, tu) - \omega(0, tv).
$$

Hence

$$
F(u+v) - F(u) - F(v) = \frac{\|\omega(0, t(u+v))\|}{t} + \\
+ \frac{\|\omega(0, tu)\|}{t} + \frac{\|\omega(0, tv)\|}{t}.
$$

Let $\varepsilon > 0$ be any positive number. Then there exists a $\delta(\varepsilon) > 0$ so that $0 < t < \delta(\varepsilon)$ implies

$$
t^{-1}\|\omega(0, t(u+v))\| < \frac{\varepsilon}{3}, \quad t^{-1}\|\omega(0, tu)\| < \frac{\varepsilon}{3}, \\
t^{-1}\|\omega(0, tv)\| < \frac{\varepsilon}{3}.
$$

Therefore

$$
\text{(2)} \qquad \|F(u+v) - F(u) - F(v)\| < \varepsilon.
$$

Being ε an arbitrary positive constant, the inequality (2) gives that $F(u+v) = F(u) + F(v)$. Hence F is additive on X. This completes the proof.

Remark 1. It is well known that a linear continuous operator $A: X \to Y$ is weakly continuous. The basic properties of weakly continuous maps were described in [6]. Among others it was shown that (a) if X is a reflexive Banach space, Y a linear normed space, $F: D_R(0) \to Y$ a weakly continuous mapping on $D_R(0) \subset X$, then F is uniformly demicontinuous and weakly compact on $D_R(0)$; (b) under the conditions of (a) there exists $u_0 \in D_R(0)$ so that $\| F(u_0) \| = \inf_{u \in D_R(0)} \| F(u) \|$.
The question under which condition $F: X \to Y$ is weakly continuous is not solved satisfactorily yet. In [6] we have found some almost obvious conditions for weak continuity of F. For instance [6, corollary 2]: Let X be a reflexive Banach space, Y a linear normed space. If either a) $F: D_R(0) \to Y$ is weakly compact and weakly closed on $D_R(0)$, or b) $F: D_R(0) \to X^*$ is locally weakly sequentially bounded and weakly closed on $D_R(0)$, then F is weakly continuous on $D_R(0)$. Theorem 5 [6] gives a necessary and sufficient condition for weak continuity of $F: X \to Y$ in separable reflexive Banach space X. Recall that it is quite analogous to Kadec's theorem [4] concerning the strong continuity of F.

One may prove similarly as Th.4.4 [1] the following assertion: Let X, Y be linear normed spaces, W a convex bounded subset of X, $F: W \to X$ a mapping having on W compact weak Fréchet derivative $\hat{F}'(u)$, Then F is weakly continuous on W.

<u>Theorem 2</u>. Let X, Y be linear normed spaces, $F: X \to Y$ so that $F(tu) = tF(u)$ for each $t \in (0,1)$ and every $u \in X$. Suppose F has the Gâteaux derivative $F'(u)$ on the segment $(0, t_0 v_0) = \{u : u = t v_0, 0 < t < t_0 < 1, v_0 \in X\}$. If $\lim_{t \to 0+} \|F'(t v_0)\| = 0$ and $\lim_{t \to 0+} t^{-1} \|\omega(t v_0, t h)\| = 0$ for every $h \in X$, then F is linear and continuous on X.

<u>Proof</u>. Let $u, v \in X$, $0 < t < t_0$, then

$$F(t(u+v)) - F(tv_0) = F'(tv_0)t(u+v-v_0) + \omega(tv_0, t(u+v-v_0)),$$

(3)

$$F(tu) - F(tv_0) = F'(tv_0)t(u-v_0) + \omega(tv_0, t(u-v_0)),$$

$$F(tv) - F(tv_0) = F'(tv_0)t(v-v_0) + \omega(tv_0, t(v-v_0)).$$

By our hypothesis and according to (3)

$$(4)\ \|F(u+v) - F(u) - F(v) + F(v_0)\| \leqq \|F'(tv_0)\| \cdot \|v_0\| +$$

$$+\frac{1}{t}\|\omega(tv_0, t(u+v-v_0))\| + \frac{1}{t}\|\omega(tv_0, t(u-v_0))\| + \frac{1}{t}\|\omega(tv_0, t(v-v_0))\|.$$

In view of (4) and our hypothesis, $F(u+v) = F(u) + F(v) - F(v_0)$ for every $u, v \in X$. Set $u = v = 0$, then $F(v_0) = 0$. Hence F is additive on X and being continuous it is homogeneous and thus linear on X. Theorem 2 is proved.

To prove the following theorem we shall use this

<u>Lemma 1</u>. Let X, Y be linear normed spaces, $A: X \to Y$ a linear compact (i.e. completely continuous) operator. Then A is strongly continuous on X.

<u>Proof</u>. Let u_0 be an arbitrary point of X, $\{u_n\} \in X$, $u_n \xrightarrow{w} u_0$. Since A is weakly continuous, $Au_n \xrightarrow{w} Au_0$.

As $u_n \xrightarrow{w} u_0$, $\{u_n\}$ is bounded ; $\|u_n\| \leq R$ $(n=0\ 1\ 2 \ldots)$. Since $AD_R(0)$ is compact set in Y and $A(u_n) \in AD_R(0)$ for every n and the weak convergence on compact set is equivalent with the strong one [1,Lemma 4.1], $Au_n \rightarrow Au_0$. This proves Lemma 1.

Remark 3. It is well-known that a linear operator $A: X \rightarrow X$ in Hilbert space X is compact $\Longleftrightarrow$ A is strongly continuous on X . For general spaces the notions of compactness and strongly continuity of A are not equivalent.For instance the identity em bedding of the space C into L_2 is strongly continuous but it is not compact / see [1,p.26] / .

Theorem 3. Let X, Y be linear normed spaces, $F: D_R(0) \rightarrow Y$ such that F possesses the first and the second Gâteaux derivatives $F'(u)$, $F''(u)$ on $D_R(0)$. Let $F'(u)h$ be compact operator in $h \in X$ for each (but fixed) $u \in D_R(0)$. Assume there exists a constant $\gamma > 0$ such that $\|F(u) - F(v)\| \geqq \gamma \|u - v\|$ for each $u, v \in D_R(0)$ and that $R \sup\limits_{u \in D_R(0)} \|F''(u)\| < \gamma$.

Then F is strongly continuous on $D_R(0)$.

Proof. Assume a contradiction that there exists a point $u_0 \in D_R(0)$ such that F is not strongly continuous at u_0 . This assumption implies the existence of $\varepsilon_0 > 0$ and the sequence $\{u_m\} \in D_R(0)$ so that $u_m \xrightarrow{w} u_0$ and $\|F(u_m) - F(u_0)\| \geqq \varepsilon_0$.

From Taylor's formula we have that

(5) $\|F(u_m) - F(u_0)\| \leqq \|F'(u_0)(u_m - u_0)\| +$

$$+\frac{1}{2}\sup_{u\in D_R(0)}\|F''(u)\|\ \|u_n-u_o\|^2 .$$

Set $k=\sup_{u\in D_R(0)}\|F''(u)\|$. Then

$$(6)\ \frac{1}{2}k\|u_n-u_o\|^2\leqq\frac{1}{2}k(\|u_n\|+\|u_o\|)\|u_n-u_o\|\leqq$$

$$\leqq Rk\|u_n-u_o\|\leqq Rk\gamma^{-1}\|F(u_n)-F(u_o)\| .$$

The relations (5),(6) give

$$\|F(u_n)-F(u_o)\|\leqq(1-\frac{Rk}{\gamma})\|F'(u_o)(u_n-u_o)\| .$$

As $u_n\xrightarrow{w}u_o$, $F'(u_o)(u_n-u_o)\longrightarrow 0$, whenever $n\longrightarrow\infty$ by our hypothesis. Hence $\|F(u_n)-F(u_o)\|\to\to 0$ as $n\longrightarrow\infty$, a contradiction. Thus F is strongly continuous on $D_R(0)$ and theorem is proved.

<u>Proposition 1.</u> Let X, Y be linear normed spaces, $F: X\to Y$ a mapping having the linear Gâteaux differential $DF(u,h)$ is some convex neighborhood $V(u_o)$ of $u_o\in X$. If $u_n\xrightarrow{w}u_o$, $h_n\xrightarrow{w}h$, $u_n\in V(u_o)$, $h_n, h\in X\Longrightarrow DF(u_n,h_n)\to D(u_o,h)$ (respectively $DF(u_n,h_n)\xrightarrow{w}D(u_o,h)$), then F is strongly continuous (resp. weakly continuous) at u_o .

<u>Proof</u>. Suppose on the contrary, for instance, F is not strongly continuous at u_o . Then there exist $\varepsilon_o>0$ and the sequence $u_n\in V(u_o)$ so that $u_n\xrightarrow{w}\xrightarrow{w}u_o\Longrightarrow\|F(u_n)-F(u_o)\|\geqq\varepsilon_o$. By the mean-value theorem

$$|\langle F(u_n)-F(u_o),e_n^*\rangle|=|\langle DF(u_o+\tau_n(u_n-u_o),u_n-u_o),e_n^*\rangle|\leqq$$

$\leqq |\langle DF(u_0+\tau_n(u_n-u_0), u_n) - DF(u_0,u_0), e_n^* \rangle| +$

$+ |\langle DF(u_0,u_0) - DF(u_0+\tau_n(u_n-u_0), u_0), e_n^* \rangle|$,

where $0 < \tau_n < 1$, $e_n^* \in Y^*$ and $\|e_n^*\| = 1$ $(n = 1, 2, \ldots)$.

Hence

$$|\langle F(u_n)-F(u_0), e_n^* \rangle| \leqq \|DF(u_0+\tau_n(u_n-u_0), u_n) - DF(u_0,u_0)\| +$$
$$+ \|DF(u_0,u_0) - DF(u_0+\tau_n(u_n-u_0), u_0)\| .$$

As $u_n \xrightarrow{w} u_0$ and $u_0+\tau_n(u_n-u_0) \xrightarrow{w} u_0$, both the terms on right side of the last inequality tend to zero by our hypothesis. According to Hahn-Banach theorem there exist $e_n^{(0)*} \in Y^*$ such that $\|e_n^{(0)*}\| = 1$, $(n = 1, 2, \ldots)$ and $|\langle F(u_n)-F(u_0), e_n^{(0)*}\rangle| = \|F(u_n)-F(u_0)\|$. Thus $\|F(u_n) - F(u_0)\| \to 0$ as $n \to \infty$ and this is a contradiction. Hence F is strongly continuous at u_0 .

Similarly one can prove the second assertion.

Let $F: X \to Y$ be a mapping having in some neighborhood $V(u_0)$ of $u_0 \in X$ the Fréchet derivative $F'(u)$. We shall say that $F'(u)$ is continuous (strongly continuous) at u_0 if $u_n \in X, u_n \to u_0$ $(u_n \xrightarrow{w} u_0) \Rightarrow \|F'(u_n) - F'(u_0)\| \to 0$ in the norm topology of the space $(X \to Y)$.

Suppose F possesses the Fréchet derivative $F'(u)$ on $B_R(0)$. Then $F'(u)$ is said to be uniformly continuous on $B_R(0)$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $u_1, u_2 \in B_R(0)$ with $\|u_1 - u_2\| < \delta$ there is $\|F'(u_1) - F'(u_2)\| < \varepsilon$. We shall say that the remainder $\omega(u, h)$ of the Fréchet derivative $F'(u)$ is uniform on $B_R(0)$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon)>0$ so that $\|\omega(u,h)\| \leqq \varepsilon \|h\|$ for each $u \in B_R(0)$ and $h \in X$, $\|h\| < \delta$.

A mapping $F: X \to Y$ is said to be uniformly smooth [10] on an open subset $M \subset X$ if for any positive number $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $0 < \|h\| < \delta$, then

(7) $$\| G(u,h)\| < \varepsilon \|h\|$$

holds for each $u \in M$, where

$$G(u,h) = F(u+h) + F(u-h) - 2F(u).$$

In [10] we have proved the following assertion: Let X be a linear normed space, f a convex functional on X. Suppose f is uniformly continuous on $B_{R+\alpha}(0)$, $(\alpha > 0)$. Then f possesses an uniformly continuous Fréchet derivative $f'(u)$ on $B_R(0) \Longleftrightarrow$ f is uniformly smooth on $B_R(0)$.

A functional f is called subadditive on a subset $G \subset X$ if $u, v \in G$, $u+v \in G$ imply $f(u+v) \leqq f(u) + f(v)$. We prove the following

<u>Theorem 4</u>. Let X be a linear normed space, f a convex functional on X such that f is subadditive and uniformly continuous on $B_{R+\alpha}(0)$, $(\alpha > 0)$. Assume either a) f possesses the Fréchet differential $df(0,h)$ at 0 and $f(0) = 0$; or b) $f(tu) = \varphi(t) f(u)$ for each $u \in D_1(0)$, $t \in (0,1)$, if $R+\alpha > 1$ and for each $u \in D_1(0)$, $t \in (0,+\infty)$ if $R+\alpha \leqq 1$, where a real function is assumed to be defined and positive in $(0,1)$ in the first case

of b) and to be defined and positive on $(0,+\infty)$ and finite on each subinterval $(0,a> \subset (0,+\infty)$ in the second case of b). Moreover, suppose that $\lim\limits_{t\to 0_+} \frac{\varphi(t)}{t} = 0$

in both cases of b).

Then f possesses the Fréchet derivative $f'(u)$ on $B_R(0)$ and $f'(u)$ is strongly continuous on $B_R(0)$.

Proof. Suppose that f is not uniformly smooth on $B_R(0)$. This denotes (see also remark 7 [10]) that there exists $\varepsilon_o > 0$ so that for any $\delta_n = \frac{1}{n}$, $(n = 1,2,\dots)$ there exist $h_n \in X$, $\|h_n\| = 1$, the numbers $t_n \in (0,\frac{1}{n})$ and $u_n \in B_R(0)$ such that

$$\frac{1}{t_n} G(u_n, t_n h_n) > \varepsilon_o , \quad (n = 1,2,\dots) , \tag{8}$$

where $G(u,h) = f(u+h)+f(u-h)-2f(u)$. Since f is convex, $G(u_n, t_n h_n) \geqq 0$ for every n $(n = 1,2,\dots)$. As $\{h_n\}$ is bounded and $t_n \to 0_+$ then for sufficiently large n $(n \geqq n_o)$ we have that $u_n \pm t_n h_n \in B_{R+\alpha}(0)$. Employing subadditivity of f on $B_{R+\alpha}(0)$ we get that

$$G(u_n, t_n h_n) \leqq f(t_n h_n) + f(t_n(-h_n)) \tag{9}$$

for $n \geqq n_o$. Assuming (a), in view of continuity of f in the neighborhood of zero it follows that f possesses the Fréchet derivative $f'(0)$ at 0 and

$$f(t_n h_n) = f'(0) t_n h_n + \omega(0, t_n h_n) , \tag{10}$$

$$f(-t_n h_n) = -f'(0)t_n h_n + \omega(0, t_n(-h_n)).$$

According to (9),(10)

$$0 \leqq \frac{1}{t_n} G(u_n, t_n h_n) \leqq \frac{1}{t_n}[\omega(0, t_n h_n) + \omega(0, t_n(-h_n))].$$

Both the terms on the right side of this inequality tend to 0 as $n \to \infty$. Hence $\frac{1}{t_n} G(u_n, t_n h_n) \to 0$ whenever $n \to \infty$ which contradicts (8).

If $R+\alpha > 1$, being f uniformly continuous on $B_{R+\alpha}(0)$, f is bounded on $D_1(0)$ (cf.[1 ,chapt. I]). Suppose $R+\alpha \leqq 1$ and that b) is fulfilled for each $u \in D_1(0)$ and $t \in (0, +\infty)$. Then continuity of f at 0 implies the existence $\delta > 0$, $c > 0$ such that $\|u\| \leqq \delta \Longrightarrow |f(u)| \leqq c$. Let $u \in D_1(0)$, there exists an integer n_0 such that $\frac{1}{n_0} \leqq \delta$. Then

$$|f(u)| = |f(\tfrac{u}{n_0} \cdot n_0)| = \varphi(n_0)|f(\tfrac{u}{n_0})| \leqq c \cdot \varphi(n_0)$$

for each $u \in D_1(0)$. Hence f is in the cases of b) bounded on $D_1(0)$; i.e. $\sup_{u \in D_1(0)} |f(u)| \leqq c_1$.

A functional f is "φ-homogeneous" in both the cases of b) for each $u \in D_1(0)$ and $t \in (0, 1)$. In view of $t_n \to 0_+$ there exists an integer n_1 so that $n \geqq n_1 \Rightarrow$ $\Rightarrow t_n \in (0, 1)$. According to (9) we have for each $n \geqq n_1$

$$0 \leqq \frac{1}{t_n} G(u_n, t_n h_n) \leqq 2c_1 \frac{\varphi(t_n)}{t_n}.$$

As $t_n \to 0_+$, $\frac{1}{t_n} G(u_n, t_n h_n) \to 0$ whenever

$n \to \infty$ which again contradicts (8). Hence in both cases a),b) f is uniformly smooth on $B_R(0)$. In view of the above mentioned result [10,Th.8], the Fréchet derivative $f'(u)$ of f exists on $B_R(0)$ and it is uniformly continuous on $B_R(0)$. This implies [see 1, the proof of the first part of Th.4.2] that the remainder $\omega(u,h)$ of the Fréchet derivative $f'(u)$ is uniform on $B_R(0)$. By corollary 2 [7] $f'(u)$ is strongly continuous on $B_R(0)$. This completes the proof.

Remark 1. It was shown in Corollary 2 [7] that under some conditions the Fréchet derivative $f'(u)$ of a convex subadditive functional f is strongly continuous on $D_R(0)$. It must be pointed out (see also remark 1 [7] that this corollary 2 [7] is valid if $D_R(0)$ is replaced by $B_R(0)$. In Theorem 1 [7] we have assumed that X is a reflexive Banach space and that f has some properties on $D_R(0)$ with the aim of the application of the Vajnberg Theorem [1,Th.1.4] which asserts that a strongly continuous operator on a closed ball $D_R(0)$ of a reflexive Banach space X is compact and uniformly continuous on $D_R(0)$. Hence the assumptions of reflexivity of X and closedness of the ball $D_R(0)$ is essential only for the second and third assertions of Theorem 1 [7] (see the end of the proof of this theorem).

From Theorem 4,[1,Th.4.1;Th.8.2] it follows the following

Corollary 3. Let X be a reflexive Banach space, f a convex functional on X so that f is subadditive and uniformly continuous on $B_{R+\alpha}(0)$, $(\alpha > 0)$. Assume either a) f possesses the Fréchet differential $df(0,h)$ at 0 and $f(0)=0$; or b) $f(tu) = \varphi(t) f(u)$ for each $u \in D_1(0)$, $t \in (0,+\infty)$, where $\varphi(t)$ is a positive function defined on $(0,+\infty)$, it is finite on each subinterval $(0, a\rangle \subset (0,+\infty)$ and so that $\lim_{t \to 0_+} \frac{\varphi(t)}{t} = 0$.

Then f possesses the Fréchet derivative $f'(u)$ and $f'(u)$ is compact and uniformly continuous on $D_R(0)$. Moreover, f is weakly continuous on $D_R(0)$.

Remarks. The result of Theorem 4 one may rewrite as follows: Under the conditions of Th.4 f possesses the gradient map $u \to f'(u)$ on $B_R(0)$ and it is strongly continuous on $B_R(0)$.

The properties of the gradient map $F(u) = f'(u)$, where f is a convex functional, have been investigated in [7]. Let us note that the existence of the Fréchet derivative $f'(u)$ on $B_R(0)$ of a convex continuous functional f defined on a (reflexive) Banach space X has been established in [11,Th.2], [13],[14].

Theorem 5. Let X be a Banach space, Y a linear normed space, $F: X \to Y$ a continuous mapping of X into Y having on a convex bounded open subset $W \subset X$, $0 \in W$, the first and the second Gâteaux differential $VF(u,h)$, $V^2F(u,h,k)$. Assume there exists a constant $M > 0$ so that $\| V^2 F(u, h, k) \| \leqq M \|h\| \cdot \|k\|$ for each $u \in W$ and every $h, k \in X$. Let $R > 0$ be such that $B_R(0) \subset W$. Suppose there exists a bounded positive function $f : \langle 0, R \rangle \longrightarrow E_1^+$ so that the remainder $\omega(0, h)$ of the Fréchet derivative $F'(0)$ of F at 0 (which exists according to Theorem 1 [12]) satisfies the inequality

$$(11) \quad \| \omega(0, h + k) - \omega(0, h) \| \leqq$$
$$\leqq f(\max(\|h\|, \|k\|)) \| \omega(0, k) \|$$

for each $h, k \in B_R(0)$ with $h + k \in B_R(0)$.

Then F possesses Lipschitzian Fréchet derivative $F'(u)$ on W. Moreover, $F'(u)$ is strongly continuous on $B_\kappa(0)$, where $\kappa < R$.

Proof. In view of Theorem 1 [12] and Remark 1 [12] F possesses Lipschitzian Fréchet derivative $F'(u)$ on W and F is uniformly differentiable on W, i.e. for given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that $0 < \|h\| \leqq \delta(\varepsilon)$ implies that $\| \omega(u, h) \| \leqq \varepsilon \|h\|$ uniformly with respect to $u \in W$. Let $B_\kappa(0) \subset B_R(0)$, it remains to prove that $F'(u)$ is strongly continuous on $B_\kappa(0)$.

Suppose there exists a point $u_0 \in B_\kappa(0)$ so that $F'(u)$ is not strongly continuous at u_0. This means that there exist $\varepsilon_0 > 0$ and the sequence $u_n \in B_\kappa(0)$, $u_n \xrightarrow{w} u_0$ and $\|F'(u_n) - F'(u_0)\| > \varepsilon_0$. Let $h \in X$ be arbitrary, $\|h\| \leqq 1$, $e_k^* \in Y^*$ $(k = 1, 2, \ldots)$, $\|e_k^*\| = 1$ and $t \in (0, 1)$. If t is small enough, $u_n + th$, $u_0 + th \in B_R(0)$ and we have that

$$\langle F(u_n+th) - F(u_n), e_n^* \rangle = \langle F'(u_n)th, e_n^* \rangle +$$
$$+ \frac{1}{2} t^2 \langle V^2 F(u_n + \tau_n th, h, h), e_n^* \rangle ,$$

$$\langle F(u_0+th) - F(u_0), e_n^* \rangle = \langle F'(u_0)th, e_n^* \rangle +$$
$$+ \frac{1}{2} t^2 \langle V^2 F(u_0 + \sigma th, h, h), e_n^* \rangle ,$$

where $\tau_n \in (0,1)$, $\sigma \in (0,1)$. From these equalities we get that

$$(12) \langle (F'(u_n) - F'(u_0))h, e_n^* \rangle = \frac{1}{t}(\langle F(u_n+th) - F(u_n), e_n^* \rangle + \langle F(u_0) - F(u_0+th), e_n^* \rangle) - \frac{1}{2} t \langle V^2 F(u_n + \tau_n th, h, h), e_n^* \rangle + \frac{1}{2} t \langle V^2 F(u_0 + \sigma th, h, h), e_n^* \rangle .$$

We have that

$$(13) \quad \begin{aligned} F(u_n+th) - F(0) &= F'(0)(u_n+th) + \omega(0, u_n+th), \\ F(u_n) - F(0) &= F'(0)u_n + \omega(0, u_n), \end{aligned}$$

$$(14) \quad F(u_0) - F(0) = F'(0)u_0 + \omega(0, u_0),$$
$$F(u_0+th) - F(0) = F'(0)(u_0+th) + \omega(0, u_0+th).$$

In view of (13) and (14) we have

$$F(u_n+th) - F(u_n) = F'(0)th + \omega(0, u_n+th) - \omega(0, u_n),$$

$$F(u_0) - F(u_0+th) = -F'(0)th + \omega(0, u_0) - \omega(0, u_0+th).$$

Hence the sum of the first and the second term in the absolute value on the right side of (12) is less than

$$J = \frac{1}{t}\,(|\langle \omega(0, u_n + th) - \omega(0, u_n), e_n^* \rangle| +$$
$$+ |\langle \omega(0, u_0 + th) - \omega(0, u_0), e_n^* \rangle|) \quad .$$

By our hypothesis f is bounded on $\langle 0, R \rangle$. Denote $K = \sup_{\alpha \in \langle 0, R\rangle} f(\alpha)$. As $u_n, u_0 \in B_R(0)$,

$u_n + th \in B_R(0)$, $u_0 + th \in B_R(0)$, employing (11) we obtain for t small enough that

$$J \leqq \frac{1}{t}\,(\|\omega(0, u_n + th) - \omega(0, u_n)\| +$$
$$+ \|\omega(0, u_0 + th) - \omega(0, u_0)\|) \leqq$$
$$\leqq \frac{2K}{t}\,\|\omega(0, th)\| \; .$$

From these inequalities and (12) it follows that

$$(15)\quad |\langle (F'(u_n) - F'(u_0))h, e_n^* \rangle| \leqq \frac{2K}{t}\,\|\omega(0, th)\| +$$
$$+ Mt\,\|h\| \cdot \|h\|$$

for each $h \in X$ with $\|h\| \leqq 1$. Since F has the Fréchet derivative $F'(0)$ at 0, there exists a number $t_0 \in (0, 1)$ such that

$$0 < t < t_0 \implies \frac{2K}{t}\,\|\omega(0, th)\| < \frac{\varepsilon_0}{3}\,\|h\| \; .$$

Then for $0 < t < \operatorname{Min}\left(t_0, \frac{\varepsilon_0}{3M}\right)$ both terms in (15) are less than $\frac{2\varepsilon_0}{3}\|h\|$. According to Hahn-Banach theorem there exist $e_n^{(0)*} \in Y^*$ $(n = 1, 2, \dots)$ so that $\|e_n^{(0)*}\| = 1$ and

$$|\langle (F'(u_n) - F'(u_0))h, e_n^{(0)*} \rangle| = \|(F'(u_n) - F'(u_0))h\| \; .$$

Hence $\sup_{\|h\| \leqq 1} \|(F'(u_n) - F'(u_0))h\| = \|F'(u_n) -$

$- F'(u_0)\| \leqq \frac{2\varepsilon_0}{3}$,

which is a contradiction. Hence the Fréchet derivative $F'(u)$ is strongly continuous on $B_{\kappa}(O)$. This concludes the proof.

Remark 3. Note that $\|V^2F(u, h, k)\| \leqq \leqq M\|h\| \|k\|$ for each $u \in W$ and h, $k \in X \Longleftrightarrow V^2F(u, h, k)$ is continuous at $(0,0)$ uniformly with respect to $u \in W$.

R e f e r e n c e s

[1] M.M. VAJNBERG: Variacionnyje metody issledovanija nelinejnych operatorov.Moskva 1956.

[2] E.S. CITLANADZE: O differencirovanii funkcionalov. Matem.sb.29(71)(1951),3-12.

[3] E.H. ROTHE: Gradient mappings in Hilbert space. Ann.of Math.47(1946),580-592.

[4] M.I. KADEC: O nekotorych svojstvach potencial´nych operatorov v refleksivnych separabel´nych prostranstvach.Izv.vysš.učeb.zaved.,mat.,15(1960),104-107.

[5] V.I. ANOSOV: Obobščenija teorem E.S.Citlanadze o svojstvach gradientov slabo nepreryvnych funkcionalov.Trudy sem.po funkc. analizu.Voroněž 1958,vyp.6,1-11.

[6] J. KOLOMÝ: A note on the continuity properties of

nonlinear operators.Comment.Math. Univ.Carolinae 8(1967),503-514.

[7] J. KOLOMÝ: Gradient maps and boundedness of Gâteaux differentials.Comment.Math.Univ. Carolinae 9(1968),613-625.

[8] J. KOLOMÝ, V. ZIZLER: Remarks on the differentiability of mappings in linear normed spaces.Comment.Math.Univ.Carolinae 8(1967),691-704.

[9] M.A. KRASNOSELSKIJ: Topologičeskije metody v teorii nelinejnych integral'nych uravnenij.Moskva 1956.

[10] J. DANEŠ, J. KOLOMÝ: On the continuity and differentiability properties of convex functionals.Comment.Math.Univ.Carolinae 9(1968),329-350.

[11] J. KOLOMÝ: On the differentiability of operators and convex functionals.Comment.Math. Univ.Carolinae 9(1968),441-454.

[12] J. KOLOMÝ: On the differentiability of mappings and convex functionals.Comment.Math. Univ.Carolinae 8(1967),735-752.A correction,ibid 9(1968),197.

[13] E. ASPLUND: Fréchet differentiability of convex functions.Acta Math.121(1968),30-47.

[14] J. KOLOMÝ: Differentiability of convex functionals

tors and functionals.Comment.Math. Univ.Carolinae 10(1969),91-114.

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