Commentationes Mathematicae Universitatis Carolinae

Ivan Kolář

On extended connections

Commentationes Mathematicae Universitatis Carolinae, Vol. 9 (1968), No. 4, 687--690

Persistent URL: http://dml.cz/dmlcz/105211

Commentationes Mathematicae Universitatis Carolinae

9, 4 (1968)

ON EXTENDED CONNECTIONS

Ivan KOLÁŘ, Brno

Dr. Bernard deduced the following result, [1], p.225: Let S be a G-structure on a manifold X, defined by a tensor t and let γ be a linear connection on X, then γ is an extension of an S-connection if and only if the absolute differential of t with respect to γ vanishes identically. - The purpose of the present note is to formulate and prove a more general

<u>Theorem</u>. Let P be a principal fibre bundle, let R be a reduction of P determined by a geometric object σ and let C be a connection on P, then C is an extension of a connection on R if and only if the absolute differential of σ with respect to C vanishes identically.

We remark that our methods differ entirely from those of D. Bernard.

1. Our consideration is in the category C^{∞}.

Let $P(B,G)$ be a principal fibre bundle with base B, structure group G and projection π and let φ be a left action of G on a manifold F. By a <u>geometric object on P of type φ</u> will be meant a mapping $\sigma : P \to F$ such that

$$\sigma(ug) = \varphi(g^{-1}) \cdot \sigma(u) \text{ for every } u \in P,\ g \in G .$$

Consider the associated fibre bundle $E(B,F,G,P)$. The elements of E are the equivalence classes $\{(u,s)\}$, $u \in P$, $s \in F$, with respect to the equivalence relation $(u,s) \sim$ $\sim (ug, \varphi(g^{-1}) \cdot s)$, $g \in G$. σ determines a global section ω of E defined by

(1) $$\omega(\pi(u)) = \{(u, \sigma(u))\}$$

and conversely, every global section of E determines a geometric object of type φ on P.

Let $s \in F$ be a given element, let H be its stable group, let A denote the orbit of s in F and let σ be a geometric object on P of type φ whose values are in A. Suppose that A is a proper submanifold of F (see [1], p. 171), then

$$R_s = \{u \in P ; \ \sigma(u) = s\}$$

is a reduction of P to $H \subset G$, which will be called the reduction of P determined by the couple (σ, s) or a reduction of P determined by σ.

2. Let Φ be a groupoid over B with projections a, b, and let 1_x be the unit of Φ over $x \in B$. Let $Q^1(\Phi)$ denote the fibre bundle of all elements of connection of the first order on Φ, see [2]. Every $X \in Q^1_x(\Phi)$ is a 1-jet of B into Φ such that $\alpha X = x$, $\beta X = 1_x$, $aX = j^1_x \hat{x}$, $bX = j^1_x$ $(= j^1_x \ \mathrm{id}_B)$, where $\hat{x}$ denotes the constant mapping $\hat{x}(t) = x$, $t \in B$. In particular, let $\Phi = PP^{-1}$ be the groupoid associated to P, let R be a reduction of P and let $\Psi = RR^{-1}$ be the groupoid associated to R, so that Ψ is a subgroupoid of Φ. A first order connection C on P (i.e. a global section of $Q^1(\Phi)$) is called an extension of a connection on R, if $C(B) \subset Q^1(\Psi)$.

The general definition of the absolute differential with respect to an element of connection was given by Ehresmann, [2]. We shall use the following particular case of this concept. Let $X \in Q^1_x(\Phi)$, then X can be expressed in the form $X = j^1_x \rho$, where $\rho(t)$ is a local mapping of B into Φ such that

(2) $$\rho(x) = 1_x, \quad a\rho(t) = x, \quad b\rho(t) = t.$$

Let E be a fibre bundle associated to P and let $\mathfrak{S}$ be a local section in E. Then the __absolute differential__ $X^{-1}(\mathfrak{S})$ __of__ $\mathfrak{S}$ with respect to X is defined by

$$X^{-1}(\mathfrak{S}) = j^1_x(\rho^{-1}(t) \cdot \mathfrak{S}(t)) \in J^1_x(B, F_x),$$

where F is the fibre over x in E. We say that $X^{-1}(\mathfrak{S})$ __vanishes__, if $X^{-1}(\mathfrak{S}) = j^1_x \widehat{\mathfrak{S}(x)}$, where $\widehat{\mathfrak{S}(x)}$ is the constant mapping $\widehat{\mathfrak{S}(x)}(t) = \mathfrak{S}(x)$, $t \in B$. In particular, if σ is a geometric object on P and ω is the corresponding section (1) in $E(B,F,G,P)$, then $X^{-1}(\omega)$ will be called the __absolute differential of__ σ __with respect to__ X.

3. Now, we prove the theorem.

Let $P, E, s, A, \sigma, \omega$ and R_s be as in item 1, then the subgroupoid $\Psi = R_s R_s^{-1}$ of $\Phi = PP^{-1}$ does not depend on the choice of $s \in A$ and is characterized by

(3) $$\Psi = \{\Theta \in \Phi \,;\, \Theta \cdot \omega(a\Theta) = \omega(b\Theta)\}.$$

I. Let $C: B \longrightarrow Q^1(\Psi)$ be a connection on Ψ, then $C(x) = j^1_x \rho(t)$, where $\rho(t)$ is a local mapping of B into Ψ satisfying (2). Then $C^{-1}(x)(\omega) = j^1_x(\rho^{-1}(t) \cdot \omega(t)) = j^1_x \widehat{\omega(x)}$ according to (3), so that $C^{-1}(x)(\omega)$ vanishes for every $x \in B$.

II. Let $X \in Q^1_x(\Phi)$ and let $X^{-1}(\omega)$ vanish, i.e. X can be represented in the form $X = j^1_x \varphi$, where $\varphi^{-1}(t) \cdot \omega(t) = \widehat{\omega(x)}(t) = \omega(x)$. Then (3) shows that the values of φ are in Ψ, which gives $X \in Q^1_x(\Psi)$, QED.

References

[1] D. BERNARD: Sur la géométrie différentielle des G-structures, Ann. Inst. Fourier, Grenoble, 10(1960), 151-270.

[2] C. EHRESMANN: Sur les connexions d'ordre supérieur, Atti del V° Congresso del l'Unione Matematica Italiana, Roma 1956, 344-346.

(Received December 2,1968)