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CONCERNING MINIMAL PRIMITIVE CLASSES OF ALGEBRAS CONTAINING ANY CATEGORY OF ALGEBRAS AS A FULL SUBCATEGORY

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Some primitive classes of algebras contain any category of algebras as a full subcategory. Such primitive classes are, e.g. the primitive class of semigroups [1], the primitive class of commutative groupoids [5], the primitive class of all algebras with two unary idempotent operations [4].

The natural question arises - are there minimal (with respect to inclusion) primitive classes of algebras of given type with the mentioned property?

For the precise formulation of the answer we need some notation.

By concrete category $(\mathcal{K}, \square)$ we mean concrete category $\mathcal{K}$ together with a fixed forgetful functor $\square$. Categories of algebras are treated as concrete categories with the usual underlying-set forgetful functor. Full embedding $\Phi : (\mathcal{K}, \square_1) \longrightarrow (\mathcal{L}, \square_2)$ is a one-to-one functor onto a full subcategory of $\mathcal{L}$. A category $(\mathcal{K}, \square)$ into which any category of algebras can be fully embedded is called binding (see [2]). A full embedding Φ is called strong if there exists a set-functor F with $\square_2 \circ \Phi = F \circ \square_1$. A category $(\mathcal{K}, \square)$ is strongly binding, if any category of algebras can be strongly embedded into it. The class $\mathfrak{A}(1,1)$ of all algebras with two unary operations is strongly binding [3], as well as the class of semigroups [6] [7], and commutative groupoids [5].

A sequence $\Delta = \{ \varkappa_\alpha \mid \alpha < \beta \}$ of ordinals, indexed by ordinals will be called a type. $\mathfrak{A}(\Delta)$ designates the class of all algebras of the type Δ. Denote $\Delta' = \{ \varkappa_\alpha \in \Delta \mid \varkappa_\alpha > 0 \}$. $\sum \Delta = \sum_{\alpha<\beta} \varkappa_\alpha$ means the usual sum of ordinal numbers.

The main result of the present note is the following assertion:

Every $\mathfrak{A}(\Delta)$ for $\sum\Delta \geqq 2$ and $card(\Delta') \geqq 2$ contains a minimal binding primitive class $\mathcal{K}_\Delta$. $\mathcal{K}_\Delta$ is strongly binding.

Note that $\mathfrak{A}(\Delta)$ is strongly binding if and only if $\sum \Delta \geqq 2$ ([2] and [3]), so that the requirement $card(\Delta') \geqq 2$ is the only essential assumption of the above assertion. Thus, we have

<u>Problem 1</u>. Are there minimal primitive classes $\mathcal{K} \subseteq$ $\subseteq \mathfrak{A}(\Delta)$ in the case of $card(\Delta') = 1$?

This problem is not yet solved.

<u>Notation</u>. Let $\mathcal{L} \subseteq \mathfrak{A}(1,1)$ be a primitive class of all the algebras $(X; \varphi, \psi)$ (X is a set, $\varphi : X \longrightarrow X$ and $\psi : X \longrightarrow X$ are unary operations) such that

$$\varphi^2(x) = \varphi^2(y) = (\varphi \circ \psi)(z)$$
$$\psi^2(x) = \psi^2(y) = (\psi \circ \varphi)(z)$$

for every x, y, z in X.

Theorem 1. The primitive class $\mathcal{L}$ is strongly binding.

Proof. A strong embedding $\Phi : \mathfrak{A}(1,1) \longrightarrow \mathcal{L}$ will be constructed as follows:

For $A = (X ; \alpha, \beta)$ - an object in $\mathfrak{A}(1,1)$ - put $\Phi(A) = (Z ; \varphi, \psi)$, where

$$Z = (X \times 7) \cup \{a_X, b_X\},$$

$$(X \times 7) \cap \{a_X, b_X\} = \emptyset .$$

The operations $\varphi : Z \longrightarrow Z$, $\psi : Z \longrightarrow Z$ are defined as follows:

$$\varphi(\langle x, 0\rangle) = \varphi(\langle x, 5\rangle) = \langle x, 3\rangle ,$$

$$\varphi(\langle x, 1\rangle) = \langle \alpha(x), 3\rangle ,$$

$$\varphi(\langle x, 2\rangle) = \langle \beta(x), 4\rangle ,$$

$$\varphi(\langle x, 3\rangle) = \varphi(\langle x, 4\rangle) = \varphi(a_X) = \varphi(b_X) = a_X ,$$

$$\varphi(\langle x, 6\rangle) = b_X ,$$

$$\psi(\langle x, 0\rangle) = \psi(\langle x, 2\rangle) = \psi(\langle x, 6\rangle) = \langle x, 4\rangle ,$$

$$\psi(\langle x, 1\rangle) = \langle x, 3\rangle ,$$

$$\psi(\langle x, 3\rangle) = \psi(\langle x, 4\rangle) = \psi(a_X) = \psi(b_X) = b_X ,$$

$$\psi(\langle x, 5\rangle) = a_X .$$

A bit of computation shows that $\Phi(A) \in \mathcal{L}$.
For $A' = (X'; \alpha', \beta')$ let us denote $\Phi(A') = (Z'; \varphi', \psi')$, $Z' = (X' \times Y) \cup \{a_{X'}, b_{X'}\}$ and for $f: X \to X'$ put

$$\Phi(f)(\langle x, i\rangle) = \langle f(x), i\rangle \text{ if } x \in X, \quad i = 0, \dots, 6,$$

$$\Phi(f)(a_X) = a_{X'},$$

$$\Phi(f)(b_X) = b_{X'}.$$

It is easy to see that for any homomorphism $f: A \to A'$ the mapping $\Phi(f)$ is a homomorphism in $\mathcal{L}$. Hence, $\Phi: \mathfrak{A}(1,1) \longrightarrow \mathcal{L}$ is a functor, it is, clearly, one-to-one.

It remains to prove that Φ is onto a full subcategory of $\mathcal{L}$.

Take $F: \Phi(A) \longrightarrow \Phi(A')$ - a homomorphism in $\mathcal{L}$.

As a_X ($a_{X'}$ resp.) is the only fixed point of φ (of φ' resp.), then $F(a_X) = a_{X'}$. Similarly, using ψ and ψ', we find that $F(b_X) = b_{X'}$. Note that $\varphi^{-1}(\{b_X\}) = X \times \{6\}$ and $\psi^{-1}(\{a_X\}) = X \times \{5\}$ analogously for $\Phi(A')$. It follows $F(X \times \{5\}) \subseteq X' \times \{5\}$, $F(X \times \{6\}) \subseteq X' \times \{6\}$. Let us denote $\langle f(x), 5\rangle = F(\langle x, 5\rangle)$, $\langle g(x,6)\rangle = F(\langle x, 6\rangle)$. Now, we have $F(\langle x, 4\rangle) = F(\psi(\langle x, 6\rangle)) = \psi'(F(\langle x, 6\rangle)) = \psi'(\langle g(x), 6\rangle) = \langle g(x), 4\rangle$, $F(\langle x, 3\rangle) =$

$= F(\varphi(\langle x, 5\rangle)) = \varphi'(\langle f(x), 5\rangle) = \langle f(x, 3\rangle$. It follows

$\langle f(x),3\rangle = F(\langle x,3\rangle) = F(\psi(\langle x,1\rangle)) = \psi'(F(\langle x,1\rangle))$.

But $(\psi')^{-1}(\{\langle f(x),3\rangle\}) = \{\langle f(x),1\rangle\}$, hence $F(\langle x,1\rangle) = \langle f(x),1\rangle$.

Further, $\varphi'(F(\langle x,0\rangle) = F(\varphi(\langle x,0\rangle)) = F(\langle x,3\rangle) = \langle f(x),3\rangle$ and $\psi'(F(\langle x,0\rangle)) = F(\psi(\langle x,0\rangle)) = \langle g(x),4\rangle$. On the other hand, $(\varphi')^{-1}(\{\langle x_1,3\rangle\} \cap (\psi')^{-1}(\{\langle x_2,4\rangle\} = \emptyset$ for $x_1 \neq x_2$ and $= \{\langle x_1,0\rangle\}$ for $x_1 = x_2$. Thus $F(\langle x,0\rangle) = \langle g(x),0\rangle = \langle f(x),0\rangle$, i.e. $f = g$, $F(\langle x,0\rangle) = \langle f(x),0\rangle$.

Finally, $\psi'(F(\langle x,2\rangle)) = F(\langle x,4\rangle) = \langle f(x),4\rangle$ and $(\psi')^{-1}(\{\langle f(x),4\rangle\}) = \{\langle f(x),0\rangle, \langle f(x),2\rangle, \langle f(x),6\rangle\}$. If $F(\langle x,2\rangle) = \langle f(x),0\rangle$, then $F(\langle\beta(x),4\rangle) = F(\varphi(\langle x,2\rangle)) = \varphi'(F(\langle x,0\rangle) = \langle f(x),3\rangle$ - a contradiction. If $F(\langle x,2\rangle) = \langle f(x),6\rangle$, then $F(\langle\beta(x),4\rangle) = \varphi'(\langle f(x),6\rangle) = b_{X'}$ - a contradiction again.

Thus, $F = \Phi(f)$, $f: X \longrightarrow X'$. Moreover, f is a homomorphism, $f: A \longrightarrow A'$, as $\langle f(\alpha(x)),3\rangle = F(\langle\alpha(x),3\rangle) = F(\varphi(\langle x,1\rangle)) = \varphi'(F(\langle x,1\rangle)) = \varphi'(\langle f(x),1\rangle) = \langle\alpha'(f(x)),3\rangle$ and $\langle f(\beta(x)),4\rangle = F(\varphi(\langle x,2\rangle)) = \varphi'(\langle f(x),2\rangle) = \langle\beta'(f(x)),4\rangle$.

This concludes the proof.

<u>Remark</u>. A binding concrete category $(\mathcal{K},\Box)$ has the following property:

$$(\mathrm{P})\begin{cases}\text{For any cardinal number } \alpha \text{ there is an object } A \in \\ \in (\mathcal{K},\Box) \text{ such that } Hom(A,A) = 1_A \text{ and} \\ card(\Box(A)) \geqq \alpha\end{cases}$$

(cf.[2]).

Theorem 2. No proper primitive subclass of $\mathcal{L}$ is binding.

Proof. Note that for any algebra $(X; \varphi, \psi)$ in $\mathcal{L}$ both φ^2 and ψ^2 are constant mappings. Denote $\varphi^2(X) = \{a\}$, $\psi^2(X) = \{b\}$. In particular, $\varphi(a) = a = \varphi(b)$, $\psi(a) = b = \psi(b)$. Let $w = w(\varphi, \psi) = \varphi^{k_1} \circ \psi^{l_1} \circ \dots \varphi^{k_n} \circ \psi^{l_n}$, $v = v(\varphi, \psi)$ be words in φ and ψ. Let us denote $l(w) = \sum k_i + \sum l_i$.

(i) Let $\mathcal{L}_1 \subseteq \mathcal{L}$ be a primitive class in which an equation $(\varphi \circ w)(x) = (\psi \circ w)(x)$ holds. Then, in particular, $(\varphi \circ w)(a) = (\psi \circ v)(a)$. It follows a = b. Every algebra $A \in \mathcal{L}_1$ has the constant mapping $const_a$ as an endomorphism. $\mathcal{L}_1$ is not binding (see (P)).

(ii) $(\varphi \circ w)(x) = (\varphi \circ v)(x)$ holds in $\mathcal{L}_1 \subseteq \subseteq \mathcal{L}$. If $l(w) > 0 < l(v)$, then the above equation follows from the equations of the class $\mathcal{L}$. If $l(w) = 0$, then $a = \varphi^2(x) = \varphi(x)$ and φ is a constant mapping. $\mathcal{L}_1$ is, in fact, some primitive class in $\mathfrak{A}(0, 1)$, hence not binding.

(iii) $(\psi \circ w)(x) = (\psi \circ v)(x)$ - the same as in (ii).

(iv) From $(\varphi \circ w)(x) = x$ it follows $a = \varphi(x)$, x = a. We have a trivial primitive class.

(v) $(\psi \circ w)(x) = x$ is quite analogous to (iv).

Proposition. In general, strong embedding does not preserve primitive classes.

Proof. Compose the trivial strong embedding $\mathcal{L} \longrightarrow$ $\longrightarrow \mathcal{O}\!\mathcal{C}(1,1)$ with strong embedding $\Phi : \mathcal{O}\!\mathcal{C}(1,1) \longrightarrow \mathcal{L}$.

Theorem 3. Let $\Delta = \{ \varkappa_\alpha \mid \alpha < \beta \}$ be a type such that $\sum \Delta \geq 2$, $card(\Delta') \geq 2$. Then $\mathcal{O}\!\mathcal{C}(\Delta)$ contains a minimal binding primitive class $\mathcal{K}$. The class $\mathcal{K}$ is strongly binding.

Proof. Algebras in $\mathcal{O}\!\mathcal{C}(\Delta)$ have, due to the assumption, at least two at least unary operations. Suppose $\varkappa_0 > 0$, $\varkappa_1 > 0$; operations will be denoted by ω_α $(\alpha < \beta)$.

Let us define a primitive class $\mathcal{K} \subseteq \mathcal{O}\!\mathcal{C}(\Delta)$ by following equations:

$$\omega_i(x, y, \dots) = \omega_i(x, z, \dots) \quad ,$$

$$\omega_i(\omega_i(x, \dots), \dots) = \omega_i(\omega_j(y, \dots), \dots)$$

for $i, j = 0,1$. Hence, any algebra $(X; \{\omega_\alpha \mid \alpha < \beta\})$ with $X \neq \emptyset$ has elements a,b such that

$$\omega_0(\omega_0(x, \dots), \dots) = a \quad ,$$

$$\omega_1(\omega_1(x, \dots), \dots) = b \quad .$$

Let us finish the list of equations:

$$\omega_\alpha(x, \dots) = \omega_0(x, \dots) \quad \text{for } \alpha > 1, \varkappa_\alpha > 0 ,$$

$$\omega_\alpha = a \quad \text{for } \varkappa_\alpha = 0 .$$

It is easy to see that $\mathcal{K}$ is an underlying-set-preserving copy of $\mathcal{L}$ in $\mathcal{O}\!\mathcal{C}(\Delta)$ and that primitive subclasses of $\mathcal{L}$ and $\mathcal{K}$ are in one-to-one correspondence.

We conclude that $\mathcal{K}$ is minimal (strongly) binding primitive class in $\mathcal{O}\!\mathcal{C}(\Delta)$.

Remark. There does not exist a smallest binding primitive class in $\mathfrak{A}(2)$ as well as in $\mathfrak{A}(1,1)$. It follows for $\mathfrak{A}(2)$ from the fact that semigroups and commutative groupoids form binding classes, but not their intersection. For $\mathfrak{A}(1,1)$ it follows from [4].

Using those facts and argument similar to this used in the proof of Theorem 3, we obtain that in no $\mathfrak{A}(\Delta)$ with $\Delta = \Delta'$ exists a smallest binding primitive class.

Problem 2. Does any binding primitive class $\mathcal{K} \subseteq \mathfrak{A}(\Delta)$ contain a minimal binding one?

Problem 3. Describe all minimal binding primitive classes in $\mathfrak{A}(1,1)$.

Problem 4. Is the class of semigroups a minimal binding primitive class ?

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