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ON PRODUCTS IN GENERALIZED ALGEBRAIC CATEGORIES

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0. Introduction.

Universal algebras of a given type $\Delta = \{\varkappa_\lambda \mid \lambda < \beta\}$ (Δ is a family - as a rule increasing - of ordinal numbers indexed by ordinal numbers) form the category $A(\Delta)$ whose objects are operational structures, the pairs $(X; \{\omega_\lambda^X \mid \lambda < \beta\})$ where X is a set and ω_λ^X are $\varkappa_\lambda$ -ary operations on X, i.e. mappings $\omega_\lambda^X : X^{\varkappa_\lambda} \longrightarrow X$, and morphisms from $(X; \{\omega_\lambda^X\})$ to $(Y; \{\omega_\lambda^Y\})$ are mappings $f : X \rightarrow Y$ compatible with operations in the sense that $\omega_\lambda^Y \circ f^{(\varkappa_\lambda)} = f \circ \omega_\lambda^X$ for every $\lambda < \beta$, where $f^{(\varkappa_\lambda)} : X^{\varkappa_\lambda} \longrightarrow Y^{\varkappa_\lambda}$ is f acting coordinate-wise on $\varkappa_\lambda$-tuples from $X^{\varkappa_\lambda}$.

Here the operations play a role of a "device selecting suitable mappings" - the morphisms of $A(\Delta)$. Now, we can let this device work in a more general situation. Take two functors F and G of the same variance from sets to sets and define the generalized algebraic category $A(F, G, \Delta)$ as follows: objects are again pairs $(X, \{\omega_\lambda^X\})$ but operations ω_λ^X range over FX and take values in GX (so they are mappings $\omega_\lambda^X : (FX)^{\varkappa_\lambda} \longrightarrow GX$), and, morphisms are in the covariant case mappings $f : X \longrightarrow Y$ such that $\omega_\lambda^Y \circ (Ff)^{(\varkappa_\lambda)} = (Gf) \circ \omega_\lambda^X$ for every

$\lambda < \beta$, so we have commutative diagrams

$$\begin{array}{ccc} (FX)^{\varkappa_\lambda} & \xrightarrow{\omega_\lambda^X} & GX \\ {\scriptstyle (Ff)^{(\varkappa_\lambda)}}\downarrow & & \downarrow{\scriptstyle Gf} \\ (FY)^{\varkappa_\lambda} & \xrightarrow{\omega_\lambda^Y} & GY \end{array}$$

(In the contravariant case the vertical arrows are reversed and compatibility of f means the fulfilment of the identities $\omega_\lambda^X \circ (Ff)^{(\varkappa_\lambda)} = (Gf) \circ \omega_\lambda^Y$ for every $\lambda < \beta$.)

We shall refer to functors F involved on the first place in $A(F, G, \Delta)$, for obvious reasons, as to domain-functors, and to functors G as to range-functors. Taking $F = G = I$ - an identical functor, we get clearly $A(\Delta)$.

It is known that $A(\Delta)$ always has products (in usual categorical sense). Unfortunately, this pleasant property is very often lost for categories $A(F, G, \Delta)$ with non-identical domain and range-functors.

It is easily seen that the existence of products in $A(F, G, \Delta)$ such that the natural forgetful functor preserves them is equivalent to the requirement that G preserves products. Much less transparent is the general problem of existence of products in categories $A(F, G, \Delta)$ - the main objective of the present paper. Then the condition that G preserves products is, of course, far from being necessary and there are many other interesting categories $A(F, G, \Delta)$ possessing products but with G not preserving products. But generally it is true that the behaviour of the range-functor with regard to products matters here, and, if it does not preserve products, then also the behaviour of the domain-functor with regard to sums (disjoint unions) becomes relevant to the problem.

Presented material is exposed in five sections. The first one brings basic definitions and facts, including conventions about notations used. In the section 2 there are given some necessary conditions for the existence of products in $A(F, G, \Delta)$. With aid of these it is proved in the section 3 that for F, G contravariant faithful and $\Sigma \Delta > 0$ $A(F, G, \Delta)$ fails to have products. Section 4 is devoted to more close study of certain properties of covariant functors. The final section 5 gives a number of theorems on products in $A(F, G, \Delta)$ with covariant functors F, G .

Some problems remain open here, nevertheless, our theorems account for most of familiar functors F and G .

In final remarks some possible generalizations are indicated.

1. Basic definitions, facts and notation

All functors throughout this paper will be functors from sets to sets (i.e. from the category $\mathcal{S}$ of all sets and mappings - including void ones - to $\mathcal{S}$). Observe that for our purposes we can consider functors only up to the natural equivalence $\cong$. When systems of functors are discussed, we use the set-theoretic symbols $\in$, $\subset$, $\cup$, $\cap$

for <u>shortness</u> sake.

Let F and G be functors of the same variance.

F is a <u>sub-functor</u> of G if there exists a monotransformation $\mu : F \longrightarrow G$;

F is a <u>factor functor</u> of G if there exists an epitransformation $\nu : G \longrightarrow F$;

F is a <u>retract</u> of G if there are a monotransformation $\mu : F \longrightarrow G$ and an epitransformation $\nu : G \longrightarrow F$ such that $\nu\mu$ is the identical transformation of F .

Recall the usual operations over functors (cf.[1]):

(a) The <u>product</u> $F \times G$,

(b) the <u>coproduct</u> (disjoint union) $F \vee G$ defined for functors of the same variance, both can be extended to an arbitrary family $\{ F_\iota \mid \iota \in \mathcal{J} \}$ over a set $\mathcal{J}$ of functors, the results written as $\prod_{\iota \in \mathcal{J}} F_\iota$ and $\bigvee_{\iota \in \mathcal{J}} F_\iota$, respectively.

(c) The <u>superposition</u> $F \circ G$ of arbitrary functors G and F written (as anywhere else) left-hand, i.e. $(F \circ G)X = F(GX)$. If F and G are of different variance, then $F \circ G$ is contravariant, otherwise it is covariant.

(d) The <u>hom -functor</u> $\langle F , G \rangle$ for functors of different variance, its variance being the same as that of G . Remind that, writing H for $\langle F , G \rangle$, we have $HX = \{ \varphi \mid \varphi : FX \longrightarrow GX \}$ and for $f : X \longrightarrow Y$ and H covariant $(Hf)(\varphi) = (Gf) \circ \varphi \circ (Ff)$.

Let us last some of the most commonly used functors:

I denotes the identical functor,

C_M - a constant functor to M ; it is both covariant and contravariant;

P^+ - the covariant power functor:

$P^+X = \{A \mid A \subset X\}$, $(P^+f)(A) = \{f(x) \mid x \in A\}$ for $f : X \to Y$;

N - a subfunctor of P^+ assigning to every set X the set NX of all its non-void subsets, evidently

$P^+ \cong N \vee C_1$;

P^- - the contravariant power functor, $P^- \cong \langle I , C_2 \rangle$;

β - a subfunctor of $(P^-)^2 = P^- \circ P^-$ assigning to every set X the set βX of all ultrafilters on X *) ;

Q_M - a cartesian power, $Q_M \cong \langle C_M , I \rangle$.

We shall often use the next fact from [2]:

Proposition 1.1. Every faithful covariant functor has I for its subfunctor. Every faithful contravariant functor has P^- for its retract.

Let $\{X_\alpha ; \alpha \in A\}$, $A \neq \emptyset$, be an arbitrary family of objects of some category $\mathcal{K}$. Any pair $\langle X , \{\pi_\alpha \mid \alpha \in A\} \rangle$ - an object X of $\mathcal{K}$ together with a family of morphisms $\pi_\alpha : X \to X_\alpha$, $\alpha \in A$ - is called an inverse bound

*) An alternative description of the functor β : if $\mathcal{T}$ is the category of all completely regular topological T_1-spaces, $\Phi : \mathcal{T} \to \mathcal{S}$ the forgetful functor, $F : \mathcal{S} \to \mathcal{T}$ the free functor and $\Psi : \mathcal{T} \to \mathcal{T}$ the functor assigning to each space its β-compactification, then $\beta = \Phi \circ \Psi \circ F$.

(further "inverse" is often omitted) of the family $\{X_\alpha \mid \alpha \in A\}$.

If every other inverse bound $\langle Y, \{\eta_\alpha \mid \alpha \in A\}\rangle$ of $\{X_\alpha \mid \alpha \in A\}$ factorizes through $\langle X, \{\pi_\alpha\}\rangle$, i.e. if there exists a morphism $h : Y \to X$ such that $\eta_\alpha = \pi_\alpha \circ h$ for all $\alpha \in A$, then $\langle X, \{\pi_\alpha\}\rangle$ is called a <u>pseudoproduct</u> of the family.

A pseudoproduct is <u>product</u> if the factorization is unique.

A category $\mathcal{K}$ is said to have (pseudo)products if every family of its objects has a (pseudo)product.

2. <u>Necessary conditions</u>

Let $\mathcal{K} = A(F, G, \Delta)$ and $\mathcal{K}_1 = A(F_1, G_1, \Delta_1)$ be two categories with all the functors F, G, F_1, G_1 of the same variance and (possibly) of different types $\Delta = \{\varkappa_\lambda \mid \lambda < \beta\}$ and $\Delta_1 = \{\varkappa'_\mu \mid \mu < \vartheta\}$. Denote the objects of $\mathcal{K}$ by $X_\sigma = (X, \{\sigma^X_\lambda \mid \lambda < \beta\})$ and the objects of $\mathcal{K}_1$ by $X_\omega = (X, \omega^X_\mu \mid \mu < \vartheta\})$. If a mapping $f : X \longrightarrow Y$ is a morphism in $\mathcal{K}$ or $\mathcal{K}_1$, write simply $f : X_\sigma \longrightarrow Y_\sigma$ or $f : X_\omega \longrightarrow Y_\omega$, respectively.

<u>Lemma 2.1</u>. Assume that there are assignments Φ and Ψ

$$\Phi X_\omega = X_\sigma \quad \text{and} \quad \Psi X_\sigma = X_\omega ,$$

between the objects of $\mathcal{K}$ and $\mathcal{K}_1$ with the following three properties:

(a) $f : X_\sigma \longrightarrow \Phi Z_\omega \implies f : \Psi X_\sigma \longrightarrow Z_\omega$,

(b) $g : Y_\omega \longrightarrow Z_\omega \implies g : \Phi Y_\omega \longrightarrow \Phi Z_\omega$,

(c) $h : \Phi Y_\omega \to X_\sigma \implies h : Y_\omega \to \Psi X_\sigma$.

Then the existence of pseudoproducts in $\mathcal{K}$ implies the existence of pseudoproducts in $\mathcal{K}_1$.

Proof. Let $\{ X_\omega^\alpha \mid \alpha \in A \}$ be an arbitrary family of objects in $\mathcal{K}_1$. The family $\{ \Phi X_\omega^\alpha \}$ has - as any other family in $\mathcal{K}$ - a pseudoproduct, say, $\langle X_\sigma, \{ f_\alpha \} \rangle$ with $f_\alpha : X_\sigma \longrightarrow \Phi X_\omega^\alpha$, $\alpha \in A$. By (a) it is $f_\alpha : \Psi X_\sigma \longrightarrow X_\omega^\alpha$, therefore $\langle \Psi X_\sigma , \{ f_\alpha \} \rangle$ is a bound of the family $\{ X_\omega^\alpha \}$.

Let $\langle Y_\omega , \{ g_\alpha \} \rangle$ be an another bound of $\{ X_\omega^\alpha \}$, i.e. $g_\alpha : Y_\omega \longrightarrow X_\omega^\alpha$ for $\alpha \in A$. By (b), $\langle \Phi Y_\omega , \{ g_\alpha \} \rangle$ is a bound of $\{ \Phi X_\omega^\alpha \}$, therefore an $h : \Phi Y_\omega \longrightarrow X_\sigma$ must exist such that $g_\alpha = f_\alpha \circ h$ for all $\alpha \in A$. By (c) it is $g_\alpha : Y_\omega \to \Psi X_\sigma$, so it is shown that $\langle \Psi X_\sigma , \{ f_\alpha \} \rangle$ is a pseudoproduct of the family $\{ X_\omega^\alpha \}$.

Theorem 2.1. Let a category $\mathcal{K} = A (F , G , \Delta)$ have (pseudo)products. Then also any category $\mathcal{K}_1 = A (F_1 , G_1 , \Delta)$ of the same type Δ but with F_1, G_1 being retracts of F and G ,resp., has pseudoproducts.

Proof. Let $\Delta = \{ \varkappa_\lambda \mid \lambda < \beta \}$.

With aid of natural transformations

$$F_1 \xrightarrow{\mu} F \xrightarrow{\nu} F_1 , \quad G_1 \xrightarrow{\varepsilon} G \xrightarrow{\pi} G_1$$

such that $\nu \circ \mu = 1_{F_1}$ and $\pi \circ \varepsilon = 1_{G_1}$ define assignments

$$\Phi : \mathcal{K}_1^{obj} \longrightarrow \mathcal{K}^{obj} \quad \text{and} \quad \Psi : \mathcal{K}^{obj} \longrightarrow \mathcal{K}_1^{obj}$$

by

$$\Phi (X , \{ \omega_\lambda^X \}) = (X , \{ \sigma_\lambda^X \}) \quad \text{with} \quad \sigma_\lambda^X = \varepsilon_X \circ \omega_\lambda^X \circ \nu_X^{(\varkappa_\lambda)}$$

and

$$\Psi(X, \{\sigma_\lambda^X\}) = (X, \{\omega_\lambda^X\}) \quad \text{with } \omega_\lambda^X = \pi_X \circ \sigma_\lambda^X \circ \mu_X^{(\varkappa_\lambda)} .$$

It is easy to show that Φ and Ψ thus defined satisfy the conditions (a),(b),(c) of lemma 2.1.

For example, the computation in the covariant case runs as follows:

(a) $(Gf)\circ\sigma_\lambda^X = \sigma_\lambda^Z \circ (Ff)^{(\varkappa_\lambda)}$ with $\sigma_\lambda^Z = \varepsilon_Z \circ \omega_\lambda^Z \circ \nu_Z^{(\varkappa_\lambda)}$ implies

$(G_1 f)\circ\omega_\lambda^X = \omega_\lambda^Z \circ (F_1 f)^{(\varkappa_\lambda)}$ for $\omega_\lambda^X = \pi_X \circ \sigma_\lambda^X \circ \mu_X^{(\varkappa_\lambda)}$:

$$(G_1 f)\circ\omega_\lambda^X = (G_1 f)\circ\pi_X\circ\sigma_\lambda^X\circ\mu_X^{(\varkappa_\lambda)} = \pi_Z\circ(Gf)\circ\sigma_\lambda^X\circ\mu^{(\varkappa_\lambda)} =$$

$$= \pi_Z\circ\sigma_\lambda^Z\circ(Ff)^{(\varkappa_\lambda)}\circ\mu_X^{(\varkappa_\lambda)} = \pi_Z\circ\sigma_\lambda^Z\circ[(Ff)\circ\mu_X]^{(\varkappa_\lambda)} =$$

$$= \pi_Z\circ\sigma_\lambda^Z\circ[\mu_Z\circ(F_1 f)]^{(\varkappa_\lambda)} = \pi_Z\circ\sigma_\lambda^Z\circ\mu_Z^{(\varkappa_\lambda)}\circ(F_1 f)^{(\varkappa_\lambda)} =$$

$$= \pi_Z\circ\varepsilon_Z\circ\omega_\lambda^Z\circ\nu_Z^{(\varkappa_\lambda)}\circ\mu_Z^{(\varkappa_\lambda)}\circ(F_1 f)^{(\varkappa_\lambda)} = \omega_\lambda^Z\circ(F_1 f)^{(\varkappa_\lambda)}.$$

(b) $(G_1 g)\circ\omega_\lambda^Y = \omega_\lambda^Z\circ(F_1 g)^{(\varkappa_\lambda)}$ implies $(Gg)\circ\sigma_\lambda^Y =$ $= \sigma_\lambda^Z\circ(Fg)^{(\varkappa_\lambda)}$ for $\sigma_\lambda^Y = \varepsilon_Y\circ\omega_\lambda^Y\circ\nu_Y^{(\varkappa_\lambda)}$ and $\sigma_\lambda^Z = \varepsilon_Z\circ\omega_\lambda^Z\circ\nu_Z^{(\varkappa_\lambda)}$:

$$(Gg)\circ\sigma_\lambda^Y = (Gg)\circ\varepsilon_Y\circ\omega_\lambda^Y\circ\nu_Y^{(\varkappa_\lambda)} = \varepsilon_Z\circ(G_1 g)\circ\omega_\lambda^Y\circ\nu_Y^{(\varkappa_\lambda)} =$$

$$= \varepsilon_Z\circ\omega_\lambda^Z\circ(F_1 g)^{(\varkappa_\lambda)}\circ\nu_Y^{(\varkappa_\lambda)} = \varepsilon_Z\circ\omega_\lambda^Z\circ\nu_Z^{(\varkappa_\lambda)}\circ(Fg)^{(\varkappa_\lambda)} = \sigma_\lambda^Z\circ(Fg)^{(\varkappa_\lambda)}.$$

(c) $(Gh)\circ\sigma_\lambda^Y = \sigma_\lambda^X\circ(Fh)^{(\varkappa_\lambda)}$ with $\sigma_\lambda^Y = \varepsilon_Y\circ\omega_\lambda^Y\circ\nu_Y^{(\varkappa_\lambda)}$ implies $(G_1 h)\circ\omega_\lambda^Y = \omega_\lambda^X\circ(F_1 h)^{(\varkappa_\lambda)}$ for $\omega_\lambda^X = \pi_X\circ\sigma_\lambda^X\circ\mu_X^{(\varkappa_\lambda)}$:

$$\omega_\lambda^X \cdot (F_1 h)^{(\varkappa_\lambda)} = \pi_X \cdot \sigma_\lambda^X \cdot \mu_X^{(\varkappa_\lambda)} \circ (F_1 h)^{(\varkappa_\lambda)} = \pi_X \cdot \sigma_\lambda^X \cdot (Fh)^{(\varkappa_\lambda)} \circ \mu_Y^{(\varkappa_\lambda)} =$$

$$= \pi_X \cdot (Gh) \cdot \sigma_\lambda^Y \circ \mu_Y^{(\varkappa_\lambda)} = (G_1 h) \circ \pi_Y \cdot \sigma_\lambda^Y \circ \mu_Y^{(\varkappa_\lambda)} =$$

$$= (G_1 h) \circ \pi_Y \cdot \varepsilon_Y \circ \omega_\lambda^Y \cdot \nu_Y^{(\varkappa_\lambda)} \circ \mu_Y^{(\varkappa_\lambda)} = (G_1 h) \cdot \omega_\lambda^Y .$$

The assertion of the theorem follows by lemma 2.1.

There is another way of "collapsing" a category $A(F, G, \Delta)$ so that pseudoproducts are preserved, namely, an essential reduction of the type Δ is possible. Before stating the next theorem assume the type $\Delta = \{\varkappa_\lambda \mid \lambda < \beta\}$ increasing $\sum \Delta > 0$ and denote by σ the first index with $\varkappa_\sigma \neq 0$. Thus, in the case $\sigma > 0$ it is $\varkappa_\lambda = 0$ for all $\lambda < \sigma$ and nullary operations enter into consideration.

<u>Theorem 2.2.</u> Let a category $A(F, G, \Delta)$ have pseudoproducts. If $\sigma > 0$, then also the category $A(F, G, \{0, 1\})$ has pseudoproducts. If $\sigma = 0$, then $A(F, G, \{1\})$ has pseudoproducts.

<u>Proof.</u> Write the objects of $\mathcal{K} = A(F, G, \Delta)$ in the form $(X, \{\sigma_\lambda^X\})$ and the objects of $\mathcal{K}_1 = A(F, G, \{0, 1\})$ - in the case $\sigma > 0$ - as $(X, \{\omega_0^X, \omega_1^X\}) = (X, \{\omega_i^X \mid i = 0, 1\})$.

For every $\lambda, \sigma \leq \lambda < \beta$, take natural transformations $\mu^\lambda : I \longrightarrow Q_{\varkappa_\lambda}$ and $\pi^\lambda : Q_{\varkappa_\lambda} \longrightarrow I$ such

that $\pi^{\lambda} \circ \mu^{\lambda} = 1_I$, and define assignments
$\Phi : \mathcal{K}_1^{obj} \longrightarrow \mathcal{K}^{obj}$ and $\Psi : \mathcal{K}^{obj} \longrightarrow \mathcal{K}_1^{obj}$ by

$$\Phi(X,\{\omega_i^X\}) = (X,\{\sigma_\lambda^X\}) \quad \text{with} \quad \sigma_\lambda^X = \omega_0^X \quad \text{for } \lambda < \sigma,$$

$$\sigma_\lambda^X = \omega_1^X \circ \pi_{FX}^{\lambda} \quad \text{for } \lambda \geq \sigma,$$

$$\Psi(X,\{\sigma_\lambda^X\}) = (X,\{\omega_i^X\}) \quad \text{with } \omega_0^X = \sigma_0^X,\ \omega_1^X = \sigma_\sigma^X \circ \mu_{FX}^{\sigma}.$$

In the case $\sigma = 0$ simply discard nullary operations ω_0^X.

Again, complete the proof by showing that Φ and Ψ satisfy the conditions of lemma 2.1. We shall content ourselves with doing this for the covariant case:

(a) Assuming $(Gf) \circ \sigma_\lambda^X = \sigma_\lambda^Z \circ (Ff)^{(\varkappa_\lambda)}$ with $\sigma_\lambda^Z = \omega_0^Z$ for $\lambda < \sigma$ and $\sigma_\lambda^Z = \omega_1^Z \circ \pi_{FZ}^{\lambda}$ for $\lambda \geq \sigma$ we must prove $(Gf) \circ \omega_i^X = \omega_i^Z \circ (Ff)^{(i)}$ for $\omega_0^X = \sigma_0^X$, $\omega_1^X = \sigma_\sigma^X \circ \mu_{FX}^{\sigma}$, but

$$(Gf) \circ \omega_0^X = (Gf) \circ \sigma_0^X = \sigma_0^Z \circ (Ff)^{(0)} = \omega_0^Z \circ (Ff)^{(0)},$$

$$(Gf) \circ \omega_1^X = (Gf) \circ \sigma_\sigma^X \circ \mu_{FX}^{\sigma} = \sigma_\sigma^Z \circ (Ff)^{(\varkappa_\sigma)} \circ \mu_{FX}^{\sigma} =$$

$$= \omega_1^Z \circ \pi_{FZ}^{\sigma} \circ (Ff)^{(\varkappa_\sigma)} \circ \mu_{FX}^{\sigma} = \omega_1^Z \circ \pi_{FZ}^{\sigma} \circ \mu_{FZ}^{\sigma} \circ (Ff) =$$

$$= \omega_1^Z \circ 1_{FZ} \circ (Ff) = \omega_1^Z \circ (Ff)^{(1)}.$$

(b) Assuming $(Gg) \circ \omega_i^Y = \omega_i^Z \circ (Fg)^{(i)}$ we must prove $(Gg) \circ \sigma_\lambda^Y = \sigma_\lambda^Z \circ (Fg)^{(\varkappa_\lambda)}$ for $\sigma_\lambda^Y = \omega_0^Y$, $\sigma_\lambda^Z = \omega_0^Z$

if $\lambda < \sigma$ and $\sigma_\lambda^Y = \omega_1^Y \circ \pi_{FY}^\lambda$, $\sigma_\lambda^Z = \omega_1^Z \circ \pi_{FZ}^\lambda$ if $\lambda \geq \sigma$ but $(Gg)\circ\sigma_\lambda^Y = (Gg)\circ\omega_0^Y = \omega_0^Z \circ (Fg)^{(\omega)} = \omega_0^Z \circ (Fg)^{(\varkappa_\lambda)}$ for $\lambda < \sigma$, and, $(Gg)\circ\sigma_\lambda^Y = (Gg)\circ\omega_1^Y\circ\pi_{FY}^\lambda =$
$= \omega_1^Z\circ(Fg)\circ\pi_{FY}^\lambda = \omega_1^Z\circ\pi_{FZ}^\lambda\circ(Fg)^{(\varkappa_\lambda)} = \sigma_\lambda^Z\circ(Fg)^{(\varkappa_\lambda)}$ for $\lambda \geq \sigma$.

(c) Assuming $(Gh)\circ\sigma_\lambda^Y = \sigma_\lambda^X \circ (Fh)^{(\varkappa_\lambda)}$ with $\sigma_\lambda^Y = \omega_0^Y$ if $\lambda < \sigma$ and $\sigma_\lambda^Y = \omega_1^Y \circ \pi_{FY}^\lambda$ if $\lambda \geq \sigma$, we are to prove $(Gh)\circ\omega_i^Y = \omega_i^X \circ (Fh)^{(i)}$ for $\omega_0^X = \sigma_0^X$ and $\omega_1^X = \sigma_\sigma^X \circ \mu_{FX}^\sigma$ but

$\omega_0^X \circ (Fh)^{(0)} = \sigma_0^X \circ (Fh)^{(\omega)} = (Gh)\circ\sigma_0^Y = (Gh)\circ\omega_0^Y$,

$\omega_1^X \circ (Fh) = \sigma_\sigma^X \circ \mu_{FX}^\sigma \circ (Fh) = \sigma_\sigma^X \circ (Fh)^{(\varkappa_\sigma)} \circ \mu_{FY}^\sigma =$
$= (Gh)\circ\sigma_\sigma^Y\circ\mu_{FY}^\sigma = (Gh)\circ\omega_1^Y\circ\pi_{FY}^\sigma\circ\mu_{FY}^\sigma =$
$= (Gh)\circ\omega_1^Y\circ 1_{FY} = (Gh)\circ\omega_1^Y$.

Both retraction of functors and reduction of type in categories $A(F, G, \Delta)$ by the above theorems can, of course, be made simultaneously and thus obtained categories are then the first ones to be considered when a negative result on products in some $A(F, G, \Delta)$ is expected.

3. Contravariant case

Theorem 3.1. No category $A(F, G, \Delta)$ with $\sum\Delta > 0$ and faithful contravariant functors F, G has products.

Proof. Since P^- is a retract of both F and G

(Proposition 1.1), we have, with regard to results of the preceding section, but to show that $A(P^-, P^-, \{0, 1\})$ fails to have pseudoproducts. In fact, unary operations do the whole job, the following proof that $A(P^-, P^-, \{1\})$ has not pseudoproducts shows it:

Suppose that $\langle (S; \omega_s), \{f_X, f_Y\} \rangle$ is a pseudoproduct of the family consisting of two objects (X, ω_X), (Y, ω_Y), where $X = \{a, b\}$, $Y = \{c, d\}$, and, ω_X and ω_Y are identical unary operations on $P^- X$ and $P^- Y$, respectively.

Take a well-ordered infinite set $Z = \{z_\alpha \mid \alpha < \vartheta\}$ with $card\ Z > card\ 2^S$ and define a bound $\langle (Z, \omega_z), \{g_X, g_Y\} \rangle$ by

$$g_X(z_0) = g_X(z_1) = a,\ g_X(z_2) = g_X(z_3) = b,\ g_X(z_\alpha) = b \quad \text{for } \alpha > 3,$$

$$g_Y(z_0) = g_Y(z_2) = c,\ g_Y(z_1) = g_Y(z_3) = d,\ g_Y(z_\alpha) = d \quad \text{for } \alpha > 3;$$

denote $Z_\beta = \{z_\alpha \mid \alpha < \beta\}$ for $\beta < \vartheta$ the segments of Z and put $\omega_z(\{z_0\}) = Z_5$, $\omega_z(Z_\beta) = Z_{\beta+1}$ for all β, $5 \leq \beta < \vartheta$, on the remaining part of $P^- Z$ take ω_Z identical.

There must exist $h : (Z, \omega_z) \longrightarrow (S, \omega_S)$ such that

$$g_X = f_X \circ h, \quad g_Y = f_Y \circ h.$$

Since $P^- h$ is a homomorphism of $(P^- S; \omega_s)$ into $(P^- Z, \omega_Z)$ and at the same time a homomorphism of the complete boolean algebra $(P^- S; \cup, \cap)$ into $(P^- Z; \cup, \cap)$, the image $\mathcal{L}$ of $P^- S$ by $P^- h$ must be closed under ω_Z and boolean operations.

Clearly $\{z_0, z_1\}$, $\{z_0, z_2\} \in \mathcal{L}$, hence $\{z_0\} \in \mathcal{L}$ and $Z_S \in \mathcal{L}$. Assume $Z_\alpha \in \mathcal{L}$ for all α, $5 \leqq \alpha < \beta$. If β is isolated, then $Z_\beta = \omega_Z(Z_{\beta-1}) \in \mathcal{L}$. If β is a limit number, then $Z_\beta = \bigcup_{\alpha<\beta} Z_\alpha \in \mathcal{L}$. Therefore $\mathrm{card}\, \mathcal{L} \geqslant \mathrm{card}\, Z$, and this, together with $\mathrm{card}\, 2^S \geqslant \mathrm{card}\, \mathcal{L}$, is a contradiction.

4. Covariant functors and their properties

It has been mentioned, that, dealing with categories $A(F, G, \Delta)$ in the covariant case, it is important to know the behaviour of F and G with regard to sums and products, respectively. From this point of view, consider first a following separation property of functors:

Definition 4.1. A covariant functor F is said to be a separating functor if for any two disjoint subsets M, N of a set X it is

(1) $$[P^+ \circ F(i_M)](FM) \cap [P^+ \circ F(i_N)](FN) = \emptyset,$$

where $i_M : M \longrightarrow X$, $i_N : N \longrightarrow X$ are the corresponding inclusions.

Denote $1 = \{0\}$ - a standard one-point set. For every non-void set X and an element x in X define $w_x X : 1 \longrightarrow X$ by $w_x^X(0) = x$, and, $u_X : X \rightarrow 1$ by $u_X(x) = 0$ for all x in X.

Statement 4.1. A functor F is separating if and only if

(2) $$w_x^X \neq w_y^X \rightarrow [P^+ \circ F(w_x^X)](F1) \cap [P^+ \circ F(w_y^X)](F1) = \emptyset.$$

Proof. Condition (2) is equivalent to (1) with $M = \{x\}, N = \{y\}$. Condition (1) reads then as

$$[P^+ \circ F(i_{\{x\}})](F\{x\}) \cap [P^+ \circ F(i_{\{y\}})](F\{y\}) = \emptyset ,$$

but $\quad F\{x\} = [P^+ \circ F(w_x^{\{x\}})](F1) \quad ,$ therefore

$$[P^+ \circ F(i_{\{x\}})](F\{x\}) = [P^+ \circ F(i_{\{x\}})] \circ [P^+ \circ F(w_x^{\{x\}})](F1) =$$

$$= [P^+ \circ F(i_{\{x\}} \circ w_x^{\{x\}})](F1) = [P^+ \circ F(w_x^X)](F1) ,$$

$[P^+ \circ F(i_{\{y\}})](F\{y\}) = [P^+ \circ F(w_y^X)](F1)$. So the condition (2) is necessary.

Assume that (2) is fulfilled, but F is not separating, that is, for some set X and two disjoint subsets M, N of X we have

(3) $\quad [P^+ \circ F(i_M)](FM) \cap [P^+ \circ F(i_N)](FN) \neq \emptyset$.

In this case both $FM \neq \emptyset$ and $FN \neq \emptyset$, hence $M \neq \emptyset$ and $N \neq \emptyset$ since otherwise it would be $F\emptyset \neq \emptyset$ and F would have a distinguished point, which contradicts (2).

Choose an element x in M and y in N and define mappings $f : X \longrightarrow M$, $g : X \longrightarrow N$ by

$$f(t) = \begin{cases} t & \text{for } t \in M \\ x & \text{for } t \in X \setminus M \end{cases} , \quad g(t) = \begin{cases} t & \text{for } t \in N \\ y & \text{for } t \in X \setminus N \end{cases} .$$

Note that

(4) $\quad i_M \circ f \circ i_N \circ g = w_x^X \circ u_X , \; i_N \circ g \circ i_M \circ f = w_y^X \circ u_X ,$

(5) $\quad f \circ i_M = 1_M , \quad g \circ i_N = 1_N .$

By (3), there exist elements p in FM and q in FN such that

(6) $$(Fi_M)(p) = (Fi_N)(q) = x \in FX \ .$$

It follows by (5) that $(Ff)(x) = (Ff)\circ(Fi_M)(p) = p$ and

$$(Fg)(x) = (Fg)\circ(Fi_N)(q) = q \ ;$$

and, by (6), $[(Fi_M)\circ(Ff)](x) = x$, $[(Fi_N)\circ(Fg)](x) = x$.

By (4) it is then $(Fw_x^X)\circ(Fu_X)(x) = (Fw_y^X)\circ(Fu_X)(x)$,

that is, $(Fw_x^X)(a) = (Fw_y^X)(a)$ for $a = (Fu_X)(x) \in F1$ – in contradiction with the fulfilment of (2).

For every functor F different from $C_\emptyset$ denote by F^* its range-domain restriction to non-void sets and mappings (such a restriction exists, since $F \neq C_\emptyset$ implies $FX \neq \emptyset$ for every non-void set X). Taking a standard two-point set $2 = \{0, 1\}$, denote

$$Q_F = [P^+\circ F(w_0^2)](F1) \cap [P^+\circ F(w_1^2)](F1) \subset F2 \ ,$$

$$A_F = [P^+\circ F(u_2)](Q_F) \ .$$

For a set X let $\vartheta_X : \emptyset \longrightarrow X$ be the empty mapping.

<u>Statement 4.2</u>. If $A_F = \emptyset$, then F is separating. If $A_F \neq \emptyset$, then $C^*_{A_F}$ is a subfunctor of F^*. It is always

$$[P^+\circ F(\vartheta_1)](F\emptyset) \subset A_F \ .$$

<u>Proof</u>. First show that a non-separating functor F

has $A_F \neq \emptyset$:

Take a set X with points $x, y, x \neq y$ such that the condition (2) does not hold for w_x^X and w_y^X, e.g. $(Fw_x^X)(c) = (Fw_y^X)(d) = s \in FX$ for some c, d in $F1$. Define an injection $d : 2 \longrightarrow X$ by $d(0) = x$, $d(1) = y$, and, let $r : X \longrightarrow 2$ be a retraction of d, i.e. $r \circ d = 1_2$. Then $w_0^2 = r \circ w_x^X$, $w_1^2 = r \circ w_y^X$, therefore

$$(Fw_0^2)(c) = (Fr)(s) = (Fw_1^2)(d) \in Q_F$$

and $A_F \neq \emptyset$.

Assume further $A_F \neq \emptyset$. The mappings Fw_0^2 and Fw_1^2 coincide on A_F : For an element a in A_F there must be elements q in Q_F and b, c in $F1$ such that $a = (Fu_2)(q)$ and $q = (Fw_0^2)(b) = (Fw_1^2)(c)$. Since $u_2 \circ w_0^2 = u_2 \circ w_1^2 = 1_1$, it is $(Fu_2)(q) = b = c = a$.

Moreover, for every non-void set X all mappings Fw_x^X for $x \in X$ coincide on A_F : Take x, y in X, $x \neq y$, and the injection $d : 2 \longrightarrow X$ as above, then $w_x^X = d \circ w_0^2$, $w_y^X = d \circ w_1^2$ and the preceding assertion applies.

Now, define a transformation $\mu : C^*_{A_F} \longrightarrow F^*$ by

$$\mu_X(a) = (Fw_x^X)(a) \quad \text{for } a \in A_F \text{ and } x \in X .$$

Clearly, μ_X does not depend on the choice of x in X, it is an injection (for w_x^X is an injection), and it is a transformation because of $f \circ w_x^X = w_{f(x)}^Y$ for every $f : X \longrightarrow Y$.

As to the last assertion of the statement 4.2 $w_0^2 \circ \vartheta_1 = \vartheta_2 = w_1^2 \circ \vartheta_1$ implies $[P^+ \circ F(\vartheta_2)](F\emptyset) \subset Q_F$, and we get the assertion using $\vartheta_1 = \mu_2 \circ \vartheta_2$

<u>Statement 4.3.</u> Every functor F $F \neq C_\emptyset$ can be written as

$$F = F_d \vee F_s ,$$

where functors F_d and F_s have following properties:

a) F_d is $C_\emptyset$ or F_d^* has a subfunctor $C_{A_F}^*$;

b) F_s is the greatest separating subfunctor of F in the sense that every separating subfunctor of F is a subfunctor of F_s.

This decomposition of F is unique up to the natural equivalence.

<u>Proof.</u> Denote $\tilde{A}_F = (F1) \setminus A_F$ - the complement of A_F in $F1$ and for every non-void set X put

$$F_d X = [P^- \circ F(\mu_X)](A_F), \quad F_s X = [P^- \circ F(\mu_X)](\tilde{A}_F).$$

For an arbitrary mapping $f : X \longrightarrow Y$ it is $\mu_X = \mu_Y \circ f$, therefore $[P^+ \circ F(f)](F_d X) \subset F_d Y$ and $[P^+ \circ F(f)](F_s X) \subset F_s Y$. Define $F_d f$ and $F_s f$, accordingly, as range-domain restrictions of Ff.

It is proved that far, that $F^* = F_d^* \vee F_s^*$.

We can now define $F_d \emptyset = F\emptyset$ and $F_d\vartheta_X$ is a domain restriction of $F\vartheta_X$ for $\vartheta_X : \emptyset \longrightarrow X$, and, $F_s\emptyset = \emptyset$, $F_s\vartheta_X = \vartheta_{F_s X} : \emptyset \longrightarrow F_s X$.

It is easily seen that $A_{F_d} = A_F$ and $A_{F_s} = \emptyset$, therefore, by statement 4.2, if $A_F \neq \emptyset$ then $C^*_{A_F}$ is a subfunctor of F^*_d and F_s is a separating functor.

Finally, let $\lambda : G \longrightarrow F$ be a monotransformation of a separating functor G into F. Then necessarily $\lambda_1(t) \in \tilde{A}_F$ for every $t \in G1$, therefore $P^+(\lambda_X))(GX) \subset F_s X$ for every $X \neq \emptyset$, and, of course, $G\emptyset = \emptyset = F_s\emptyset$.

This property of F_s secures uniqueness of the decomposition.

Corollary (to Statement 4.1). Every separating functor F is faithful and $F\emptyset = \emptyset$.

Proof. Assume $Ff = Fg$ for some mappings $f, g : X \to \to Y$. Then $Fw^Y_{f(x)} = F(f \circ w^X_x) = F(g \circ w^X_x) = Fw^Y_{g(x)}$ for all x in X, therefore, by (2), $f(x) = g(x)$ for all x in X, i.e. $f = g$.

Definition 4.2. A functor F is said to be tight on X, $X \neq \emptyset$, if

$$(7) \qquad \bigcup_{x \in X} [P^+ \circ F(w^X_x)](F1) = FX .$$

If this identity does not hold, then F is loose on X.

If F is tight on every X, $X \neq \emptyset$, then it is a tight functor, otherwise it is a loose functor.

Statement 4.4. If F is loose on Y; $Y \neq \emptyset$, and $Y \subset X$, then F is loose on X.

Proof. Denote $i_Y : Y \to X$ an inclusion of Y into X and choose some retract $r : X \to Y$ of i_Y. Then $r \circ w_x^X = w_{r(x)}^Y$ for every x in X. Now, assume that F is tight on X, that is, (7) holds. Since r is a surjection, we get

$$FY = [P^+ \circ F(r)](FX) = [P^+ \circ F(r)](\bigcup_{x \in X} [P^+ \circ F(w_x^X)](F1)) =$$

$$= \bigcup_{x \in X} [P^+ \circ F(r)] \circ [P^+ \circ F(w_x^X)](F1) =$$

$$= \bigcup_{x \in X} [P^+ \circ F(w_{r(x)}^Y)](F1) = \bigcup_{y \in Y} [P^+ \circ F(w_y^Y)](F1)$$

in contradiction with looseness of F on Y.

Corollary. If F is loose on a set X, $X \neq \emptyset$, then it is loose on every set Y with $card\ Y \geq card\ X$.

Equivalently, if F is tight on X, then it is tight on every Y, $Y \neq \emptyset$ with $card\ Y \leq card\ X$.

Proof. - Immediate consequence of Statement 4.4.

Define $w_{x,y}^X : 2 \to X$ by $w_{xy}^X(0) = x$, $w_{xy}^X(1) = y$. For a given functor F denote $W_{xy}^X = [P^+ \circ F(w_{xy}^X)](F2)$, $W_x^X = [P^+ \circ F(w_x^X)](F1)$.

Statement 4.5. Let a functor F be loose on a given set X with $card\ X > 2$, i.e.

$$FX \setminus \bigcup_{x \in X} W_x^X \neq \emptyset .$$

Then

$$FX \setminus \bigcup_{x \in X} W_{ax}^X \neq \emptyset \text{ for arbitrary } a \text{ in } X .$$

Proof. First note that $W^X_{xx} = W^X_x$ for x in X and for any mapping $f : X \longrightarrow X$ it is $[P^+ \circ F(f)](W^X_{xy}) = W^X_{f(x)f(y)}$.

Assume, now, that $\bigcup_{x \in X} W_{ax} = FX$ for some a in X.

Choose an element p in $FX \setminus \bigcup_{x \in X} W^X_{xx}$. Then for some x, $x \neq a$, $p \in W^X_{ax}$. Take an element b in X so that $b \neq a$, $b \neq x$, and a bijection $f :$ $: X \longrightarrow X$ such that $f(b) = a$. Then

$$[P^+ \circ F(f)](\bigcup_{x \in X} W^X_{bx}) = \bigcup_{x \in X} [P^+ \circ F(f)](W_{bx}) =$$

$$= \bigcup_{x \in X} W_{af(x)} = \bigcup_{x \in X} W_{ax} = FX ,$$

therefore $\bigcup_{x \in X} W_{bx} = FX$, and, for some $y \neq b$, it is $p \in W^X_{by}$.

It remains to show that $p \in W^X_{ax} \cap W^X_{by}$ leads to a contradiction: Take a mapping $g : X \longrightarrow X$ such that

$$g(a) = a, \quad g(x) = x, \quad g(b) = g(y) = \begin{cases} a & \text{if } y = a , \\ x & \text{if } y \neq a . \end{cases}$$

Then

$$p = (Fg)(p) \in [P^+ \circ F(g)](W^X_{ax} \cap W^X_{by}) \subset$$

$$\subset W^X_{ax} \cap W^X_{g(b)g(y)} \subset W^X_{aa} \cup W^X_{xx} .$$

Statement 4.6. If F is tight, then for every set X and for its arbitrary two subsets M, N it holds

(8) $[P^+ \circ F(i^X_M)](FM) \cup [P^+ \circ F(i^X_N)](FN) = [P^+ \circ F(i^X_S)](FS)$

where $S = M \cup N$, and, $i_M^X : M \to X$, $i_N^X : N \to X$, $i_S^X : S \to X$ are the respective inclusions of M, N, S into X.

<u>Proof</u>. Denote $i_M^S : M \to S$, $i_N^S : N \to S$ the inclusions of M, N into S, respectively. Then we have

$$(9) \qquad i_M^X = i_S^X \circ i_M^S \,, \quad i_N^X = i_S^X \circ i_N^S \,.$$

It is easy to see that (8) holds, if one of the sets M, N, S is void. Assume further that $M \neq \emptyset$, $N \neq \emptyset$. Then, by tightness of F,

$$FM = \bigcup_{x \in M} [P^+ \circ F(w_x^M)](F\mathbb{1}), \quad FN = \bigcup_{x \in N} [P^+ \circ F(w_x^N)](F\mathbb{1}).$$

Using (9), we get

$$[P^+ \circ F(i_M^X)](FM) = [P^+ \circ F(i_M^X)]\Big(\bigcup_{x \in M} [P^+ \circ F(w_x^M)](F\mathbb{1})\Big) =$$

$$= \bigcup_{x \in M} [P^+ \circ F(i_M^X \circ w_x^M)](F\mathbb{1}) = \bigcup_{x \in M} [P^+ \circ F(w_x^X)](F\mathbb{1}) \,,$$

and, similarly

$[P^+ \circ F(i_N^X)](FN) = \bigcup_{x \in N} [P^+ \circ F(w_x^X)](F\mathbb{1})$, therefore

$$[P^+ \circ F(i_M^X)](FM) \cup [P^+ \circ F(i_N^X)](FN) = \bigcup_{x \in S} [P^+ \circ F(w_x^X)](F\mathbb{1}),$$

but $w_x^X = i_S^X \circ w_x^S$ for x in S, so it is, finally,

$$\bigcup_{x \in S} [P^+ \circ F(w_x^X)](F\mathbb{1}) = [P^+ \circ F(i_S^X)]\Big(\bigcup_{x \in S} [P^+ \circ F(w_x^S)](F\mathbb{1})\Big) =$$

$$= [P^+ \circ F(i_S^X)](FS)$$

by tightness of F.

Tight separating functors are exactly the functors preserving sums. Let us formulate this as

Statement 4.7. If F does not preserve sums, then F is either loose or it is not separating.

Remark. Denote by $\mathcal{P}$, $\mathcal{T}$, $\mathcal{L}$ the systems of all separating, tight, loose functors, respectively. Each of these systems is closed under $\vee, \times, \circ$ for functors, $\mathcal{P}$ is closed on subfunctors, $\mathcal{T}$ is closed on subfunctors and factor-functors, $\mathcal{L}$ is closed on extensions ($F \in \mathcal{L}$, $F \xrightarrow{\mu} F' \Longrightarrow F' \in \mathcal{L}$). Every F in $\mathcal{T}$ splits by statement 4.3 into $F_d \vee F_s$ such that $F_d^* \cong C^*_{F_A}$ and F_s preserves sums.

It is $I \in \mathcal{P} \cap \mathcal{T}$, constant functors C_M are in $\mathcal{T}$, N, P^*, $\beta \in \mathcal{L}$, $Q_M \in \mathcal{L}$ for $\operatorname{card} M \geq 2$.

Turn now to range functors.

Statement 4.8. If G does not preserve the product of a family $\{X_\alpha \mid \alpha \in A\}$, then it does not preserve the product of any family $\{Y_\alpha \mid \alpha \in A\}$ with $\operatorname{card} Y_\alpha \geq \operatorname{card} X_\alpha$ for all α in A.

Proof. Choose for each α in A mappings $i_\alpha : X_\alpha \to Y_\alpha$, $\kappa_\alpha : Y_\alpha \to X_\alpha$ such that $\kappa_\alpha \circ i_\alpha = 1_{X_\alpha}$. Denote $\langle X, \{\pi^X_\alpha\}\rangle$ and $\langle Y, \{\pi^Y_\alpha\}\rangle$ the products of $\{X_\alpha\}$ and $\{Y_\alpha\}$, respectively. Define mappings $i : X \to Y$ and $\kappa : Y \to X$ by

$$(10) \qquad i_\alpha \circ \pi^X_\alpha = \pi^Y_\alpha \circ i, \qquad \kappa_\alpha \circ \pi^Y_\alpha = \pi^X_\alpha \circ \kappa .$$

It is then $\kappa \circ i = 1_X$.

Assume that G preserves the product of $\{Y_\alpha\}$ and show that then it preserves the product of $\{X_\alpha\}$ too:

For an arbitrary family $\{x_\alpha\}$, $x_\alpha \in GX_\alpha$ for α in A, there must exist y in GY such that $(G\pi_\alpha^Y)(y) = (Gi_\alpha)(x_\alpha)$, and, using (10), we get $(G\pi_\alpha^X)(x) = x_\alpha$ for $x = (Gr)(y)$ by easy calculation. The element x with $(G\pi_\alpha^X)(x) = x_\alpha$ must be unique, since $(G\pi_\alpha^X)(x_1) = (G\pi_\alpha^X)(x_2)$ implies $(G\pi_\alpha^Y)(y_1) = (G\pi_\alpha^Y)(y_2)$ for $y_1 = (Gi)(x_1)$, $y_2 = (Gi)(x_2)$ by simple calculation using (10).

Next three definitions reflect certain properties of the functors not preserving products.

Let $\mathfrak{X} = \{X_\alpha \mid \alpha \in A\}$ be a family of sets. Denote by $\langle X, \{\pi_\alpha^X\}\rangle$ its product $X = \prod_{\alpha \in A} X_\alpha$ with $\pi_\alpha^X : X \longrightarrow X_\alpha$ - the ordinary projections. If a functor G does not preserve the product of the family $\mathfrak{X}$, then either

(I) there exists a family $\{x_\alpha\}$, $x_\alpha \in GX_\alpha$ for $\alpha \in A$, such that there is no x in GX with $(G\pi_\alpha^X)(x) = (x_\alpha)$ for all α in A,

or

(II) there exist two points x, y in GX, $x \neq y$, such that $(G\pi_\alpha^X)(x) = (G\pi_\alpha^X)(y)$ for all α in A.

<u>Definition 4.3.</u> A functor G not preserving products is said to <u>blow up</u> products if for some family of sets the alternative (II) takes place. If, moreover, the alterna-

tive (I) takes place for no family, then G is said to inflate products.

Definition 4.4. A functor G not preserving products is said to filtrate products, if for an arbitrary family $\{X_\alpha \mid \alpha \in A\}$ with the product $\langle X, \{\pi_\alpha\}\rangle$ the family of mappings $\{G\pi_\alpha \mid \alpha \in A\}$ is separating on GX in the sense that

(11) $$\forall \alpha \in A\, ((G\pi_\alpha)(x) = (G\pi_\alpha)(y)) \Rightarrow x = y$$

for x, y in GX .

Remark. The system of all functors with the property (11) is closed under $\vee\ \times$, $\circ$ and subfunctors. We obtain the system $\mathcal{F}$ of filtrating functors by removing functors preserving products.

Definition 4.5. A functor G superinflates products if there exists a family $\{X_\alpha \mid \alpha \in A\}$ of non-void sets with the following property:

There exist x_α in X_α and y_α in GX_α for all α in A such that, denoting $\langle X, \{\pi_\alpha\}\rangle$ the product of $\{X_\alpha\}$, for an arbitrary set S and mappings $\sigma_\alpha : X \vee S \longrightarrow X_\alpha$ such that $\sigma_\alpha \mid X = \pi_\alpha$ and $\sigma_\alpha(s) = x_\alpha$ for all s in S , it holds

$\operatorname{card}\{z \in G(X \vee S) \mid (G\sigma_\alpha)(z) = y_\alpha$ for all α in $A\} > 1 + \operatorname{card} S$.

Statement 4.9. The functors $N, \beta, \langle P^-, I\rangle$ superflate products. For the system $\mathcal{N}$ of functors superinflating products it holds:

(α) G has a subfunctor belonging to $\mathcal{N} \Rightarrow G \in \mathcal{N}$,

(β) $F \times \mathcal{N} \subset \mathcal{N}$ for any functor F,

(γ) F is a covariant faithful functor $\Rightarrow F \circ \mathcal{N} \subset \mathcal{N}$, $\mathcal{N} \circ F \subset \mathcal{N}$,

(δ) F, G are contravariant faithful $\Rightarrow F \circ G \in \mathcal{N}$,

(ε) F is contravariant faithful or constant, $G \in \mathcal{N} \Rightarrow$
$\Rightarrow \langle F, G \rangle \in \mathcal{N}$.

Proof.

1) N superinflates products; choose $X_1 = \{a, b\}$, $X_2 = \{c, d\}$, $x_1 = a$, $x_2 = c$, $y_1 = \{a, b\}$, $y_2 = \{c, d\}$, then the family $\{X_1, X_2\}$ and points x_1, x_2, y_1, y_2 meet the requirements of the definition 4.5.

2) β superinflates products; choose $X_n = \{a_n, b_n\}$, $n = 1, 2, 3, \ldots$, $x_n = a_n$, $y_n = \{\{a_n\}, \{a_n, b_n\}\}$, then the (countable) system $\{X_n \mid n = 1, 2, \ldots\}$ and points x_n, y_n meet the requirements. (If $card\, S < \aleph_0$ use the fact that $\{x \in \beta X \mid (\beta \pi_n)(x) = y_n$ for $n = 1, 2, \ldots\} \geq 2^{2^{\aleph_0}}$, if $card\, S \geq \aleph_0$, then use $card\, \beta S = 2^{2^{card\, S}}$.)

3) $\langle P^-, I \rangle$ superinflates products; again choose the family $\{X_1, X_2\}$ where $X_1 = \{a, b\}$, $X_2 = \{c, d\}$, $x_1 = a$, $x_2 = c$, $y_1 : P^- X_1 \longrightarrow X_1$ is the constant mapping to a, $y_2 : P^- X_2 \longrightarrow X_2$ is the constant mapping to c.

The assertions (α) - (ε) can be easily proved with aid of the Proposition 1.1.

5. Covariant case. We suppose always $F \neq C_{\emptyset}$, $G \neq C_{\emptyset}$.

Theorem 5.1. Let $A(F, G, \Delta)$ be a category whose type $\Delta = \{ \varkappa_\lambda \mid \lambda < \beta \}$ contains zeros, say, $\varkappa_0 = 0$. Then $A(F, G, \Delta)$ has products if and only if G preserves products.

Proof. If G preserves products, then, clearly, $A(F, G, \Delta)$ has products, so we have to show the converse implication.

Take an arbitrary family $\{ X_\alpha \mid \alpha \in A \}$ of non-void sets and choose a family $\{ x_\alpha \in GX_\alpha \mid \alpha \in A \}$. Denote $\langle X, \{ \pi_\alpha \mid x \in A \} \rangle$ the product of $\{ X_\alpha \}$ with π_α - the ordinary projections. We must show that

(a) there exists an element x in GX such that $(G\pi_\alpha)(x) = x_\alpha$ for all α in A,

(b) if for some x, y in GX it is $(G\pi_\alpha)(x) = (G\pi_\alpha)(y) = x_\alpha$ for all α in A, then $x = y$.

By theorem 2.2, the category $A(F, G, \{0, 1\})$ *) has pseudoproducts. To show (a), take the family $\{ (X_\alpha, \{ \sigma_0^\alpha, \sigma_1^\alpha \}) \mid \alpha \in A \}$ of objects of $A(F, G, \{0, 1\})$ with operations defined so that for each α, $\alpha \in A$, σ_0^α selects x_α in GX_α and σ_1^α carries the whole FX_α into x_α.

Let $\langle (S, \{ \sigma_0^S, \sigma_1^S \}), \{ \sigma_\alpha \} \rangle$ be a pseudoproduct of this family. There exists a mapping $h : S \to X$

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*) Unary operations play no role in our proof and it works in the case $\varkappa_\lambda = 0$ for all λ, $\lambda < \beta$, as well.

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such that

(1) $$\sigma_\alpha = \pi_\alpha \circ h \quad \text{for all } \alpha \text{ in } A .$$

Denote s the element in GS selected by σ_o^S . For $x = (Gh)(s)$ it is $(G\pi_\alpha)(x) = (G\pi_\alpha)\circ(Gh)(s) = (G\sigma_\alpha)(s) = x_\alpha$ for all α in A , as required.

To prove (b), assume $(G\pi_\alpha)(x) = (G\pi_\alpha)(y) = x_\alpha$ for all α in A , and, take inverse bounds $\langle (X, \{\sigma_o^X, \sigma_1^X\}), \{\pi_\alpha\} \rangle$ with σ_o^X selecting x and σ_1^X carrying FX into x and $\langle (X, \{\omega_o^X, \omega_1^X\}), \{\pi_\alpha\} \rangle$ with ω_1^X carrying FX into y selected by ω_o^X .

Let $f, g : X \longrightarrow S$ be the respective factoring morphisms, that is

(2) $$\pi_\alpha = \sigma_\alpha \circ f = \sigma_\alpha \circ g \quad \text{for all } \alpha \text{ in } A ,$$

and, in particular,

(3) $$(Gf)(x) = (Gg)(y) = s .$$

By (1) and (2) we get $h \circ f = h \circ g = 1_X$ which applied to (3) gives $x = y = (Gh)(s)$.

Consider further only categories $A(F, G, \Delta)$ with a completely positive type $\Delta = \{ \varkappa_\lambda \mid \lambda < \beta \}$, i.e. $\varkappa_\lambda > 0$ for all λ, $\lambda < \beta$. As a corollary of theorem 5.1 we get

<u>Theorem 5.2</u>. If $F\emptyset \neq \emptyset$ and G does not preserve products, then a category $A(F, G, \Delta)$ has not products.

<u>Proof</u>. Assume that $A(F, G, \Delta)$ has products. Then $A(C_1, G, \{1\})$ has pseudoproducts, by theorems 2.1 and 2.2, since $F\emptyset \neq \emptyset$ means that C_1 is a retract of F .

Now, unary operations $\sigma^X : C_1 X \longrightarrow GX$ just select a point in GX, therefore $A(C_1, G, \{1\})$ coincides with $A(C_1, G, \{0\})$ which fails to have pseudoproducts by theorem 5.1, in contradiction with our assumption.

<u>Theorem 5.3</u>. Let $A(F, G, \Delta)$ be a category of a type $\Delta = \{ \varkappa_\lambda \mid \lambda < \beta \}$ with a range-functor G not preserving products.

If the functor $Q_{\varkappa_\lambda} \circ F$ is loose for some λ, $\lambda < \beta$, then $A(F, G, \Delta)$ has not products.

<u>Proof</u>. Assume $Q_{\varkappa_\gamma} \circ F$ loose. Combining statements 4.4 and 4.8 of the preceding section find a set X such that $Q_{\varkappa_\gamma} \circ F$ is loose on X and G does not preserve a power $\langle X^A, \{\pi_\alpha \mid \alpha \in A\} \rangle$ for a suitable set A.

(I) Denote $P = X^A$ and first assume that for some family $\{ x_\alpha \in GX \mid \alpha \in A \}$ there is no point v in GP with $((G\pi_\alpha)(v) = x_\alpha$ for all α in A.

Using the notation introduced in statement 4.5, define operations $\sigma_\lambda^\alpha : (FX)^{\varkappa_\lambda} \longrightarrow GX$, $\alpha \in A$, $\lambda < \beta$, as follows:

Choose an element a in X and an element d in the part $[P^+ \circ G(w_\circ^2)](G1)$ of $G2$, denote $D_X^\lambda = \bigcup_{x \in X} [P^+ \circ Q_{\varkappa_\lambda} \circ F(w_{ax}^X)](F2)$, $d_X = (G w_{aa}^X)(d)$, and put

$$\sigma_\lambda^\alpha(t) = \begin{cases} d_X & \text{for } t \in D_X^\lambda \\ x_\alpha & \text{for } t \in (FX)^{\varkappa_\lambda} \setminus D_X^\lambda \end{cases} \tag{4}$$

Define $(2, \{\sigma_\lambda^2\})$ by $\sigma_\lambda^2(t) = d$ for all t in

$(F\mathbf{2})^{\varkappa_\lambda}$ and note that every w^X_{ax}, $x \in X$, is a morphism of $(\mathbf{2}, \{\sigma^{\mathbf{2}}_\lambda\})$ into $(X, \{\sigma^\alpha_\lambda\})$, since $(G w^X_{ax})(d) = (G w^X_{aa})(d)$ for every x in X. Therefore $\langle(\mathbf{2}, \{\sigma^{\mathbf{2}}_\lambda\}), \{w^X_{a\varphi(\alpha)}\}\rangle$ with an arbitrary $\varphi : A \longrightarrow X$ is an inverse bound of the family $\{(X, \{\sigma^\alpha_\lambda\}) \mid \alpha \in A\} = \mathfrak{X}$.

Suppose that $\langle(S, \{\sigma^S_\lambda\}), \{\sigma_\alpha\}\rangle$ is a product of $\mathfrak{X}$ and denote $h : S \longrightarrow P$ the mapping uniquely determined by

(5) $$\sigma_\alpha = \pi_\alpha \circ h \quad \text{for all } \alpha \text{ in } A .$$

Denote $f_x : \mathbf{2} \longrightarrow S$, $x \in X$, factoring morphisms of inverse bounds $\langle(\mathbf{2}, \{\sigma^{\mathbf{2}}_\lambda\}), \{w^X_{a\varphi(a)}\}\rangle$ with $\varphi(\alpha) = x$ for all α in A, i.e. $w^X_{ax} = \sigma_\alpha \circ f_x$ for all α in A.

Then for a mapping $\tau : X \longrightarrow S$ defined by $\tau(x) = f_x(1)$ it is $x = w^X_{ax}(1) = \sigma_\alpha \circ f_x(1) = \sigma_\alpha \circ \tau(x)$, hence

(6) $$\sigma_\alpha \circ \tau = 1_X \quad \text{for all } \alpha \text{ in } A .$$

Now, by statement 4.5, choose t in $(FX)^{\varkappa_\gamma} \setminus D^\gamma_X$, denote $s = (F\tau)^{(\varkappa_\gamma)}(t)$, $v = (Gh)(\sigma^S_\gamma(s))$, and, using (5) and (6), get

$$(G\pi_\alpha)(v) = (G\pi_\alpha)\cdot(Gh)(\sigma^S_\gamma(s)) = (G\sigma_\alpha)(\sigma^S_\gamma(s)) =$$

$$= \sigma^\alpha_\gamma \circ (F\sigma_\alpha)^{(\varkappa_\gamma)}(s) = \sigma^\alpha_\gamma \circ (F\sigma_\alpha)^{(\varkappa_\gamma)} \circ (F\tau)^{(\varkappa_\gamma)}(t) = \sigma^\alpha_\gamma(t) = x_\alpha$$

for all α in A, in contradiction with our assumption.

(II) Assume further that (I) happens for no family in GX, but for a family $\{ x_\alpha \in GX \mid \alpha \in A \}$ there are v, v' in GP, $v \neq v'$, such that $(G\pi_\alpha)(v) = (G\pi_\alpha)(v') = x_\alpha$ for all α in A.

Take again the family $\{ (X, \{\sigma_\lambda^\alpha\}) \mid \alpha \in A \}$ with operations defined by (4) and suppose that it has a product $\langle (S, \{\sigma_\lambda^S\}), \{\sigma_\alpha\} \rangle$.

Define inverse bounds $\langle (P, \{\sigma_\lambda^P\}), \{\pi_\alpha\} \rangle$ and $\langle (P, \{\omega_\lambda^P\}), \{\pi_\alpha\} \rangle$ as follows:

Define $\mu : X \to P$ by $\pi_\alpha \circ \mu = 1_X$ for all α in A, denote $D_P^\lambda = \bigcup_{p \in P} [P^+ \circ Q_{\varkappa_\lambda} \circ F(w^P_{\mu(a)p})](F\mathbf{2})$, $d_p = (Gw^P_{\mu(a)\mu(a)})(d)$, and put $\sigma_\lambda^P(u) = \omega_\lambda^P(u) = d_p$ for $u \in D_P^\lambda$, $\sigma_\lambda^P(u) = v$, $\omega_\lambda^P(u) = v'$ for $u \in [P^+ \circ Q_{\varkappa_\lambda} \circ F(\mu)]((FX)^{\varkappa_\lambda} \setminus D_X^\lambda)$, on the rest of $(FP)^{\varkappa_\lambda}$ define σ_λ^P and ω_λ^P so that all π_α become morphisms, which is possible by our assumption.

Note that all $w^P_{\mu(a)p}$, $p \in P$, are morphisms of $(\mathbf{2}, \{\sigma_\lambda^{\mathbf{2}}\})$ into both $(P, \{\sigma_\lambda^P\})$ and $(P, \{\omega_\lambda^P\})$. Let $f, f' : P \to S$ be the respective morphisms of $(P, \{\sigma_\lambda^P\})$ and $(P, \{\omega_\lambda^P\})$ into $(S, \{\sigma_\lambda^S\})$ with $\pi_\alpha = \sigma_\alpha \circ f = \sigma_\alpha \circ f'$ for all α in A.

Together with (5) we get $h \circ f = h \circ f' = 1_P$, so f and f' are injections, and, it cannot be $f = f'$, since then it would be $\sigma_\gamma^S \circ (Ff)^{(\varkappa_\gamma)}(u) = (Gf)(v) = (Gf)(v')$ for any u in $[P^+ \circ Q_{\varkappa_\gamma} \circ F(\mu)]((FX)^{\varkappa_\gamma} \setminus D_X^\gamma)$.

Therefore it is $f(p^*) \neq f'(p^*)$ for some p^* in P.

Now, $\langle (2, \{\sigma_\lambda^2\}), \{\pi_\alpha \circ w^P_{\mu(a)p^*}\} \rangle$ is an inverse bound of $\mathfrak{X}$ with two different factoring morphisms through $\langle (S, \{\sigma_\lambda^S\}), \{6_\alpha\} \rangle$, namely, $f \circ w^P_{\mu(a)p^*}$ and $f' \circ w^P_{\mu(a)p^*}$

As a simple corollary we have

<u>Theorem 5.4.</u> If F is faithful, G not preserving products, and, Δ contains a number $\varkappa_\lambda$ different from 1, then $A(F, G, \Delta)$ has not products.

<u>Proof.</u> $Q_{\varkappa_\lambda} \circ F$ has a subfunctor $Q_{\varkappa_\lambda}$ which is loose for $\varkappa_\lambda > 1$.

<u>Theorem 5.5.</u> If F is not separating and G blows up products, then $A(F, G, \Delta)$ has not products.

<u>Proof.</u> Assume that for a family $\{X_\alpha \mid \alpha \in A\}$ with the product $\langle P, \{\pi_\alpha\} \rangle$ there are v, v' in GP, $v \neq v'$, such that $(G\pi_\alpha)(v) = (G\pi_\alpha)(v')$ for all α in A.

Take a family $\{(X_\alpha, \sigma_\alpha) \mid \alpha \in A\}$ of objects of $A(F, G, \{1\})$ with $\sigma_\alpha(t) = (G\pi_\alpha)(v)$ for all t in FX_α, $\alpha \in A$, and, suppose that the family has a pseudoproduct $\langle (S, \sigma_s), \{6_\alpha\} \rangle$.

Define inverse bounds $\langle (P, \sigma_p), \{\pi_\alpha\} \rangle$ and $\langle (P, \sigma_p'), \{\pi_\alpha\} \rangle$ by $\sigma_p(t) = v$, $\sigma_p'(t) = v'$

for all t in FP , denote $f, f' : P \longrightarrow S$ the corresponding morphisms such that $\pi_\alpha = \sigma_\alpha \circ f = \sigma_\alpha \circ f'$ for all α in A .

If F is not separating, then there exists an element t in FP such that $(Ff)(t) = (Ff')(t) = u$.

It is then

(7) $$(Gf)(v) = (Gf')(v') = \sigma_S(u) \, .$$

Now, f and f' have a common retraction $h : S \longrightarrow P$ defined by $\sigma_\alpha = \pi_\alpha \circ h$, $\alpha \in A$, that is, $h \circ f = h \circ f' = 1_P$. Applying to the identity (7) we get $v = v'$ - a contradiction.

Let us call a type $\Delta = \{ \varkappa_\lambda \mid \lambda < \beta \}$ with $\varkappa_\lambda = 1$ for all $\lambda, \lambda < \beta$, a unary type.

Theorem 5.6. A category $A(F, G, \Delta)$ with G not preserving products and whose type is not unary has products if and only if $F\emptyset = \emptyset$, $F^* \cong C^*_M$ and G filtrates products (F^* is a range-domain restriction to non-void sets and mappings).

Proof. If $A(F, G, \Delta)$ has products, then F is neither loose nor faithful. Therefore $Fw_x^X = Fw_y^X$ for arbitrary x, y in X and Fw_x^X is - by tightness - a bijection between $F1$ and FX independent of choice of x in X . Putting $\varepsilon^X = Fw_x^X$ we obtain a nat. equivalence $\varepsilon : C^*_{F1} \longrightarrow F^*$.

Since F is not separating, G must then, by theorem 5.5, filtrate products.

The condition $F\emptyset = \emptyset$ has been established by theorem 5.2.

Assume, conversely, that the conditions imposed on F and G are fulfilled. Let $\mathfrak{X} = \{(X_\alpha, \{\sigma_\lambda^\alpha \mid \lambda < \beta\}) \mid \alpha \in A\}$ be an arbitrary family of objects of $A(F, G, \Delta)$. Let $\langle P, \{\pi_\alpha\}\rangle$ be the product $P = \prod_{\alpha \in A} X_\alpha$ with ordinary projections.

If, for some m in $M^{\varkappa_\lambda}$, there is no u in GP such that $(G\pi_\alpha)(u) = \sigma_\lambda^\alpha(m)$ for all α in A, then every inverse bound $\langle (Y, \{\sigma_\lambda^Y\}), \{\eta_\alpha\}\rangle$ of $\mathfrak{X}$ must be void and is, in fact, a product of $\mathfrak{X}$.

If, for every m in $M^{\varkappa_\lambda}$, $\lambda < \beta$, there exists some u in GP such that $(G\pi_\alpha)(u) = \sigma_\lambda^\alpha(m)$ for all α in A, then $\langle (P, \{\sigma_\lambda^P\}), \{\pi_\alpha\}\rangle$ with σ_λ^P defined by

$$(G\pi_\alpha)\sigma_\lambda^P = \sigma_\lambda^\alpha \qquad \text{for all } \alpha \text{ in } A$$

is a product of $\mathfrak{X}$.

Theorem 5.7. A category $A(F, G, \Delta)$ with a unary type Δ and G filtrating products has products if and only if F is a tight functor with $F\emptyset = \emptyset$, in particular if F preserves sums.

Proof. The condition is necessary by theorem 5.3 and 5.2.

Let $\mathfrak{X} = \{(X_\alpha, \{\sigma_\lambda^\alpha\}) \mid \alpha \in A\}$ be an arbitrary family of objects of $A(F, G, \Delta)$, let $\langle X, \{\pi_\alpha \mid \alpha \in A\}\rangle$ be the product $X = \prod_{\alpha \in A} X_\alpha$ with ordinary projections π_α, $\alpha \in A$.

Define a system $\mathcal{U}$ of admissible subsets of X

by the condition that $M \in \mathcal{U}$ if and only if for every t in FM, there exists a family $\{u_\lambda \in GM \mid \lambda < \beta\}$ such that

(1) $\sigma_\lambda^\alpha \circ [F(\pi_\alpha \circ i_M^X)](t) = [G(\pi_\alpha \circ i_M^X)](u_\lambda)$ for all α in A,

where $i_M^X : M \longrightarrow X$ is the inclusion of M into X.

Since G filtrates products, the family $\{u_\lambda\}$ is uniquely determined by t and $\langle (M, \{\sigma_\lambda^M\}), \{\pi_\alpha \circ i_M^X\} \rangle$ with $\sigma_\lambda^M(t) = u_\lambda$ for t in FM becomes an inverse bound of $\mathcal{X}$.

Denote $S = \bigcup \mathcal{U}$ = the union of all admissible subsets of X, $i_M^S : M \longrightarrow S$, $M \in \mathcal{U}$, - the inclusion of M into S. Since F is tight, we have by statement 4.6

$$\bigcup_{M \in \mathcal{U}} [P^+ \circ F(i_M^S)](FM) = FS ,$$

therefore, for every s in FS, we have $(Fi_S^X)(s) = (Fi_M^X)(t)$ for some admissible set M and t in FM. Putting $v_\lambda = (Gi_M^S)(\sigma_\lambda^M(t))$ we get

$$\sigma_\lambda^\alpha \circ [F(\pi_\alpha \circ i_S^X)](s) = \sigma_\lambda^\alpha \circ [F(\pi_\alpha \circ i_M^X)](t) =$$

$$= [G(\pi_\alpha \circ i_M^X)] \circ \sigma_\lambda^M(t) = [G(\pi_\alpha \circ i_S^X)] \circ (Gi_M^S) \circ \sigma_\lambda^M(t) = [G(\pi_\alpha \circ i_S^X)](v_\lambda),$$

therefore S is admissible. Moreover, it is easily seen that i_M^S is a morphism of $(M, \{\sigma_\lambda^M\})$ into $(S, \{\sigma_\lambda^S\})$.

It remains to show that $\langle (S, \{\sigma_\lambda^S\}), \{\pi_\alpha \circ i_S^X\} \rangle$ is a product of $\mathcal{X}$.

Let $\langle (Y, \{\sigma_\lambda^Y\}), \{\eta_\alpha\} \rangle$ be an inverse bound of $\mathfrak{X}$, i.e. $\sigma_{\lambda}^{\alpha} \circ (F\eta_{\alpha}) = (G\eta_{\alpha}) \circ \sigma_{\lambda}^{Y}$ for all α in A , and let $h : Y \rightarrow X$ be the mapping uniquely determined by $\pi_{\alpha} \circ h = \eta_{\alpha}$, $\alpha \in A$. Denote $M = (P^{+}h)(Y)$ and let $\hat{h} : Y \longrightarrow M$ be the range restriction of h . Then we have $h = i_{M}^{X} \circ \hat{h}$ and $\eta_{\alpha} = \pi_{\alpha} \circ i_{M}^{X} \circ \hat{h}$, therefore

$$(2) \quad \sigma_{\lambda}^{\alpha} \circ [F(\pi_{\alpha} \circ i_{M}^{X})] \circ (F\hat{h}) = [G(\pi_{\alpha} \circ i_{M}^{X})] \circ (G\hat{h}) \circ \sigma_{\lambda}^{Y} .$$

Now, for every t in FM there exists an y in FY such that $(F\hat{h})(y) = t$. By (1) and (2) it must be

$$(G\hat{h}) \circ \sigma_{\lambda}^{Y}(y) = \sigma_{\lambda}^{M}(t) = \sigma_{\lambda}^{M} \circ (F\hat{h})(y) ,$$

therefore $\hat{h}$ is a morphism of $(Y, \{\sigma_{\lambda}^{Y}\})$ onto $(M, \{\sigma_{\lambda}^{M}\})$, M is admissible, and $f : i_{M}^{S} \circ \hat{h}$ is the unique factoring morphism of $(Y, \{\sigma_{\lambda}^{Y}\})$ into $(S, \{\sigma_{\lambda}^{S}\})$ such that $\eta_{\alpha} = (\pi_{\alpha} \circ i_{S}^{X}) \circ f$ for all α in A .

As a corollary we have

<u>Theorem 5.8.</u> A category $A(F, G, \Delta)$ of a unary type and with F not preserving sums has products if and only if F is tight with $F\emptyset = \emptyset$ and G filtrates or preserves products.

<u>Proof</u>. If $A(F, G, \Delta)$ has products, then F must be tight by theorem 5.3, $F\emptyset = \emptyset$ by theorem 5.2, therefore it cannot be separating and G then cannot blow up products by theorem 5.5.

The converse has been asserted in theorem 5.7.

Theorem 5.9. If G superinflates products, then $A(F, G, \Delta)$ has not products.

Proof. Having in view the theorems 5.1, 5.6, 5.8, we shall have only to prove that $A(F, G, \Delta)$ has not products in the case of a unary type Δ and the functor F preserving sums. Then it is $F \simeq I \times C_M$ and thus $A(F, G, \Delta)$ is isomorphic to some $A(I, G, \Delta')$ with a suitable unary type Δ'. Therefore to prove the theorem, it will do to show that $A(I, G, \{1\})$ has not pseudoproducts. The proof then runs as follows.

Let $\{X_\alpha \mid \alpha \in A\}$, $x_\alpha \in X_\alpha$, $y_\alpha \in G X_\alpha$ enjoy the properties stated in the definition 4.5. Let $\sigma_\alpha : X_\alpha \to$ $\to G X_\alpha$, $\alpha \in A$, be the constant mapping assigning to every x from X_α the element y_α. We shall show that the family $\mathfrak{X} = \{(X_\alpha, \sigma_\alpha) \mid \alpha \in A\}$ of objects of $A(I, G, \{1\})$ fails to have a pseudoproduct in this category.

Assume that the family $\mathfrak{X}$ has a pseudoproduct, say, $\langle (P, \sigma_P); \{p_\alpha \mid \alpha \in A\}\rangle$. Let m be an arbitrary infinite cardinal number. It will be shown that $card\, P \geq m$.

Let $\langle X, \{\pi_\alpha \mid \alpha \in A\}\rangle$ be the cartesian product of the family $\{X_\alpha \mid \alpha \in A\}$. Let S be a set with $card\, S \geq m$. Define an inverse bound $\mathcal{Z} = \langle (Z, \sigma_Z); \{\varrho_\alpha \mid \alpha \in A\}\rangle$ of the family $\mathfrak{X}$ as follows:

$Z = X \vee S$, $\varrho_\alpha : Z \longrightarrow X_\alpha$ is a mapping such that $\varrho_\alpha \mid X = \pi_\alpha$, $\varrho_\alpha(s) = x_\alpha$ for all s in S.

To define the operation σ_Z denote $M = \{x \in G(X \vee S) \mid (G\sigma_\alpha)(x) = y_\alpha$ for all α in $A\}$. Let $\prec$ be a well-ordering of S, for a given s in S, denote $S_s = \{t \in S \mid t \prec s\}$. For an s in S denote further $M_s = M \cap [(P^+ \circ G)(i_s)](G(X \vee S_s))$, where $i_s : X \vee S_s \longrightarrow Z$ is the inclusion. Since G superinflates products we have $\mathrm{card}\, M_s > 1 + \mathrm{card}\, S_s$. Therefore, we can now define $\sigma_Z : Z \longrightarrow GZ$ by the transfinite induction in such a way that $\sigma_Z(X) \cap \sigma_Z(S) = \emptyset$, σ_Z is one-to-one on S and for every s in S it is $\sigma_Z(X \vee S_s) \subset M_s$. Then, clearly, $(G\sigma_\alpha) \circ \sigma_Z = \sigma_\alpha \circ \sigma_\alpha$ so Z really is an inverse bound.

Let $f : Z \longrightarrow P$ be a factoring morphism, i.e.

(1) $$p_\alpha \circ f = \sigma_\alpha \quad \text{for all } \alpha \text{ in } A,$$

(2) $$\sigma_P \circ f = (Gf) \circ \sigma_Z .$$

We shall show that f is one-one. On X it follows immediately by (1), further procede by transfinite induction. Let $s \in S$ and let $f \circ i_s$ be one-to-one, $i_s : X \vee S_s \to$ $\longrightarrow Z$ being the inclusion. Then also $G(f \circ i_s)$ is one-to-one, therefore Gf is one-to-one on M_s. It remains to show f to be one-to-one on $X \vee S_s \vee \{s\}$. But it would be, otherwise, $f(s) = f(s')$ for some s' in $X \vee S_s$, and, by (2), $(Gf) \circ \sigma_Z(s) = (Gf) \circ \sigma_Z(s')$, in contradiction with $\sigma_Z(s) \neq \sigma_Z(s')$, $\sigma_Z(s), \sigma_Z(s') \in M_s$ and Gf being one-to-one on M_s.

Appendix

A. Although the problem of products in $A(F, G, \Delta)$ is not solved completely in the present paper, we can nevertheless show that the theorems proved here clear up many situations. Let $\mathcal{D}$ denote the least system of functors containing I, N, β, C_M with $M \neq \emptyset$, closed with regard to operations $\vee$, $\times$ (over sets), $\circ$, $\langle -, - \rangle$ (whenever defined) and to natural equivalence. From this recursive definition of $\mathcal{D}$ and with aid of the results of the section 4 we can prove easily:

If F in $\mathcal{D}$ is covariant, then either $F\emptyset \neq \emptyset$ or $F \simeq I \times C_M$ or F is loose;

if G in $\mathcal{D}$ is covariant, then either G preserves products and $G \simeq Q_M$, or G filtrates products and $G \simeq \bigvee_{\iota \in \mathcal{J}} Q_{M_\iota}$, or G has for a subfunctor one of the functors β, $\beta \times I$, N, $N \times I$, $\langle P^-, I \rangle$ and hence superinflates products.

Therefore, from the theorems stated in the paper it follows that:

If F, G are covariant functors belonging to the system $\mathcal{D}$, then $A(F, G, \Delta)$ has products exactly in the following two distinct cases:

1) $G \simeq Q_M$;

2) Δ is unary, $F \simeq I \times C_M$, $G \simeq \bigvee_{\iota \in \mathcal{J}} Q_{M_\iota}$.

B. Beside categories $A(F, G, \Delta)$ treated in the text it is but natural to study also the categories

$P(F, G, \Delta)$ whose objects are all pairs (X, σ) with X - a set and σ - a system of partial operations of the type Δ from the set FX into GX, or, the categories $R(F, G, \Delta)$ with objects (X, σ) - a set with a relational system, i.e. the system of multivalued partial operations of the type Δ from FX into GX (see also [3]).

The authors have chosen for study the categories $A(F, G, \Delta)$ since the behaviour of categories $P(F, G, \Delta)$ and $R(F, G, \Delta)$ with regard to products is essentially simpler. The theorem 3.1 is valid - after some quite formal modifications - for categories $P(F, G, \Delta)$ and $R(F, G, \Delta)$. Therefore, for faithful contravariant F, G and $\Sigma\Delta > 0$ the categories $P(F, G, \Delta)$ and $R(F, G, \Delta)$ have not products. If F, G are covariant, then $R(F, G, \Delta)$ always has products and the forgetful functor preserves them.

In situations treated in the paper, the behaviour of $P(F, G, \Delta)$ differs from that of $A(F, G, \Delta)$ only in the following case: If G filtrates products then $P(F, G, \Delta)$ always has products and the forgetful functor preserves them. All other results and their proofs brought in the text can be with just formal changes transformed to $P(F, G, \Delta)$.

C. It is, of course, possible to regard a system of structures simultaneously. If $\mathfrak{I}$ is a set, then categories

$A(\{F_\iota, G_\iota, \Delta_\iota\} \mid \iota \in \mathcal{J})$, $P(\{F_\iota, G_\iota, \Delta_\iota\} \mid \iota \in \mathcal{J})$, $R(\{F_\iota, G_\iota, \Delta_\iota\} \mid \iota \in \mathcal{J})$ are defined in an obvious way. It is clear that all proofs of non-existence of products are of that kind that, as soon as for some ι_o in $\mathcal{J}$ the category $A(F_{\iota_o}, G_{\iota_o}, \Delta_{\iota_o})$ has not products by some of the stated theorems, then $A(\{F_\iota, G_\iota, \Delta_\iota\} \mid \iota \in \mathcal{J})$ has not products either. Further, we can assert the following: Let for every ι in $\mathcal{J}$ G preserves products, or, for every $\iota \in \mathcal{J}$, Δ_ι be unary, G_ι filtrate products and F_ι be tight with $F_\iota \emptyset = \emptyset$. Then $A(\{F_\iota, G_\iota, \Delta_\iota\} \mid \iota \in \mathcal{J})$ has products.

We do not bring explicitly the results for categories $P(\ldots)$ and $R(\ldots)$.

D. Let $A^*(F, G, \Delta)$ be a full subcategory of the category $A(F, G, \Delta)$ whose objects are exactly the objects of $A(F, G, \Delta)$ with a non-void underlying set. All the results in the text claiming the non-existence of products in $A(F, G, \Delta)$ are without any changes valid in $A^*(F, G, \Delta)$ as well. The positive results on the existence of products are slightly different. Completing in a simple way the proof of the theorem 5.6 we can for example prove: If the type Δ is not unary, then $A^*(F, G, \Delta)$ has products if and only if G preserves products.

If G filtrates or superinflates products, then

$A^*(F, G, \Delta)$ has not products even for a unary type Δ .

The same problems on products as in $A(F, G, \Delta)$ remain open for categories $A^*(F, G, \Delta)$.

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