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# A REMARK ON THE THEORY OF LATTICE POINTS IN ELLIPSOIDS II

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The aim of this remark [1)] is to refer to the use of a certain "dual" relation in the theory of lattice points in ellipsoids. Combining the basic identity (see Theorem 1) with some author's previous results it is possible to deduce a number of interesting O-estimations. In this paper there are made use of certain ideas, which can be originally found in Landau [2].

In the following let $r$ be a natural number, $r \geqq 2$. $Q$ let be a positive definite quadratic form in $r$ variables whose determinant is denoted by $D$, $\tilde{Q}$ be the form conjugated with $Q$. Let further $\alpha_1, \alpha_2, \dots, \alpha_r$ and $b_1, b_2, \dots, b_r$ be systems of real numbers and $M_1, M_2, \dots, M_r$ a system of positive real numbers. For $x \geqq 0$ let us define the function $A(x)$ as follows

$$A(x) = A(x; Q, \alpha_j, b_j, M_j) = \sum e^{2\pi i \sum_{j=1}^{r} \alpha_j u_j}$$

where the summation runs over all systems $u_1, u_2, \dots, u_r$ of real numbers, which satisfy the relations $Q(u_j) = Q(u_1, u_2, \dots, u_r) \leqq x$ and $u_j \equiv b_j \pmod{M_j}$, $j = 1, 2, \dots, r$.

1) As part I of the presented work (which is independent) is considered the paper [5].

If we put as usual

$$V(x) = \frac{\pi^{k/2} x^{k/2} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\sqrt{D} \prod_{j=1}^{k} M_j \, \Gamma(\frac{k}{2}+1)} \, \delta$$

( $\delta = 1$ if all numbers $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_k M_k$ are integers, $\delta = 0$ otherwise) then for the function

$$P(x) = A(x) - V(x)$$

hold as known (see [2] pp.11 and 71) the estimates

(1) $$P(x) = O(x^{k/2 - k/(k+1)}) \text{ and } P(x) = \Omega(x^{\frac{k-1}{4}}),$$

(we shall exclude from our considerations the case where $A(x) = 0$ identically).

Let further $0 < \lambda_1 < \lambda_2 < \dots$ be the sequence of all values of the form $Q(m_j M_j + b_j) > 0$ with integer $m_1, m_2, \dots, m_k$, $\lambda_0 = 0$, and for integer $n \geqq 0$ let

$$a_0 = A(\lambda_0), \quad a_{n+1} = A(\lambda_{n+1}) - A(\lambda_n).$$

Thus

$$A(x) = \sum_{\lambda_n \leqq x} a_n .$$

For $\rho$ complex, $\operatorname{Re}\rho > 0$ let us put

$$A_\rho(x) = \frac{1}{\Gamma(\rho)} \int_0^x A(t)(x-t)^{\rho-1}\,dt$$

and analogously let us define the functions $V_\rho(x)$ and $P_\rho(x)$. If we put $A_0(x) = A(x)$, $V_0(x) = V(x)$, $P_0(x) = P(x)$ then for nonnegative $\rho$ obviously

$$P_{\rho+1}(x) = \int_0^x P_\rho(t)\,dt, \quad V_\rho(x) = \frac{\pi^{k/2} x^{k/2+\rho} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\sqrt{D} \prod_{j=1}^{k} M_j \, \Gamma(k/2+\rho+1)} \, \delta$$

etc.

Let the letter c denote (generally various) positive constants, which depend at most on $Q$ , $\alpha_j$ , $b_j$ , $M_j$ $(j = 1,2,\dots, r)$ . The relation $A \ll B$ means that $|A| \leqq c B$ . The symbols $O$, $o$ and $\Omega$ are meant in the usual sense. For $s$ complex, $\operatorname{Re} s > 0$ put

$$\Theta(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s} .$$

As known, the function $\Theta(s)$ is a holomorphic function in the half plane $\operatorname{Re} s > 0$ .

In the introduced way the functions $A(x)$, $V(x)$, $P(x)$ , $A_{\wp}(x)$ , $\Theta(s)$ etc. and the numbers $\sigma$ , $\lambda_n$ , $a_n$ $(n = 0,1,2,\dots)$ correspond to the form $Q$ and to the systems of numbers $\alpha_j$ , $b_j$ , $M_j$ $(j = 1,2,\dots, r)$ (in this order). The functions $\tilde{A}(x)$ , $\tilde{V}(x)$ , $\tilde{P}(x)$ , $\tilde{A}_{\wp}(x)$, $\tilde{\Theta}(s)$ etc. and the numbers $\tilde{\sigma}$ , $\tilde{\lambda}_n$ , $\tilde{a}_n$ $(n = 0,1,2,\dots)$ we shall design for the form $\tilde{Q}$ and systems of numbers $b_j$ , $-\alpha_j$ , $1/M_j$ $(j = 1,2,\dots,r)$ (in this order) analogously. If we choose for $s$ complex, $\operatorname{Re} s > 0$ the branch $s^{r/2}$ in such a way that it will be positive for positive values of $s$ , then as known (see [1] p.108) for the $s$ considered holds

$$\Theta(s) = \frac{\pi^{r/2} e^{2\pi i \sum_{j=1}^{r} \alpha_j b_j}}{\sqrt{D} \prod_{j=1}^{r} M_j \, s^{r/2}} \, \tilde{\Theta}\left(\frac{\pi^2}{s}\right) . \tag{2}$$

Let us note, that obviously $a_0 = \tilde{\sigma}$, $\tilde{a}_0 = \sigma$ and

$$\tilde{V}(x) = \frac{\pi^{r/2} x^{r/2} e^{-2\pi i \sum_{j=1}^{r} \alpha_j b_j} \sqrt{D} \prod_{j=1}^{r} M_j}{\Gamma(r/2 + 1)} \, \tilde{\sigma} .$$

The dual relation referred to in the introduction is given in the following theorem.

Theorem 1. For $\rho$ complex, $\operatorname{Re}\rho > k/2$, $x > 0$ holds

$$(P(x)-a_0)_\rho = P_\rho(x) - \frac{a_0 x^\rho}{\Gamma(\rho+1)} =$$

(3)

$$= \frac{x^{k/4+\frac{\rho+1}{2}} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\pi^{\rho-1}\sqrt{D}\prod_{j=1}^{k} M_j} \int_0^\infty (\tilde{P}(\xi)-\tilde{a}_0) \frac{J_{k/2+\rho+1}(2\pi\sqrt{\xi x})}{\xi^{k/4+\frac{\rho+1}{2}}}\, d\xi \,,$$

where $J_\nu(x)$ is the Bessel function of the 1st kind and in the integrand we put $\arg x = \arg \xi = 0$.

Proof. If $\rho$ is complex, $\operatorname{Re}\rho > 0$, $a > 0$ then obviously

$$A_\rho(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{xs}}{s^{\rho+1}}\,\Theta(s)\,ds \,,$$

where the integration is to be taken over the line $\operatorname{Re}\rho = a$. If we now use the relation (2) we find that for $\operatorname{Re}\rho > k/2$ we can according to the trivial estimate

$$\sum_{\tilde{\lambda}_n \le x} |\tilde{a}_n| \ll (x+1)^{k/2}$$

interchange the summation and integration. For $A > 0$, $B \geqq 0$ however

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{As-\frac{B}{s}}}{s^{\frac{k}{2}+\rho+1}}\,ds = \begin{cases} \dfrac{A^{k/2+\rho}}{\Gamma(k/2+\rho+1)} & \text{for } B = 0 \\[2ex] \left(\dfrac{A}{B}\right)^{k/4+\frac{\rho}{2}} J_{k/2+\rho}(2\sqrt{AB}) & \text{for } B > 0 \,. \end{cases}$$

For $\rho$ complex, $\operatorname{Re}\rho > k/2$ we thus obtain a general form of Landau's relation, as deduced in [5]

$$A_\rho(x) = \frac{\pi^{k/2} x^{k/2+\rho} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\sqrt{D} \prod_{j=1}^{k} M_j \Gamma(k/2+\rho+1)} \sigma + \tag{4}$$

$$+ \frac{x^{k/4+\rho/2} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\pi^\rho \sqrt{D} \prod_{j=1}^{k} M_j} \sum_{n=1}^{\infty} \tilde{a}_n \frac{J_{k/2+\rho}(2\pi\sqrt{\tilde{\lambda}_n x})}{\tilde{\lambda}_n^{k/4+\rho/2}} .$$

Now let us consider that for an arbitrary function $g$ with a continuous derivation on the interval $\langle \tilde{\lambda}_1, T \rangle$ $(T > \tilde{\lambda}_1)$ holds

$$\sum_{\tilde{\lambda}_1 \le \tilde{\lambda}_n \le T} \tilde{a}_n g(\tilde{\lambda}_n) = (\tilde{A}(T) - \tilde{a}_0) g(T) - \int_{\tilde{\lambda}_1}^{T} (\tilde{A}(\xi) - \tilde{a}_0) g'(\xi) d\xi .$$

If we choose

$$g(\xi) = \xi^{-k/4-\rho/2} J_{k/2+\rho}(2\pi\sqrt{x\xi})$$

and consider that $\tilde{A}(\xi) - \tilde{a}_0 = 0$ for $\xi \in \langle 0, \tilde{\lambda}_1)$,

$\tilde{A}(\xi) - \tilde{a}_0 << \xi^{k/2}$ we get, using the limit for $T \to +\infty$ and substituting in (4) immediately

$$P_\rho(x) = \frac{x^{k/4+\frac{\rho+1}{2}} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\pi^{\rho-1} \sqrt{D} \prod_{j=1}^{k} M_j} \int_0^\infty (\tilde{A}(\xi) - \tilde{a}_0) \cdot \frac{J_{k/2+\rho+1}(2\pi\sqrt{\xi x})}{\xi^{k/4+\frac{\rho+1}{2}}} d\xi . \tag{5}$$

If $\alpha$ is real, $\alpha > -1$, then for $\rho$ complex, $\mathrm{Re}\,\rho > 2\alpha - k/2$ following well known relation for the Hankel transform holds (see [6]p.435)

$$\int_0^\infty \frac{J_{k/2+\rho+1}(2\pi\sqrt{\xi x})}{\xi^{k/4+\frac{\rho+1}{2}-\alpha}} d\xi = 2(4\pi^2 x)^{k/4+\frac{\rho-1}{2}-\alpha} \cdot \int_0^\infty J_{k/2+\rho+1}(t) \sqrt{t}\, t^{2\alpha-\rho-\frac{k+1}{2}} dt = \tag{6}$$

$$= \pi^{r/2+\rho-2\alpha-1} x^{r/4+\frac{\rho-1}{2}-\alpha} \frac{\Gamma(\alpha+1)}{\Gamma(r/2+\rho-\alpha+1)} .$$

Using this relation for $\alpha = r/2$ we obtain

$$\frac{x^{r/4+\frac{\rho+1}{2}} e^{2\pi i \sum_{j=1}^{r} \alpha_j b_j}}{\pi^{\rho-1} \sqrt{D} \prod_{j=1}^{r} M_j} \int_0^\infty \widetilde{V}(\xi) \frac{J_{r/2+\rho+1}(2\pi\sqrt{\xi x})}{\xi^{r/4+\frac{\rho+1}{2}}} d\xi = \frac{x^\rho}{\Gamma(\rho+1)} a_0$$

and thus from (5) immediately follows (3). Using (6) for $\alpha = 0$, we can rewrite (5) in the form

$$A_\rho(x) =$$

(6')

$$= \frac{x^{r/4+\frac{\rho+1}{2}} e^{2\pi i \sum_{j=1}^{r} \alpha_j b_j}}{\pi^{\rho-1} \sqrt{D} \prod_{j=1}^{r} M_j} \int_0^\infty \widetilde{A}(\xi) \frac{J_{r/2+\rho+1}(2\pi\sqrt{\xi x})}{\xi^{r/4+\frac{\rho+1}{2}}} d\xi .$$

Using Theorem 1 we can now deduce a basic relation for the O-estimates:

**Theorem 2.** Let $\sigma = 1$, $r \geq 5$ [2] and

(7) $$\widetilde{P}(x) = O(x^\alpha) \quad [3]$$

then

(8) $$P(x) = O(x^{r/2-1+\frac{r-3-2\alpha}{r-3-4\alpha}}) \quad \text{for } \alpha > r/4 - 1/4 ,$$

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2) Let us note that, for $2 \leq r \leq 4$ and for $\alpha \geq \frac{r}{2} - \frac{r}{r-1}$, $r = 5, 6, \ldots$, we cannot obtain on the base of this method any better result than Landau's estimation $O(x^{r/2 - r/r+1})$.

3) According to (1) is $\alpha \geq r/4 - 1/4$ as $A(x) \not\equiv 0$ and thus according to (6') also $\widetilde{A}(x) \not\equiv 0$. Obviously (see (1)) we can assume that $\alpha \leq r/2 - r/r + 1$.

(9) $$P(x) = O(x^{r/4+1/4}\lg x) \text{ for } \alpha = r/4 - 1/4 .$$

If (7) holds with symbol $o$ 4), also (8) holds with the symbol $o$ .

<u>Proof</u>. We shall use the usual Landau's procedure (see [2],pp.25-29). Let $x > c$ and $\tilde{P}(x) << x^{\alpha}\varphi(x)$, where we consider the following cases:

a) $\varphi(x) \equiv 1$ (if (7) holds and $\alpha > r/4 - 1/4$ ),

b) $\varphi(x) \equiv 1$ (if (7) holds with $\alpha = r/4 - 1/4$ ),

c) $\varphi(x)$ is a positive function, $\varphi(x) = o(1)$ (if (7) holds with symbol $o$ - let us note, that we can assume that the function $\varphi(x)$ is defined for $x > c$ and is continuous and decreasing).

Let $\rho = [2\alpha + 1/2] + 1$ and let $z = z(x)$ be a positive function defined for $x > c$ , $z \leqq \sqrt{x}$ (for $x > c$ ) and $z(x) = o(\sqrt{x})$ . Thus, $\rho > r/2$ , $\alpha - r/4 - 3/4 - \rho/2 < -1$ and (for $x > c$ ) $0 < \rho z < x$ , $\lim_{x\to+\infty} t(x) = +\infty$ , where $t = t(x) = \sqrt{x/z^2}$ . We put

$$\Delta_z f(x) = \sum_{j=0}^{\rho}(-1)^{\rho-j}\binom{\rho}{j} f(x+jz) .$$

It is easy to ascertain that for $y > 0$ holds

$$\Delta_z x^{r/4+\frac{\rho+1}{2}} J_{\frac{r}{2}+\rho+1}(2\pi\sqrt{yx}) << \frac{x^{\frac{r+1}{4}}}{y^{1/4}}(\min(x, z^2 y))^{\rho/2}$$

(see [2],p.25). From (3) we now obtain

$$\Delta_z P_\rho(x) - a_0 z^{\rho} << \int_0^\infty |\tilde{P}(\xi) - \tilde{a}_0| \frac{x^{\frac{r+1}{4}}\min^{\rho/2}(x, z^2\xi)}{\xi^{r/4+3/4+\rho/2}} d\xi <<$$

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4) According to (1) then $\alpha > r/4 - 1/4$ .

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$$<< \int_0^{\tilde{\lambda}_1} \cdots d\xi + \int_{\tilde{\lambda}_1}^{t} \cdots d\xi + \int_t^{t^2} \cdots d\xi + \int_{t^2}^{\infty} \cdots d\xi <<$$

$$<< x^{\frac{r+1}{4}} z^{\rho} \int_0^{\tilde{\lambda}_1} \xi^{r/4 - 3/4} d\xi + x^{\frac{r+1}{4}} z^{\rho} \int_{\tilde{\lambda}_1}^{t} \xi^{\alpha - r/4 - 3/4} d\xi +$$

$$+ x^{\frac{r+1}{4}} z^{\rho} \varphi(t) \int_t^{t^2} \xi^{\alpha - r/4 - 3/4} d\xi + x^{\frac{r+1}{4}} \varphi(t^2) x^{\rho/2} \int_{t^2}^{\infty} \xi^{\alpha - r/4 - 3/4 - \rho/2} d\xi$$

(for $0 < \xi < \tilde{\lambda}_1$ we used the estimate $\tilde{P}(\xi) - \tilde{a}_0 << \xi^{r/2}$, for $\xi \geqq \tilde{\lambda}_1$ the estimate $\tilde{P}(\xi) - \tilde{a}_0 << \xi^{\alpha} \varphi(\xi)$ ). Thus, we can write

(10) $$\Delta_z P_{\rho}(x) - a_0 z^{\rho} << z^{\rho} x^{\frac{r+1}{4}} t^{2(\alpha - r/4 + 1/4)} \lambda(t)$$

where

a) $\lambda(t) \equiv 1$ , b) $\lambda(t) \equiv \lg t$ , c) $\lambda(t)$ is a positive continuous and decreasing function, $\lambda(t) = o(1)$ for $t \to +\infty$ .

For a suitable $\xi \in \langle x, x + \rho z \rangle$ holds

$$\Delta_z x^{\frac{r}{2}+\rho} = z^{\rho}(\tfrac{r}{2}+\rho)(\tfrac{r}{2}+\rho-1)\cdots(\tfrac{r}{2}+1)\xi^{\frac{r}{2}} = z^{\rho}(\tfrac{r}{2}+\rho)(\tfrac{r}{2}+\rho-1)\cdots$$

$$\cdots(\tfrac{r}{2}+1)x^{\frac{r}{2}} + O(x^{\frac{r}{2}-1} z^{\rho+1}) = z^{\rho}(\tfrac{r}{2}+\rho)(\tfrac{r}{2}+\rho-1)\cdots(\tfrac{r}{2}+1)(x+\rho z)^{r/2} + O(x^{\frac{r}{2}-1} z^{\rho+1})$$

and thus

(11)
$$\Delta_z V_{\rho}(x) = z^{\rho} V(x) + O(x^{r/2-1} z^{\rho+1})$$
$$\Delta_z V_{\rho}(x) = z^{\rho} V(x + \rho z) + O(x^{r/2-1} z^{\rho+1}) .$$

The function $\eta\, A(x)$ is nonnegative and nondecreasing ($\eta = e^{-2\pi i \sum_{j=1}^{r} \alpha_j k_j}$).

For $x_\wp \in \langle x, x+\wp z\rangle$ thus holds

$$\eta A(x) \leqq \eta A(x_\wp) \leqq \eta A(x+\wp z)$$

and as well

$$z^\wp \eta A(x) \leqq \Delta_z \eta A_\wp(x) =$$

(12)

$$= \int_x^{x+z}[\int_{x_1}^{x_1+z}[\cdots\int_{x_{\wp-1}}^{x_{\wp-1}+z}\eta A(x_\wp)\,dx_\wp]\,dx_{\wp-1}\cdots]\,dx_1 \leqq \eta z^\wp A(x+\wp z).$$

If we now use (10) and (11) we obtain from the relation (12)

$$(13)\quad \eta A(x) \leqq \eta V(x) + O(x^{\kappa/2-1} z) + O(x^{\alpha+1/2} z^{\kappa/2-1/2-2\alpha}\lambda(\sqrt{x/z^2}))$$

and

$$(14)\quad \eta A(x+\wp z) \geqq \eta V(x+\wp z) + O(x^{\kappa/2-1} z) + O(x^{\alpha+1/2} z^{\kappa/2-1/2-2\alpha}\lambda(\sqrt{x/z^2})).$$

Put $z = x^{\frac{\kappa-3-2\alpha}{\kappa-3-4\alpha}}\psi(x)$, where $\psi(x) = \lambda^{\frac{2}{4\alpha+3-\kappa}}(x)^{\frac{\kappa-3}{8\alpha+6-2\kappa}})$ .

According to remark 3) is for $x > c$ certainly $0 < z \leqq \sqrt{x}$, $z(x) = o(\sqrt{x})$ $(4\alpha + 3 - \kappa > 0$, $\frac{\kappa-3-2\alpha}{\kappa-3-4\alpha} < 1/2)$. For simplicity, let us write $y = y(x) = x + \wp z$. From (13) and (14) we obtain

$$\eta A(x) \leqq \eta V(x) + O(x^{\frac{\kappa}{2}-1+\frac{\kappa-3-2\alpha}{\kappa-3-4\alpha}}\psi(x))$$

$$\eta A(y) \geqq \eta V(y) + O(x^{\frac{\kappa}{2}-1+\frac{\kappa-3-2\alpha}{\kappa-3-4\alpha}}\psi(x)) \geqq$$

$$\geqq \eta V(y) + O(y^{\frac{\kappa}{2}-1+\frac{\kappa-3-2\alpha}{\kappa-3-4\alpha}}\psi(x)) .$$

If we consider that for $x > c$ is $y$ a continuous function of $x$, $y \longrightarrow +\infty$ for $x \longrightarrow +\infty$ we obtain immediately all the assertions of Theorem.

On the base of the Landau's identity (4) the estimation (8) of Theorem 2 may be slightly improved in some special cases.

<u>Theorem 3</u>. Let $\sigma = 1$, $\varkappa \geq 4$, $n << \tilde{\lambda}_n << n$ (n = 1,2,...) and

(15) $$\tilde{P}(x) = O(x^{\alpha}) .$$

Then

(16) $$P(x) = O(x^{\frac{\varkappa}{2}-1+\frac{\varkappa-3-2\alpha}{\varkappa-5-4\alpha}}) .$$

If (15) holds with symbol $o$, (16) also holds with symbol $o$.

<u>Proof</u>. If $\tilde{P}(x) = O(x^{\alpha})$, where $\alpha \geq \frac{\varkappa}{2} - 1$ or $\tilde{P}(x) = o(x^{\alpha})$, where $\alpha > \frac{\varkappa}{2} - 1$ then, according to (1), the assertion is trivially satisfied. Let $\tilde{P}(x) = o(x^{\varkappa/2-1})$ and $\tilde{\sigma} \neq 0$. First ($\tilde{\lambda}_n >> n$, $\tilde{\lambda}_n = \sum_{k=2}^{n}(\tilde{\lambda}_k - \tilde{\lambda}_{k-1}) + \tilde{\lambda}_1$), there exists such a constant c that the inequality $\tilde{\lambda}_{n+1} - \tilde{\lambda}_n > c$ is valid for infinitely many natural n ; i.e. for infinitely many n holds $\tilde{A}(\tilde{\lambda}_n + c) = \tilde{A}(\tilde{\lambda}_n)$,

$$|\tilde{P}(\tilde{\lambda}_n + c) - \tilde{P}(\tilde{\lambda}_n)| = |\tilde{V}(\tilde{\lambda}_n + c) - \tilde{V}(\tilde{\lambda}_n)| >> \tilde{\lambda}_n^{\frac{\varkappa}{2}-1} >> n^{\frac{\varkappa}{2}-1} .$$

This is a contradiction with $|\tilde{P}(\tilde{\lambda}_n + c) - \tilde{P}(\tilde{\lambda}_n)| = o(n^{\frac{\varkappa}{2}-1})$ (for $n \to \infty$) i.e. $\tilde{\sigma} = 0$, $\tilde{A}(x) = \tilde{P}(x)$. If (15) holds, then $\tilde{a}_n = \tilde{A}(\tilde{\lambda}_n) - \tilde{A}(\tilde{\lambda}_{n-1}) = O(n^{\alpha})$ (for $n \to \infty$) and similarly with the symbol $o$ i.e. we have

$\tilde{a}_n << n^{\alpha}\varphi(n)$ , where $\varphi(x) \equiv 1$ or $\varphi(x)$ is a positive continuous and decreasing function, $\varphi(x) = \sigma(1)$ . Then ( $\rho = [2\alpha + \frac{1}{2}] + 1 > \frac{r}{2}$ , $z = z(x)$ is a positive function, $z(x) = \sigma(\sqrt{x})$, $t = \sqrt{\frac{x}{z^2}}$ ) according to (4)

$$\Delta_z P_\rho(x) << x^{\frac{r}{4}-\frac{1}{4}} \sum_{n=1}^{\infty} n^{\alpha-\frac{r}{4}-\frac{\rho}{2}-\frac{1}{4}} \varphi(n) \min(x, n z^2)^{\frac{\rho}{2}} <<$$

$$<< x^{\frac{r}{4}-\frac{1}{4}} \Big( z^{\rho} \sum_{n<t} n^{\alpha-\frac{r}{4}-\frac{1}{4}} + \varphi(t) \sum_{t \leq n \leq t^2} z^{\rho} n^{\alpha-\frac{r}{4}-\frac{1}{4}} +$$

$$+ \varphi(t^2) x^{\frac{\rho}{2}} \sum_{n>t^2} n^{\alpha-\frac{r}{4}-\frac{\rho}{2}-\frac{1}{4}} \Big) << x^{\frac{r}{4}-\frac{1}{4}} z^{\rho} t^{2(\alpha-\frac{r}{4}+\frac{3}{4})} \lambda(t) ,$$

where $\lambda(x) \equiv 1$ (for $\varphi(x) \equiv 1$ ) or $\lambda(x)$ is a positive continuous and decreasing function, $\lambda(x) = \sigma(1)$ (in the second case). If we put

$$z = x^{\frac{r-3-2\alpha}{r-5-4\alpha}} \lambda^{\frac{1}{2\alpha+5/2-r/2}} \Big( x^{\frac{r-1}{8\alpha+10-2r}} \Big)$$

we obtain easily, that $z$ satisfies the conditions mentioned above and

$$x^{\frac{r}{2}-1} z + x^{\frac{r}{4}-\frac{1}{4}} t^{2(\alpha-\frac{r}{4}-\frac{3}{4})} \lambda(t) << x^{\frac{r}{2}-1+\frac{r-3-2\alpha}{r-5-4\alpha}} \lambda^{\frac{2}{4\alpha+5-r}} \Big( x^{\frac{r-1}{8\alpha+10-2r}} \Big).$$

Analogously as in proof of Theorem 2 we obtain now immediately the assertions of Theorem 4.

<u>Remark</u>. Theorem 3 gives better results than Theorem 2 (than Landau's estimation (1)) only for $\alpha > \frac{r}{2} - \frac{3}{2}$ ( $\alpha < \frac{r}{2} - 1$ or $\tilde{P}(x) = \sigma(x^{\frac{r}{2}-1})$ ). For $r = 2$ and $r = 3$ the Theorem 3 does not give new results.

Remark. If the assumption $\sigma = 1$ does not hold, the transition from the function $A_{\wp}(x)$ to the function $A(x)$ is not so simple. Let us denote $\tilde{A}^{\circ}(x) = \tilde{A}(x; \tilde{Q}, \ell_j, 0, 1/M_j)$, $A^{\circ}(x) = A(x; Q, 0, \ell_j, M_j)$ and let $\tilde{V}^{\circ}(x)$, $\tilde{P}^{\circ}(x)$ etc. have the same meaning. Let $\alpha > \frac{r}{4} - \frac{1}{4}$ and

$$(17) \qquad \tilde{P}^{\circ}(x) = O(x^{\alpha}), \quad \tilde{P}(x) = O(x^{\alpha}) .$$

From the proof of Theorem 2 we obtain (all the time we preserve the notation from the corresponding theorem and its proof) $P^{\circ}(x) = O(x^{\beta})$, where $\beta = \frac{r}{2} - 1 + \frac{r-3-2\alpha}{r-3-4\alpha}$ ; from (10) (derived without assuming $\sigma = 1$ ) and (11) we obtain

$$(18) \qquad \Delta_z A_{\wp}(x) = z^{\wp} V(x) + O(x^{\beta} z^{\wp}) .$$

However

$$|\Delta_z A_{\wp}(x) - z^{\wp} A(x)| = \left|\int_x^{x+z} \left[\int_{x_1}^{x_1+z} \cdots \int_{x_{\wp-1}}^{x_{\wp-1}+z} (A(x_{\wp}) - A(x))\, dx_{\wp}\right] dx_{\wp-1} \cdots \right] dx_1 \Big| \leq z^{\wp} \sum_{x < Q(m_j M_j + \ell_j) \leq x + \wp z} 1 =$$

$$= z^{\wp}(A^{\circ}(x+\wp z) - A^{\circ}(x)) << z^{\wp}(x^{r/2-1} z + x^{\beta}) << z^{\wp} x^{\beta}$$

and thus using (18)

$$P(x) = O(x^{\beta}) .$$

We procede analogously if (17) takes place with the symbols $\sigma$ , for $\alpha = \frac{r}{4} - \frac{1}{4}$ and or for Theorem 3.

In the papers [3] and [4] were - as well as some others - derived the following results:

Let $r > 4$ and let the form $Q$ have integer coefficients, let $b_1, b_2, \dots, b_r$ be integers, $M_1, M_2, \dots, M_r$ natural numbers. Then holds:

a) $$P(x) = O(x^{\frac{r}{2}-1}) .$$

b) If at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ is irrational, then

$$P(x) = o(x^{\frac{r}{2}-1}) .$$

c) For almost all systems $\alpha_1, \alpha_2, \dots, \alpha_r$ (in the sense of Lebesgue measure in r-dimensional Euclidean space $E_r$) there is

$$P(x) = O(x^{r/4+\varepsilon})$$

for every $\varepsilon > 0$.

d) If $\gamma$ is the supremum of all numbers $\beta > 0$, for which the inequalities

$$|\alpha_j M_j k - p_j| \leq k^{-\beta} , \quad j = 1, 2, \dots, r$$

have an infinite number of solution in integers $k > 0$, $p_1, p_2, \dots, p_r$, $f = (\frac{r}{4} - \frac{1}{2})\frac{2\gamma+1}{\gamma+1} + \frac{1}{2(\gamma+1)}$ )

(for $\gamma = +\infty$ let $f = r/2 - 1$ ) then for every $\varepsilon >$ $> 0$ holds the estimate

(19) $$P(x) = O(x^{f+\varepsilon}) .$$

e) Let $r > 5$, $\alpha_1 = \alpha_2 = \dots = \alpha_r$ and let $\gamma$ be the supremum of all numbers $\beta > 0$, for which the inequality

$$|\alpha_1 k - p| \leq k^{-\beta}$$

has an infinite number of solutions in integers $k > 0$, p ; $f = (\frac{r}{4} - \frac{1}{2})\frac{2\gamma+1}{\gamma+1}$ (for $\gamma = +\infty$ let $f = \frac{r}{2} - 1$ ). Then for every $\varepsilon > 0$ holds (19) and the value of f in in this estimate cannot be generally decreased: e.g. for

$b_1 = b_2 = \dots = b_r = 0$ we have for every $\varepsilon > 0$ also

$$P(x) = \Omega\,(x^{q-\varepsilon}) .$$

If we consider that for $t > 0$ is

$$(20)\quad A(x; Q, \alpha_j, b_j, M_j) = A(t^3 x; tQ, \frac{\alpha_j}{t}, t b_j, t M_j)$$

it is possible (we interchange $Q$, $\alpha_j$, $b_j$, $M_j$ and $\tilde{Q}$, $b_j$, $-\alpha_j$, $1/M_j$) from the assertions a) - d) derive the same estimates for the function $\tilde{P}(x)$ assuming that $\sigma = 1$, $M_1, M_2, \dots, M_r$ natural; $r > 4$ (for d) $r > 5$) and for forms $Q$ with integer coefficients [5] and thus using Theorem 2 or Theorem 3 [6] to prove the following results:

Theorem 4. Let $r > 4$, $\sigma = 1$ and let the coefficients of the form $Q$ be integers and $M_1, M_2, \dots, M_r$ natural numbers. Let at least one of numbers $b_1, b_2, \dots, b_r$ be irrational. Then

$$P(x) = \sigma\,(x^{\frac{r}{2} - \frac{r}{r+1}}) .$$

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5) According to (20) it is possible to generalize these assumptions.

6) Under the assumptions of Theorem 4 it is clear that $\tilde{\lambda}_n >> n$. According to (20) and to assertion a) is $B(x) = A(x; \tilde{Q}, 0, -\alpha_j, \frac{1}{M_j}) = c\,x^{\frac{r}{2}} + O(x^{\frac{r}{2}-1})$ and thus $B(\tilde{\lambda}_n) - B(\tilde{\lambda}_n -) = B(\tilde{\lambda}_n) - \lim_{\varepsilon \to 0+} B(\tilde{\lambda}_n - \varepsilon) << \tilde{\lambda}_n^{r/2-1}$. Herefrom we immediately obtain $B(\tilde{\lambda}_n) = c\,\tilde{\lambda}_n^{r/2} + O(\tilde{\lambda}_n^{r/2-1}) << n\,\tilde{\lambda}_n^{r/2-1}$ i.e. $\tilde{\lambda}_n << n$. We can conclusively use Theorem 3 and assertion c). Theorem 5 follows from assertion b) and Theorem 2; the consequences of assertions d) and e) are not explicitely presented.

Theorem 5. Let $r > 5$, $\sigma = 1$ and let the coefficients of the form $Q$ be integers, $M_1, M_2, \ldots, M_r$ natural numbers. Then for almost all systems $b_1, b_2, \ldots, b_r$ (in the sense of Lebesgue measure in the r-dimensional Euclidean space $E_r$) is

$$P(x) = O(x^{r/3 + \varepsilon})$$

for every $\varepsilon > 0$.

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