Commentationes Mathematicae Universitatis Carolinae

Lev Bukovský

∇-model and distributivity in Boolean algebras

Commentationes Mathematicae Universitatis Carolinae, Vol. 9 (1968), No. 4, 595--612

Persistent URL: http://dml.cz/dmlcz/105203

Commentationes Mathematicae Universitatis Carolinae
9,4 (1968)

∇ - MODEL AND DISTRIBUTIVITY IN BOOLEAN ALGEBRAS

Lev BUKOVSKÝ, Košice

§ 0. Preliminaries. The main purpose of this paper is to study the addition of "new" sets in a ∇-model constructed over a complete Boolean algebra and to investigate connections between it and the distributive laws in the Boolean algebra. A main part of this paper was presented on the 3rd Congress for Logic in Amsterdam 1967 (see [2]). Independently, Prof.D. Scott has presented similar results at the Summer Institute on Axiomatic Set Theory in Los Angeles 1967.

The reader is supposed to be familiar with the paper [12]. We remind some definitions, facts and introduce some notations.

A couple $\langle c, t \rangle$ is called a topological space iff a) $t \subseteq \mathcal{P}(c)$ ($\mathcal{P}(x)$ is the set of all subsets of x), b) $\emptyset, c \in t$, c) t is closed under finite intersections and arbitrary unions. t is called a topology on c. Other topological terminology is used in an obvious way (see e.g.[5]).

If B is a Boolean algebra (see [10]), then $0, 1$ denote the zero and unit element of B respectively. The symbols $-$, $\vee$, $\bigvee$, $\wedge$, $\bigwedge$ are used for the Boolean complement, join and meet. $\mathcal{S}(B)$ denotes the set of all ultrafilters on B and s is the isomorphism of B onto $\mathcal{F}(B)$, a field of subsets of $\mathcal{S}(B)$. t(B) is the weakest topology on $\mathcal{S}(B)$ containing $\mathcal{F}(B)$. The topological space $\langle \mathcal{S}(B), t(B) \rangle$ is called the Stone space of the Boolean algebra B. A Boolean algebra B is called complete iff there exists the join of any subsets of B.

For a topological space $\langle c,t \rangle$, let $\mathcal{R}(c,t)$ denote the set of all regular open subsets of c, i.e. the set of all $u \in t$ for which $u = \operatorname{Int} \overline{u}$. It is well known that the set $\mathcal{R}(c,t)$ ordered by inclusion is a complete Boolean algebra. Moreover, for a system $u_\xi \in \mathcal{R}(c,t)$, $\xi \in T$ we have:

$$\bigwedge_{\xi\in T} u_\xi = \operatorname{Int} \bigcap_{\xi\in T} u_\xi , \ \bigvee_{\xi\in T} u_\xi = \operatorname{Int} \overline{\bigcup_{\xi\in T} u_\xi} , \ -u = \operatorname{Int}(c-u),$$

$$0 = \emptyset , \quad 1 = c .$$

(see [10],p.66). For other consultations the theory of Boolean algebras, see [10].

All our considerations concern the Gödel-Bernays set theory Σ^* with the axioms of groups A - E (see [3]). From the text, it will be clear when our considerations are mathematical (i.e. we construct a proof in Σ^*) and when they are metamathematical (i.e. we investigate properties of the theory Σ^*).

The set theoretical notations are used in an obvious way. An ordinal is the set of all less ordinals, a cardinal is an initial ordinal, $\operatorname{card}(x) = \overline{\overline{x}}$ is a cardinal -

the power of the set x, ${}^{x}y$ is the set of all functions defined on x with values in y. Small greek letter always denotes an ordinal, the letters m,n,k (with indices eventually) denote cardinals. $m^{\underline{n}} = \mathrm{card}(\bigcup_{k<n} {}^{k}m) = \sum_{k<n} m^k$, $\mathrm{cf}(m)$ is the least cardinal n such that m is confinal with n.

If φ is a normal formula (see [3]), then

$$z \in \{x : \varphi(x, X_1, \dots, X_n)\} \equiv \varphi(z, X_1, \dots, X_n) .$$

The existence of the class $\{x : \varphi(x, X_1, \dots, X_n)\}$ is proved in [3].

The notion of a constructive set is defined in [3]. The axiom of constructivity $V = L$ is sometimes used (V is the class of all sets, L is the class of all constructive sets). GCH stands for the formula $(\forall\alpha)(2^{\aleph_\alpha} = \aleph_{\alpha+1})$, i.e. the Generalized Continuum Hypothesis.

Let $f \in {}^{x}y$, $g \in {}^{x}\mathcal{P}(y)$. We define $f \in\in g \equiv (\forall z)(z \in x \rightarrow f(z) \in g(z))$. Let $\langle x_i, r_i \rangle$, $i = 1,2$ be partial ordered sets, i.e. x_i is a set and r_i is a binary relation on x_i; which is a partial ordering. We denote

$$Map(x_1, r_1, x_2, r_2) = \{f : f \in {}^{x_1}x_2 \;\&\; \langle u v\rangle \in r_1 \rightarrow \langle f(u) f(v)\rangle \in r_2\},$$

i.e. $Map(x_1, r_1, x_2, r_2)$ (shortly $Map(x_1, x_2)$) is the set of all non decreasing functions from x_1 into x_2.

The notations and results of the paper [12] will be used without references. One notion will be denoted in different way only, namely, we shall write $\check{x}$ instead of k_x, i.e. $\check{x}$ is defined as follows: $\check{x} \in C(B)$ and $D(\hat{x}) = \{\check{y} : y \in x\}$ and $\check{x}(\check{y}) = 1$ for $y \in x$. Let us remember, that φ^* denotes the translation of the formula φ in the model $\nabla(B,z)$ (see [12],p.157).

1. Distributive laws in a complete Boolean algebra

In this paragraph, B denotes always a complete Boolean algebra.

Definition 1.1. Let $\langle x_i, r_i \rangle$ be partially ordered set, $i = 1,2$, $\mathcal{F} \subseteq {}^{x_1}\mathcal{P}(x_2)$. Let z be a filter on B. The algebra B is called $(z, \mathcal{F})$-distributive (or more precisely $(z, \mathcal{F}, x_1, r_1, x_2, r_2)$-distributive iff for every system $a_{ij} \in B$, $i \in x_1$, $j \in x_2$ such that

(1.1) $\bigwedge_{i \in x_1} \bigvee_{j \in x_2} a_{ij} \in z$,

(1.2) if $\langle i_1, i_2 \rangle \in r_1$, $a_{i_1 j_1} \wedge a_{i_2 j_2} \neq 0$, then $\langle j_1, j_2 \rangle \in r_2$, the following condition holds true

(1.3) $\bigvee_{f \in \mathcal{F}} \bigwedge_{i \in x_1} \bigvee_{j \in f(i)} a_{ij} \in z$.

The $(\{1\}, \mathcal{F})$-distributivity is called simply $\mathcal{F}$-distributivity.

Remarks: a) By (1.2) we have $a_{ij_1} \wedge a_{ij_2} = 0$ for $j_1 \neq j_2$.

b) Always $\bigwedge_{i \in x_1} \bigvee_{j \in x_2} a_{ij} \geq \bigvee_{f \in \mathcal{F}} \bigwedge_{i \in x_1} \bigvee_{j \in f(i)} a_{ij}$.

c) If $\mathcal{F} \subseteq \mathcal{F}'$, B is $(z, \mathcal{F})$-distributive, then B is also $(z, \mathcal{F}')$-distributive.

Let us remember that B is called homogenous iff B is isomorphic to the Boolean algebra $\{x : x \in B \ \& \ \& \ x \leq a\}$ for every $a \in B$, $a \neq 0$.

Theorem 1.2. Let B be a homogenous complete Boolean algebra. The following conditions are equivalent:

(i) B is $(\{1\}, \mathcal{F})$-distributive,

(ii) B is $(z, \mathcal{F})$-distributive for every filter z on B,

(iii) there is no system $a_{ij} \in B$ satisfying (1.2) such that

a) $\bigwedge_{i \in X_1} \bigvee_{j \in X_2} a_{ij} = 1$,

b) $\bigvee_{f \in \mathcal{F}} \bigwedge_{i \in X_1} \bigvee_{j \in f(i)} a_{ij} = 0$.

Proof. Evidently (ii) $\longrightarrow$ (i) $\longrightarrow$ (iii). Let us suppose that (ii) does not hold, i.e. there is a filter z and a system $a_{ij} \in B$ satisfying (1.2) such that

$$u = \bigwedge_{i \in X_1} \bigvee_{j \in X_2} a_{ij} \in z ,$$

$$v = \bigvee_{f \in \mathcal{F}} \bigwedge_{i \in X_1} \bigvee_{j \in f(i)} a_{ij} \notin z .$$

Let $b = u - v \ (\neq 0)$. We define $b_{ij} = b \wedge a_{ij}$. This system possesses properties (1.2) and (iii)a), but

$$\bigvee_{f \in \mathcal{F}} \bigwedge_{i \in X_1} \bigvee_{j \in f(i)} b_{ij} = 0 .$$

However, $B \mid b = \{x : x \in B \ \& \ x \leq b\}$ is isomorphic to B, thus B does not fulfil the condition (iii).

Q.E.D.

Now, we shall consider some special cases of distributive laws.

(m,n,k)-distributivity. Let r_h be the trivial ordering of the cardinal h, i.e. $\langle \xi\eta \rangle \in r_h \equiv . \xi, \eta \in h \,\&\, \xi = \eta$. Let $\mathcal{F} = \{f : f \in {}^m\mathcal{P}(n) \,\&\, (\forall \xi)(\xi \in m \rightarrow \overline{\overline{f(\xi)}} < k)\}$.

In this case, the $\mathcal{F}$-distributivity is called (m,n,k)-distributivity (m,n are ordered by r_m, r_n respectively).

(m,n,2)-distributivity is the obvious (m,n)-distributivity, (m,n,ω_o)-distributivity is the weak (m,n)-disstributivity in the sense of [10].

$(m \neq n)$-distributivity is the $\mathcal{F}$-distributivity, where m, n are oredered by r_m, r_n respectively and

$\mathcal{F} = \{f \in {}^m\mathcal{P}(n) : (\exists \eta)(\forall \xi)(\xi \in m \rightarrow f(\xi) \in n - \{\eta\})\}$.

Lebesgue distributivity. Let $\mathcal{X}$ be the set of all closed segments $[a,b]$,where $0 \leq a < b \leq 1$ are rational numbers. Let $\mathcal{X}_o = \{f : f \in \bigcup_{n \in \omega_o} {}^n\mathcal{X}$ & Lebesgue measure of $\bigcup W(f)$ be less than one half$\}$. $\mathcal{X}_o$ is ordered by inclusion, ω_o is ordered by " $\in$ ". Let $\mathcal{F} = \{\Phi_x : x \in [0,1]\}$, where

$\Phi_x(n) = \{f : D(f) = n \,\&\, f \in \mathcal{X}_o \,\&\, x \notin \bigcup w(f)\}$ $(\mathcal{F} \subseteq {}^{\omega_o}\mathcal{P}(\mathcal{X}_o))$.

In this case, $\mathcal{F}$-distributivity is called Lebesgue distributivity.

For the next, the following notation will be useful. B is called $(< m)$-distributive iff B is (n,k)-distributive for every $n < m$ and every k .

The definition of z - Lebesgue distributivity, (z, m,n,k)-distributivity ect. is clear.

§ 2. Some criteria for distributivity. In this paragraph we prove some theorems which allow us to show some distributive laws in a Boolean algebra.

In the paper [11], P.Vopěnka has proved that

$\mu(B) = \min\{\aleph_\alpha :$ B does not have a subset of pairwise disjoint elements of power $\aleph_\alpha \}$

is a regular cardinal. Vopěnka's proof was given for topological spaces, but the fact mentioned above follows from it directly. A direct proof may be given using some results of R.S. Pierce (see [7]). μ is a cardinal property, thus B may be decomposed in μ - homogenous factors. For μ - homogenous complete Boolean algebra, this proof is trivial.

The characteristic μ may be used for proving some distributive laws (almost the same results are proved in [12],p.161).

Theorem 2.1. If $n \geq \mu(B)$, then B is $(m \neq n)$-distributive.

Proof. Let $n \geq \mu(B)$. By remark a), $\xi_1 \neq \xi_2 \in n$ implies $a_{\eta\xi_1} \wedge a_{\eta\xi_2} = 0$. Therefore, $A_\eta = \{\xi : a_{\eta\xi} \neq 0\}$ has cardinality less than μ(B). We define $f(\eta) = A_\eta$. Evidently $f \in \mathcal{F}$ (since μ(B) is regular) and

$$\bigwedge_{\eta \in m} \bigvee_{\xi \in n} a_{\eta\xi} = \bigwedge_{\eta \in m} \bigvee_{\xi \in f(\eta)} a_{\eta\xi} \ . \qquad \text{Q.E.D.}$$

In a similar way, one can prove

Theorem 2.2. Every complete Boolean algebra B is (m,n, μ(B))-distributive for any m,n .

Definition 2.3. Let $\langle c,t \rangle$ be a topological space. We define:

$ND_\alpha(c,t) = \{x : x \subseteq c$ and $x \subseteq \bigcup_{\xi \in \omega_\alpha} x_\xi$, x_ξ closed and

$\mathrm{Int}\, x_\xi = \emptyset\}$,

$\mathcal{B}_\alpha(c,t) = \{(x \cup y) - z : x \in t \;\&\; y, z \in ND_\alpha(c,t)\}$,

$\nu(c,t) = \min\{\aleph_\alpha : ND_\alpha(c,t) \cap t \neq \{\emptyset\}\}$,

$ND(c,t) = \bigcup_{\omega_\alpha < \nu(c,t)} ND_\alpha(c,t)$,

$\mathcal{B}(c,t) = \bigcup_{\omega_\alpha < \nu(c,t)} \mathcal{B}_\alpha(c,t)$.

A concept $\square$ is defined for a Boolean algebra B as $\square(B) = \square(\mathcal{S}(B), t(B))$, (compare [11]) .

Lemma 2.4. Let $\omega_\beta < cf(\nu(c,t))$. Then $\mathcal{B}(c,t)$ is an ω_β-field of sets, ND(c,t) is an ω_β-ideal. Moreover, $\mathcal{R}(c,t)$ is isomorphic to the factor algebras $\mathcal{B}_\beta(c,t)/ND_\beta(c,t)$ and $\mathcal{B}(c,t)/ND(c,t)$.

Proof. The first part is trivial. To prove the second part, it suffices to prove that $\mathcal{R}(c,t)$ is isomorphic to $\mathcal{B}_\alpha(c,t)/ND_\alpha(c,t)$ for any $\omega_\alpha < \nu(c,t)$.

Let π be the natural homomorphism of $\mathcal{B}_\alpha(c,t)$ onto $\mathcal{B}_\alpha(c,t)/ND_\alpha(c,t)$. For every $x \in \mathcal{R}(c,t)$ we define $h(x) = \pi(x)$. By simple computation we have: $h(\emptyset) = 0$, $h(c) = 1$, $h(\mathrm{Int}(c-u)) = 1 - h(u)$, $h(\mathrm{Int}\,\overline{\bigcup x_\xi}) = \bigvee h(x_\xi)$. If $h(x) = 0$, then $x = \emptyset$ since $ND_\alpha(c,t) \cap t = \{\emptyset\}$. For every $u \in \mathcal{B}_\alpha(c,t)$ there is a $v \in \mathcal{R}(c,t)$ such that $v = \mathrm{Int}\,\overline{x}$, where $u = (x - y) \cup z$,

$y, z \in ND_{\alpha}(c,t)$, thus $h(v) = \pi(u)$.

Q.E.D.

Theorem 2.5. If $\omega_{\beta}^{\omega_{\alpha}} < \nu(c,t)$, then $\mathcal{R}(c,t)$ is $(\omega_{\alpha}, \omega_{\beta})$-distributive (compare [6], [8]).

Proof. Let $\omega_{\gamma} = \omega_{\beta}^{\omega_{\alpha}}$. By the lemma 2.4, $\mathcal{B}_{\gamma}(c,t)$ is $(\omega_{\alpha}, \omega_{\beta})$-distributive (as ω_{γ}-field) and $ND_{\gamma}(c,t)$ is an ω_{γ}-additive ideal. Therefore, the factoralgebra $\mathcal{B}_{\gamma}(c,t)/ND_{\gamma}(c,t)$ is also $(\omega_{\alpha}, \omega_{\beta})$-distributive.

Q.E.D.

Theorem 2.6. If $\langle c,t \rangle$ is an ω_{γ}-additive topological space (i.e. the intersection of less than ω_{γ} open set is an open set, see [9],p.125), then $\mathcal{R}(c,t)$ is $(< m)$-distributive, $m = \min\{\omega_{\gamma}, \nu(c,t)\}$.

Proof. Let $a_{\xi\eta} \in \mathcal{R}(c,t)$, $\bigwedge_{\xi\in\omega_{\alpha}} \bigvee_{\eta\in\omega_{\beta}} a_{\xi\eta} = 1$, i.e.

$$\operatorname{Int} \bigcap_{\xi\in\omega_{\alpha}} \operatorname{Int} \overline{\bigcup_{\eta\in\omega_{\beta}} a_{\xi\eta}} = c .$$

Evidently $\overline{\bigcup_{\eta\in\omega_{\beta}} a_{\xi\eta}} = c$ for any $\xi \in \omega_{\alpha}$. Let $\omega_{\alpha} < m$. Since $m \le \nu(c,t)$, we have $\overline{\bigcap_{\xi\in\omega_{\alpha}} \bigcup_{\eta\in\omega_{\beta}} a_{\xi\eta}} = c$.

Thus

$$\bigvee_{\varphi\in\omega_{\alpha}^{\omega_{\beta}}} \bigwedge_{\xi\in\omega_{\alpha}} a_{\xi\varphi(\xi)} = \operatorname{Int} \overline{\bigcup_{\varphi\in\omega_{\alpha}^{\omega_{\beta}}} \operatorname{Int} \bigcap_{\xi\in\omega_{\alpha}} a_{\xi\varphi(\xi)}} =$$

$$= \operatorname{Int} \overline{\bigcup_{\varphi\in\omega_{\alpha}^{\omega_{\beta}}} \bigcap_{\xi\in\omega_{\alpha}} a_{\xi\varphi(\xi)}} = \operatorname{Int} \overline{\bigcap_{\xi\in\omega_{\alpha}} \bigcup_{\eta\in\omega_{\beta}} a_{\xi\eta}} = c .$$

Q.E.D.

On the other hand, one can easily prove

Theorem 2.7. Let B be a complete Boolean algebra.

If B is $(<m)$-distributive, then $\nu(\mathcal{S}(B), t(B)) \geq m$.

The essential part of the theorems 2.5 - 2.7 is known (see [4],[6],[8],[11],[10]). Now, we prove two criteria for the Lebesgue distributivity.

Theorem 2.8. If B contains a regular subalgebra isomorphic to $\mathcal{R}(c_0, t_0)$, where $\langle c_0, t_0 \rangle$ is the Cantor set with the obvious topology, the B is not Lebesgue distributive.

Proof. Evidently, it suffices to prove that $\mathcal{R}(c_0, t_0)$ is not Lebesgue distributive.

We may assume $c_0 = {}^{\omega_0}2$. If $\varphi \in {}^{n}2$, $n \in \omega_0$, then $u_\varphi = \{f \in c_0 : \varphi \subseteq f\}$ is a regular open subset of c_0. Let $\varkappa_{n,k} = [\frac{k}{2^{n+2}}, \frac{k+1}{2^{n+2}}]$, $k = 0,1,\dots$ $\dots, 2^{n+2} - 1$; $s_n = (n+1)(n+4)/2$, $s_{-1} = 0$. Let φ^n_k, k 2^{n+2} be an enumeration of all functions from $(s_n - s_{n-1})$ into 2. We define

$$b(n, \varkappa_{n,k}) = u_{\varphi^n_k},$$

$$b(n, \varkappa) = \emptyset \quad \text{for other } \varkappa \in \mathcal{X}.$$

For $\varphi \in \mathcal{X}_0$ we denote $a_{n\varphi} = \bigwedge_{k<n} b(k, \varphi(k))$. The conditions (1.1) and (1.2) are evidently satisfied (with $z = \{1\}$). By simple computation we obtain

$$\bigwedge_{n\in\omega_0} \bigvee_{\varphi\in\Phi_x(n)} a_{n\varphi} = \emptyset \quad \text{for any } x \in [0,1].$$

Q.E.D.

A strictly positive finite measure m on a complete Boolean algebra B (see [10],p.73) is a σ-additive measure on B such that $m(x) \neq 0$ for $x \neq 0$ and $m(1) = 1$.

Theorem 2.9. If there exists a strictly positive finite measure on B, then B is Lebesgue distributive.

Proof. Let m be a strictly positive finite measure on B. B is isomorphic to $\mathcal{B}_0(B)/ND_0(B)$. Let

m_0 be the induced measure on $\mathcal{B}_0(B)$ then $ND_0(B)$ is the set of all sets from $\mathcal{B}_0(B)$ of m_0-measure zero). Let m_1 be the product measure of m_0 and the Lebesgue measure on $[0,1]$. Thus, m_1 is a finite σ-additive measure on $\mathcal{S}(B) \times [0,1]$. If $x \in \mathcal{B}_0(B)$, then $[x]$ denotes the corresponding element of B. Let

$$\bigwedge_{n\in\omega_0} \bigvee_{\varphi\in\mathcal{X}_0} a_{n\varphi} = 1 \quad \text{and} \quad a = 1 - \bigvee_{x\in[0,1]} \bigwedge_{n\in\omega_0} \bigvee_{\varphi\in\Phi_x(n)} a_{n\varphi} \neq 0.$$

Let $A, A_{n\varphi} \in \mathcal{B}_0(B)$, $a = [A]$, $a_{n\varphi} = [A_{n\varphi}]$. Evidently $m_0(A) = m(a) > 0$. We denote

$$B_n = \bigcup_{\varphi\in\mathcal{X}_0} \bigcup_{k<n} [(A \cap A_{n\varphi}) \times \varphi(k)].$$

By simple computation we have $m_1(B_n) < \frac{1}{2} m_0(A)$,

$B_n \subseteq B_{n+1}$. Therefore $m_1(\bigcup_{n\in\omega_0} B_n) \leq \frac{1}{2} m_0(A)$.

Since $m_1(A \times [0,1]) = m_0(A)$, there exists an $x \in [0,1]$ and a set $C \subseteq A$ such that $m_0(C) > 0$ and $(C \times \{x\}) \cap \bigcup_{n\in\omega_0} B_n = \emptyset$. This fact implies

$[C] \leq \bigvee_{\varphi \in \Phi_x(n)} a_{n\varphi}$, what is a contradiction.

Q.E.D.

§ 3. <u>The main theorem</u>. Let B_0 denote the two-elements Boolean algebra consisting of zero and unit elements only. If z is an ultrafilter on B , then $\Psi_{C(B), R_{B,z}}(\overline{C(B_0)})$ is a model class in the sense of the model $\nabla(B,z)$ (see [12]), which is "elementary equivalent to the whole theory", i.e. for an elementary formula (formula without class variables) $\Sigma^* \vdash \varphi(x_1, \ldots, x_k)$ if and only if $\varphi^*(\check{x}_1, \ldots, \check{x}_k)$ holds true in the model $\Psi_{C(B), R_{B,z}}(\overline{C(B_0)})$.

Thus, e.g. if the axiom of constructivity $V = L$ is assumed to hold true, then $\Psi(\overline{C(B_0)})$ is the class of all constructive sets in the sense of the model $\nabla(B,z)$.

A condition for the existence of a "new set" of the model $\nabla(B,z)$ (a set which does not belong to $\Psi(\overline{C(B)})$), is given by the following

<u>Theorem 3.1</u>. Let $\langle x_i, r_i \rangle$ be partially ordered sets, $i = 1,2$. Let $\mathcal{F} \subseteq {}^{x_1}\mathcal{P}(x_2)$. Let B be a complete Boolean algebra, z an ultrafilter on B . Then B is $(z,\mathcal{F})$-distributive if and only if in the model $\nabla(B,z)$, for every $g \in^* Map^*(\check{x}_1, \check{r}_1, \check{x}_2, \check{r}_2)$ there exists an $f \in^* \check{\mathcal{F}}$ such that $g \subseteq \subseteq^* f$.

<u>Proof</u>. a) Let B be $(z, \mathcal{F})$-distributive, $g \in^*$ $\in^* Map^*(\check{x}_1, \check{x}_2)$. We denote $a_{ij} = F^{\ulcorner} \langle \check{j}\check{i} \rangle \in g^{\urcorner}$. The system a_{ij} , $i \in x_1$, $j \in x_2$ satisfies the condi-

tions (1.1) and (1.2). By simple computation we have

$$(3.1)\ F\ulcorner(\exists f)(f \in \check{\mathcal{F}} \,\&\, g \subseteq\subseteq f)\urcorner = \bigvee_{f \in \mathcal{F}} \bigwedge_{i \in x_1} \bigvee_{j \in f(i)} a_{ij} \quad .$$

Now the existence of such a function $f \in^* \check{\mathcal{F}}$ follows from $(z, \mathcal{F})$-distributivity.

b) Let B be not $(z, \mathcal{F})$-distributive, i.e. there is a system a_{ij} , $i \in x_1$, $j \in x_2$ satisfying (1.1) and (1.2) such that $\bigvee_{f \in \mathcal{F}} \bigwedge_{i \in x_1} \bigvee_{j \in f(i)} a_{ij} \notin Z$.

We define a function $g \in C(B)$ as follows: $g(\langle \check{j}\check{i} \rangle) = a_{ij}$ for $i \in x_1$, $j \in x_2$. It is easy to see that

$$F\ulcorner g \in Map^*(\check{x}_1, \check{\kappa}_1, \check{x}_2, \check{\kappa}_2)\urcorner \in z \quad , \qquad \text{i.e.}$$

$$g \in^* Map^*(\check{x}_1, \check{\kappa}_1, \check{x}_2, \check{\kappa}_2) .$$

The theorem follows by (3.1). Q.E.D.

<u>Corollary 3.2</u>. In the model $\nabla(B,z)$, $\check{m}$ and $\check{n}$ are of the same power if and only if B is not $(z, m \neq n)$-distributive.

<u>Proof</u>. In the case of $(m \neq n)$-distributivity, $f \subseteq\subseteq^* g \in^* \check{\mathcal{F}}$ is equivalent to the condition: f is not a mapping of m onto n in the model $\nabla(B,z)$.

<u>Corollary 3.3</u>. In the model $\nabla(B,z)$, $\check{m}^{\check{n}} =^* \widecheck{m^n}$ if and only if B is (z,m,n)-distributive.

<u>Proof</u>. In the case of (m,n)-distributivity, $f \subseteq\subseteq^* \subseteq\subseteq^* g \in^* \check{\mathcal{F}}$ is equivalent to the condition $f \in^* \widecheck{m^n}$.

<u>Corollary 3.4</u>. B is not z - Lebesgue distributive, if and only if the set $\widecheck{[0,1]}$ (i.e. the set of real numbers between zero and one, which are in the model class

$\Psi(\overline{C(B_o)}))$ has the Lebesguqe measure $\leq \frac{1}{2}$ in the model $\nabla(B,z)$.

Proof. It suffices to note that elements of $\mathcal{X}_o$ are finite coverings of measure less than $\frac{1}{2}$ and an infinite covering belongs to $\check{\Phi}_x$ if and only if it does not cover the real x .

Corollary 3.5. In the model $\nabla(B,z)$, for every function $f \in {}^{*\check{n}}\check{m}$ there are two functions $h, g \in {}^{*\check{n}}\check{m}$ such that

$$(\forall\xi)(\xi\in^*\check{n} \rightarrow . h(\xi) \leq^* f(\xi) \leq^* g(\xi) \,\&\, \overline{\overline{g(\xi) - h(\xi)}}^* <^* \check{k})$$

if and only if B is (z,n,m,k)-distributive.

§ 4. Applications to the Boolean algebras. Using the results of § 3, many theorems concerning complete Boolean algebras may be proved or reproved, e.g. the well known theorem of Pierce-Smith-Tarski for complete Boolean algebras is a direct consequence of the corollary 3.3(at the Summer Institute in Los Angeles 1967, prof.D. Scott also announced similar proof).

Now we prove some theorems concerning distributive laws in a complete Boolean algebra.

Theorem 4.1. Let B be a complete Boolean algebra. If B is $(\omega_o, 2)$-distributive, then B is Lebesgue distributive.

Proof. Let us suppose that B is not Lebesgue distributive. Then there is an ultrafilter z on B such that the set $\widecheck{[0,1]}$ has the Lebesgue measure less than one in the model $\nabla(B,z)$. Therefore there is a real number which is not in the set $\check{[0,1]}$ and, by corollary 3.3, B is not $(\omega_0, 2)$-distributive.

Q.E.D.

The author knows nothing about the connection between Lebesgue distributivity and the weak ω_0-distributivity (may be they are equivalent).

In a similar way as the theorem 4.1, one can prove

Theorem 4.2. Let B be a complete Boolean algebra.

a) B is $(m \neq n)$-distributive if and only if B is $(n \neq m)$-distributive.

b) If B is $(m,2)$-distributive, then B is $(m \neq n)$-distributive for any $n \neq m$.

Using properties of model classes we can prove

Theorem 4.3. Let B be a complete Boolean algebra.

a) Let $\omega_\beta > \omega_\alpha \geq cf(\omega_\beta)$. If B is $(\omega_\xi \neq \omega_{\xi+1})$-distributive for every $\xi : \alpha \leq \xi < \beta$, then B is $(\omega_\alpha \neq \omega_\beta)$-distributive.

b) Let $\omega_\beta > \omega_\alpha$ be the first cardinal which is confinal with a cardinal $\leq \omega_\alpha$. Let $\xi < \beta$. If B is $(\omega_\alpha, 2)$-distributive, $(m \neq n)$-distributive for every m, n, $\omega_\alpha \leq m, n \leq \omega_\xi$, $m \neq n$, then B is $(\omega_\alpha, \omega_\xi)$-distributive.

Proof. We prove the part b) only. Let ω_ξ be the smallest cardinal for which b) does not hold true. Thus, there is a complete Boolean algebra which is not $(\omega_\alpha, \omega_\xi)$-distributive. It is easy to see that there is an ultrafilter z on B such that, in the model $\nabla(B,z)$, ${}^{\check{\omega}_\alpha}\check{\omega}_\xi \neq^* {}^{\omega_\alpha}\check{\omega}_\xi$. Since $\check{\omega}_\xi$ is not confinal with $\check{\omega}_\alpha$, then also

$\check{\omega}_\alpha \check{\omega}_\eta \neq^* \check{{}^{\omega_\alpha}\omega_\eta}$ for some $\eta < \xi$. That contradicts to the assumption.

Q.E.D.

§ 5. A consistency result. Using the corollaries 3.3 and 3.4, we can prove a consistency result concerning the Lebesgue measure. The first part of this theorem was announced by the author in [1], the second part was presented on the Prague Set Theory Seminarium in October 1967. As the author was informed (January 1968), R. Solovay has announced similar results.

Theorem 5.1. Let the axiom of constructivity hold true.

a) Let D be the Boolean algebra of regular open subsets of the Cantor set. Let z be an ultrafilter on D . In the model $\nabla(D, z_0)$ the following hold true:

(i) there is a nonconstructive real number,

(ii) the set of constructive real numbers is of power $\aleph_1$ and has the Lebesgue measure zero.

b) Let C be the Boolean algebra of Borel subsets of the unit segment factorized by the ideal of all sets of the Lebesgue measure zero. Let z_1 be an ultrafilter on C . In the model $\nabla(C, z_1)$ the following holds true:

(i) there is a nonconstructive real number,

(ii) the set of all constructive real numbers is of power $\aleph_1$ and is not Lebesgue measurable.

Proof Both D and C are not $(\omega_0, 2)$-distributive and fulfil the countable chain condition. Thus, by theorem 2.1, corollaries 3.2 and 3.3, a) (i), b) (i) hold true and the set $[\check{0,1}]$ is of power $\check{\aleph}_1$ in both models.

If there is a nonconstructive real number and the set of constructive real numbers is of power $\aleph_1$, then this set is either of Lebesgue measure zero or is not Lebesgue measurable (see [1]). Using this fact, the theorem follows by the theorems 2.8 and 2.9 and the Corollary 3.4.

Q.E.D.

References

[1] L. BUKOVSKÝ: The consistency of some theorems concerning Lebesgue measure, Comment.Math.Univ.Carol.6,2(1965),179-180.

[2] L. BUKOVSKÝ: ∇-models and distributivity in Boolean algebras, Abstracts of papers 3rd ICLMPS, Amsterdam 1967, p.27.

[3] K. GÖDEL: The consistency of the continuum hypothesis, Annals of Math.Studies, No.3, Princeton 1940.

[4] K. HRBÁČEK: Measurable cardinals in some Gödelian set theory, Comment.Math.Univ.Carol.7,3(1966), 343-358.

[5] K. KURATOWSKI: Topologie I, Warsaw 1958.

[6] R.S. PIERCE: Distributivity in Boolean algebras, Pacific. J. Math.7(1957),983-992.

[7] R.S. PIERCE: A note on complete Boolean algebras, Proc. Am.Math.Soc.9(1958),892-896.

[8] D. SCOTT: The independence of certain distributive laws in Boolean algebras,Trans.Am.Math. Soc.84(1957),258-261.

[9] R. SIKORSKI: Remarks on some topological spaces of high power,Fund.Math.37(1950),125-136.

[10] R. SIKORSKI: Boolean algebras,Springer Verlag,Berlin 1964.

[11] P. VOPĚNKA: Properties of ∇-model,Bull.Acad.Polon. Sci.,Sér.Mat.,Astr.et Phys.XIII(1965),189-192.

[12] P. VOPĚNKA: General theory of ∇-models,Comment. Math.Univ.Carol.8,1(1967),145-170.

(Received October 1,1968)