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A NOTE ON INTEGRALS OF THE CAUCHY TYPE

(Preliminary communication)

Jaroslav LUKEŠ, Praha

In this note we investigate the behaviour of the integrals of Cauchy's type. The main result are the theorems 6 and 7 .

0. Notations.

Let us first introduce some notations. The term path will be used to denote a continuous complex - valued function ψ defined on an interval $\langle a,b \rangle$. The symbol $[\psi]$ means the set $\psi(\langle a,b \rangle)$. In the following always we suppose that the path ψ is defined on $\langle a,b \rangle$ and $\operatorname{var}[\psi; \langle a,b \rangle] < +\infty$.

For every $\Delta > 0$, $\alpha_1, \alpha_2 \in E_1$ (= the set of finite real numbers), $\alpha_1 \leq \alpha_2$, $\eta \in E_2$ (= the Euclidean plane) we put

$$P_\Delta^\eta(\langle \alpha_1, \alpha_2 \rangle) = \{z \in E_2;\ z = \eta + r \exp i\gamma\ ,$$

$$r \in \langle 0, \Delta \rangle\ ,\quad \gamma \in \langle \alpha_1, \alpha_2 \rangle \}\ .$$

The symbols O, o are used in the usual sense.

1. In [1] there are introduced the cyclic and radial variations of a plane path; their further properties are

investigated in [2] -[5] and [8] mainly in connexion with the potential of the double distribution. The general radial and cyclic variations corresponding to the weight function p defined on the $[\psi]$ appear in [6] .

If $z \in E_2$, $\alpha, \nu \in E_1$, G open, $G \subset \langle a,b \rangle$ and if p is a nonnegative real-valued Baire function on $[\psi]$ we denote

$$T_\alpha = \{t \in G; |\psi(t)-z| > 0, \psi(t)-z = |\psi(t)-z| \exp i\alpha\},$$

$$S_\nu = \{t \in G; |\psi(t)-z| = \nu\}$$

and define

$$v(\alpha, z, p, G) = \sum_{t \in T_\alpha} p(\psi(t)), \quad u(\nu, z, p, G) = \sum_{t \in S_\nu} p(\psi(t)),$$

$$V(\alpha, z, p, G) = \sum_{t \in T_\alpha} p(\psi(t)) \cdot |\psi(t) - z| ,$$

$$U(\nu, z, p, G) = \nu^{-1} \cdot u(\nu, z, p, G) .$$

Since the quantities $v(\alpha, z, p, G)$, $V(\alpha, z, p, G)$ and $u(\nu, z, p, G)$, $U(\nu, z, p, G)$ are Lebesgue measurable with respect to α on $(0, 2\pi)$ and with respect to ν on $(0, +\infty)$, respectively, we may finally define

$$v(z, p, G) = \int_0^{2\pi} v(\alpha, z, p, G)\, d\alpha, \quad u(z, p, G) = \int_0^{\infty} u(\nu, z, p, G)\, d\nu,$$

$$V(z, p, G) = \int_0^{2\pi} V(\alpha, z, p, G)\, d\alpha, \quad U(z, p, G) = \int_0^{\infty} U(\nu, z, p, G)\, d\nu .$$

2. Lemma.

Let $z \in E_2$ and denote by $\mathcal{G}$ the system of all components of $\{t \in G;\ |\psi(t) - z| > 0\}$. With every $I \in \mathcal{G}$ we associate a continuous single-valued argument ϑ_z^I of $\psi(t) - z$ on I .

Then

$$v(z,p,G) = \sum_{I\in\sigma} \int_I p(\psi(t)) \cdot d\,var_t\, \vartheta_z^I(t) ,$$

$$V(z,p,G) = \sum_{I\in\sigma} \int_I p(\psi(t)) \cdot |\psi(t) - \\ - z|\, d\,var_t\, \vartheta_z^I(t) ,$$

$$u(z,p,G) = \int_G p(\psi(t))\, d\,var_t\, |\psi(t) - z| ,$$

$$U(z,p,G) = \int_G p(\psi(t)) \cdot |\psi(t) - \\ - z|^{-1} d\,var_t\, |\psi(t) - z| .$$

3. Put $G_R = \{t \in \langle a, b \rangle;\ 0 < |\psi(t) - z| < R\}$ for every $R > 0$ and let

$$v_R(z,p) = v(z,p,G_R), \qquad u_R(z,p) = u(z,p,G_R),$$

$$V_R(z,p) = V(z,p,G_R), \qquad U_R(z,p) = U(z,p,G_R),$$

$$v(z,p) = v_{+\infty}(z,p), \qquad u(z,p) = u_{+\infty}(z,p),$$

$$V(z,p) = V_{+\infty}(z,p), \qquad U(z,p) = U_{+\infty}(z,p).$$

The functions $v_R(z,p)$, $u_R(z,p)$, $V_R(z,p)$, $U_R(z,p)$ have a number of interesting properties. At least we observe the following one, on which the proofs of theorems 6 and 7 are based.

Let $\alpha_1, \alpha_2 \in E_1$, $\sigma \in (0, \frac{\pi}{4})$, $\Delta > 0$, $\eta \in [\psi]$. Let p be a nonnegative lower semicontinuous function on $[\psi]$. Suppose that the following conditions are fulfilled

1) $\alpha_1 \leq \alpha_2 < \alpha_2 + \delta < \alpha_1 + 2\pi - \delta$,

2) $P_\Delta^\eta(\langle \alpha_1 - \delta, \alpha_2 + \delta \rangle) \cap [\psi] = \eta$.

Then U(z,p) is bounded on $P_\Delta^\eta(\langle \alpha_1, \alpha_2 \rangle)$ provided

3_1) $u(\eta, p) + \sup_{x>0} x^{-1} V_x(\eta, p) < +\infty$,

v(z,p) is bounded on $P_\Delta^\eta(\langle \alpha_1, \alpha_2 \rangle)$ provided

3_2) $v(\eta, p) + \sup_{x>0} x^{-1} u_x(\eta, p) < +\infty$.

4. Integrals of the Cauchy type

Let us keep the notations and assumptions of the lemma 2, let $G = \{ z \in \langle a, b \rangle;\ 0 < |\psi(t) - z| \}$ and denote by $\mathfrak{G}$ the system of all components of G .

Given a real-valued function F on $[\psi]$ we define

$$P(F, z) = \int_G F(\psi(t)) \cdot |\psi(t) - z|^{-1} d_t |\psi(t) - z| ,$$

$$W(F, z) = \sum_{I \in \mathfrak{G}} \int_I F(\psi(t))\, d_t\, \vartheta_z^I(t)$$

provided the Lebesgue-Stieltjes integrals on the right-hand sides exist and the sum is meaningful.

We put further $I(F,z) = P(F,z) + iW(F,z)$.

The following assertion is well-known.

5. Proposition

Suppose that $\int_a^b F(\psi(t))\, d\, \mathrm{var}\, \psi(t)$ exists.

Then

1) $P(F,z)$, $W(F,z)$ exist for every $z \in E_2 - [\psi]$,

2) $I(F,z) = \int_{\psi} \frac{F(\xi)}{\xi - z} d\xi \left(= \int_a^b \frac{F(\psi(t))}{\psi(t) - z} d\psi(t)\right)$

for every $z \in E_2 - [\psi]$.

6. <u>Theorem</u>

Let $\eta \in [\psi]$ (ψ is a path on $\langle a,b \rangle$) , $a = \psi^{-1}(\eta)$. Suppose that

1/ p is a nonnegative lower semicontinuous function on $[\psi]$,

2/ $\alpha_1, \alpha_2 \in E_1$, $\sigma \in (0, \frac{\pi}{4})$, $\Delta > 0$ and

a) $\alpha_1 \leq \alpha_2 < \alpha_2 + \sigma < \alpha_1 + 2\pi - \sigma$,

b) $P_\Delta^\eta(\langle \alpha_1 - \sigma, \alpha_2 + \sigma \rangle) \cap [\psi] = \eta$,

3/ Θ is a continuous nonnegative nondecreasing function on $(0, +\infty)$,

4/ F is a real-valued bounded Baire function on $[\psi]$ with $F(\eta) = 0$ fulfilling one of the following conditions

4_1/ $F(z) = O(p(z) \cdot \Theta(|z - \eta|))$, $z \in [\psi]$, $z \to \eta$,

or

4_2/ $F(z) = o(p(z) \cdot \Theta(|z - \eta|))$, $z \in [\psi]$, $z \to \eta$,

5/ $v(\eta, p) + U(\eta, p) < +\infty$.

For every $R \in (0, \Delta)$ we denote

$I_F(R) = \sup |I(F,z) - I(F,\eta)|$, $z \in P_\Delta^\eta(\langle \alpha_1, \alpha_2 \rangle)$, $|z - \eta| = R$.

Then

$$I_F(R) = O(R(\int_R^\Delta \theta(x)x^{-2}dx)), \quad R \to 0_+ \quad ,$$

provided 4_1/ or

$$I_F(R) = o(R(\int_R^\Delta \theta(x)x^{-2}dx)), \quad R \to 0_+ \quad ,$$

provided 4_2/.

The following assertion is a simple consequence of this theorem.

7. <u>Theorem</u>

Let us keep the notations and assumptions introduced in the preceding theorem 6 with validity of 4_1/.

Then

1) $I_F(R) = O(R \log \frac{1}{R})$, $R \to 0_+$, provided $\theta(\rho)\cdot\rho^{-1}$ is nondecreasing on $(0,+\infty)$ and, moreover, $I_F(R) = O(R)$, $R \to 0_+$ provided, in addition, $\int_R^\Delta \theta(x)x^{-2}dx = O(1)$, $R \to 0_+$;

2) $I_F(R) = O(\theta(R)\log\frac{1}{R})$, $R \to 0_+$ provided $\theta(\rho)\cdot\rho^{-1}$ is nondecreasing on $(0,+\infty)$ and, moreover, $I_F(R) = O(\theta(R))$, $R \to 0_+$ provided, in addition, $\int_R^\Delta \theta(x)x^{-2}dx = O(R^{-1}\theta(R))$, $R \to 0_+$.

We may obtain the analogous theorem for the symbol o with the interchange of the assumption 4_1/ for 4_2/.

8. Remarks

A) We may generalize the theorems 6 and 7 to the case $\psi^{-1}(\eta) \in (a, b\rangle$ or to the case where the set $\psi^{-1}(\eta)$ is not a single point.

B) The assumption $F(\eta) = 0$ is not essential, it is adopted here only to make the formulations simpler.

C) We may give the similar theorems for the function $P(F,z)$, resp. $W(F,z)$, too, we must only replace the condition 5/ by

$$5_1/ \quad U(\eta, r) + \sup_{\alpha > 0} \alpha^{-1} V_\alpha(\eta, r) < +\infty ,$$

and

$$5_1/ \quad v(\eta, r) + \sup_{\alpha > 0} \alpha^{-1} u_\alpha(\eta, r) < +\infty ,$$

respectively.

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