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# PRINCIPAL DUAL IDEALS IN LATTICES OF PRIMITIVE CLASSES

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Consider a type $\Delta$ of universal algebras and the lattice $\mathcal{L}_\Delta$ of all primitive classes of algebras of type $\Delta$. J. Rebane [2] has shown that if $\Delta$ contains at least one at least unary operation, then each proper principal dual ideal $J$ of $\mathcal{L}_\Delta$ is infinite. It will be shown in the present paper that if $\Delta$ contains either at least two unary operations or at least one at least binary operation, then each $J$ is uncountable. (It will follow that if, in addition, $\Delta$ is finite, then each $J$ has exactly $2^{\aleph_0}$ elements; the continuum hypothesis is not used here.) Let us remark that if $\Delta$ consists of one unary and a finite number of nullary operations, then (as it is shown in [1]) $\mathcal{L}_\Delta$ and hence each $J$ is countable; if $\Delta$ consists of one unary and an infinite number of nullary operations, then it is easy to prove that $\mathcal{L}_\Delta$ contains both countable and uncountable proper principal dual ideals.

Some terminology will be given in § 1. However, the reader is supposed to know the definitions and fundamental properties of absolutely free algebras and primitive classes. See Słomiński [3].

## § 1. Lattices of primitive classes

By a type we mean an arbitrary family $\Delta = (n_i)_{i \in I}$ of non-negative integers. Let us make a convention: if a type is denoted by $\Delta$, then its definition set is denoted by $I$ and the integer corresponding to $i \in I$ by $n_i$.

Algebra of type $\Delta$ is a set $A$ together with a family $(f_i)_{i \in I}$ where $f_i$ is an $n_i$-ary operation in $A$. We call $f_i$ the $i$-th fundamental operation of this algebra. If $n_i = 0$, then $f_i$ is simply an element of $A$.

Let us fix an infinitely countable set $X$; its elements are called variables. For each type $\Delta$ let us fix an absolutely free algebra $W_\Delta$ of type $\Delta$ freely generated by $X$. If $i \in I$, then the $i$-th fundamental operation of $W_\Delta$ is denoted by $f_i$.

Let us define a set $S(w)$ for each $w \in W_\Delta$: if $w \in X$, then $S(w) = \{w\}$; if $i \in I$, $w = f_i(w_1, \ldots, w_{n_i})$, then $S(w) = \{w\} \cup S(w_1) \cup \ldots \cup S(w_{n_i})$. The elements of $S(w)$ are called subwords of $w$. It is easy to prove that if $w_1$ is a subword of $w_2$, then $\varphi(w_1)$ is a subword of

$\varphi(w_2)$ for any endomorphism $\varphi$ of $W_\Delta$.

Let us define a non-negative integer $\varkappa(w)$ for each $w \in W_\Delta$: if $w \in X$ or $w = f_i$ for some $i \in I$, $n_i = 0$, then $\varkappa(w) = 0$; if $i \in$ $\in I$, $n_i \neq 0$, $w = f_i(w_1, \ldots, w_{n_i})$, then $\varkappa(w) = 1 + \varkappa(w_1) + \ldots$ $\ldots + \varkappa(w_{n_i})$. It is easy to prove $\varkappa(w) \leq \varkappa(\varphi(w))$ for any endomorphism $\varphi$ of $W_\Delta$.

By a $\Delta$-equation we mean an ordered pair $\langle w_1, w_2 \rangle$ of elements of $W_\Delta$. By a $\Delta$-theory we mean any set of $\Delta$-equations, i.e. any binary relation in $W_\Delta$. A $\Delta$-equation $e$ is identified with the $\Delta$-theory $\{e\}$. A $\Delta$-equation $\langle w_1, w_2 \rangle$ is called trivial if $w_1 = w_2$.

By a fully invariant congruence relation (shortly: FI-congruence relation) of $W_\Delta$ we mean a congruence relation $E$ such that $\langle w_1, w_2 \rangle \in E$ implies $\langle \varphi(w_1), \varphi(w_2) \rangle \in E$ for any endomorphism $\varphi$ of $W_\Delta$.

Lemma 1. Let $m$ be a non-negative integer. The set of all $\Delta$-equations $\langle w_1, w_2 \rangle$ such that either $w_1 = w_2$ or $\varkappa(w_1) \geq m$ & $\varkappa(w_2) \geq m$ is a FI-congruence relation of $W_\Delta$.

The proof is evident.

For any $\Delta$-theory $E$, the least FI-congruence relation of $W_\Delta$ containing $E$ is denoted by $Cn(E)$.

We shall write $E_1 \vdash E_2$ instead of $E_2 \subseteq \subseteq Cn(E_1)$.

The set of all FI-congruence relations of $W_\Delta$ is a complete lattice with respect to the set-theoretic inclusion. The dual of this lattice is denoted by $\mathcal{L}_\Delta$. This is the set of all FI-congruence relations of $W_\Delta$ with the relation $\leq_\Delta$ defined by $E_1 \leq_\Delta E_2$ if and only if $E_2 \subseteq E_1$.

A $\Delta$-equation $\langle w_1, w_2 \rangle$ is called valid in an algebra $A$ of type $\Delta$ if $\varphi(w_1) = \varphi(w_2)$ for all homomorphisms $\varphi$ of $W_\Delta$ into $A$. If $E$ is a $\Delta$-theory, then $\mathcal{M}od(E)$ denotes the primitive class of all algebras of type $\Delta$ in which all equations from $E$ are valid. If $\mathcal{O}$ is a class of algebras of type $\Delta$, then $\mathcal{E}q(\mathcal{O})$ denotes the set of all $\Delta$-equations that are valid in each $A \in \mathcal{O}$.

The following three properties are well-known:

i) If $E_1$ and $E_2$ are two different elements of $\mathcal{L}_\Delta$, then the primitive classes $\mathcal{M}od(E_1)$ and $\mathcal{M}od(E_2)$ are different, too.

ii) Any primitive class of algebras of type $\Delta$ can be expressed as $\mathcal{M}od(E)$ for some $E \in \mathcal{L}_\Delta$.

iii) If $E_1, E_2 \in \mathcal{L}_\Delta$, then $E_1 \leq_\Delta E_2$ if and only if $\mathcal{M}od(E_1) \subseteq \mathcal{M}od(E_2)$.

This shows that the name "lattice of primitive classes" for $\mathcal{L}_\Delta$ is available.

Let us denote by $\iota_\Delta$ the greatest element of $\mathcal{L}_\Delta$.

If $E \in \mathcal{L}_\Delta$, then the set of all $H \in \mathcal{L}_\Delta$ such that $E \leq_\Delta H$ is called the principal dual ideal (of $\mathcal{L}_\Delta$) generated by $E$. It is called proper if $E \neq \iota_\Delta$.

## § 2. The uncountability of proper principal dual ideals of $\mathcal{L}_\Delta$ for large types $\Delta$

Let us call a type $\Delta$ large if either

(1) $n_i \leq 1$ for all $i \in I$; there exist two different elements $i_1, i_2$ of $I$ such that $n_{i_1} = n_{i_2} = 1$ or

(2) there exists an $i_1 \in I$ such that $n_{i_1} \geq 2$.

In my paper [1] it is shown that for each finite type $\Delta$, the lattice $\mathcal{L}_\Delta$ is uncountable if and only if $\Delta$ is large. Here we shall prove this

__Theorem__. Let $\Delta$ be a large type. Then each proper principal dual ideal of $\mathcal{L}_\Delta$ is uncountable. Moreover, it contains a subset which (considered as partially ordered by $\leq_\Delta$) is isomorphic to the lattice of all subsets of an infinite set.

First a definition. A $\Delta$-theory $E$ is called separated if it is infinite and $Cn(E_1) = Cn(E_2)$ implies $E_1 = E_2$ for all $E_1, E_2 \subseteq E$.

It is easy to prove that if $E$ is separated, then the mapping $\varphi$ defined by $\varphi(E_1) = Cn(E - E_1)$ is an order-isomorphism of the lattice of all subsets of $E$ onto a subset of the principal dual ideal of $\mathcal{L}_\Delta$ generated by $Cn(E)$.

We have further evidently: if $H \in \mathcal{L}_\Delta$, $H \neq \iota_\Delta$, then there exists in $H$ at least one non-trivial equation $e$, and it is $H \vdash e$.

Hence, to prove the Theorem, it is enough to prove that for each non-trivial $\Delta$-equation $e$ there exists a separated $\Delta$-theory $E$ such that $e \vdash E$. This will be proved in the following two lemmas.

Lemma 2. If a type $\Delta$ satisfies (1), then for each non-trivial $\Delta$-equation $e$ there exists a separated $\Delta$-theory $E$ such that $e \vdash E$.

Proof. The elements $i \in I$ such that $n_i = 1$ are called unary symbols. If $s = s_1 s_2 \dots s_n$ is a finite (not necessarily non-empty) sequence of unary symbols and $w \in W_\Delta$, then $w^s$ is defined in this way: if $s$ is empty, then $w^s = w$; further, $w^{s_1 \dots s_n s_{n+1}} = f_{s_{n+1}}(w^{s_1 \dots s_n})$. The special unary symbols $i_1$ and $i_2$ (see (1)) are denoted by |

and $+$ , respectively. We shall denote by $\overset{n}{+}$ the sequence consisting of $n$ symbols $+$ .

Let $N$ be the set of all positive integers. Put $e = \langle u, v \rangle$ , so that $u \neq v$ . For each $n \in N$ put $e_n = \langle u^{1\overset{n}{+}1}, v^{1\overset{n}{+}1} \rangle$ . For each $M \subseteq N$ let $E_M$ be the set of all $e_m$ with $m \in M$ . Put $E = E_N$ . We have evidently $e \vdash E$ .

Put $H = Cn(e)$ . Let us define a relation $R_M$ in $W_\Delta$ for each $M \subseteq N$ in this way: $\langle w_1, w_2 \rangle \in R_M$ if and only if either $w_1 = w_2$ or there exists an equation $\langle u_1, u_2 \rangle \in H$ , a number $m \in M$ and a finite (not necessarily non-empty) sequence $s$ of unary symbols such that $w_1 = u_1^{1\overset{m}{+}1s}$ and $w_2 = u_2^{1\overset{m}{+}1s}$ . Let us prove that $R_M$ is a FI-congruence relation of $W_\Delta$ . It is evidently enough to prove transitivity. Let $\langle w_1, w_2 \rangle \in R_M$ and $\langle w_2, w_3 \rangle \in R_M$ . If $w_1 = w_2$ or $w_2 = w_3$ , then $\langle w_1, w_3 \rangle \in R_M$ evidently. In the opposite case there exist equations $\langle u_1, u_2 \rangle \in H$ , $\langle v_2, v_3 \rangle \in H$ , numbers $m, n \in M$ and sequences $s, \bar{s}$ such that $w_1 = u_1^{1\overset{m}{+}1s}$, $w_2 = u_2^{1\overset{m}{+}1s} = v_2^{1\overset{n}{+}1\bar{s}}$, $w_3 = v_3^{1\overset{n}{+}1\bar{s}}$ .

It follows from the expression of $w_2$ that either $|\overset{m}{+}|s$ is an end of $|\overset{n}{+}|\overline{s}$ or $|\overset{n}{+}|\overline{s}$ is an end of $|\overset{m}{+}|s$. We shall consider the first case; the second could be handled similarly. There exists a sequence $t$ such that $|\overset{n}{+}|\overline{s}$ is equal to $t\,|\overset{m}{+}|s$. Clearly $\langle v_2^t, v_3^t \rangle \in H$ and $v_2^t = u_2$; we get $\langle u_1, v_3^t \rangle \in H$. As $w_1 = u_1^{|\overset{m}{+}|s}$ and $w_3 = v_3^{t|\overset{m}{+}|s}$, we get $\langle w_1, w_3 \rangle \in R_M$. The assertion on $R_M$ is thus proved. We have evidently $R_M \supseteq E_M$ and hence $R_M \supseteq Cn(E_M)$.

To prove the Lemma, it is evidently enough to prove that if $n \in N - M$, then $e_n \notin Cn(E_M)$. Suppose on the contrary that $e_n \in C_n(E_M)$; we get $e_m \in R_M$. There exists an equation $\langle u_1, v_1 \rangle \in H$, a number $m \in M$ and a sequence $s$ such that $u^{|\overset{n}{+}|} = u_1^{|\overset{m}{+}|s}$ and $v^{|\overset{n}{+}|} = v_1^{|\overset{m}{+}|s}$. We shall go on under the assumption $\kappa(u_1) \leq \kappa(v_1)$; in the contrary case the proof would be analogous. Evidently $\kappa(u) \leq \kappa(v)$, too. As $\langle u_1, v_1 \rangle \in Cn\langle u, v \rangle$ and $u_1 \neq v_1$, we get $\kappa(u_1) \geq$

$\geq r(u)$ evidently applying Lemma 1. From this and from $u^{|f^n|} = u_1^{|f^m|s}$ it follows easily that $s$ is empty, $n = m$ and $u = u_1$.
But $n = m$ is in a contradiction with the assumption $n \notin M$.

**Lemma 3.** If a type $\Delta$ satisfies (2), then for each non-trivial $\Delta$-equation $e$ there exists a separated $\Delta$-theory $E$ such that $e \vdash E$.

**Proof.** Let us fix an $i_1 \in I$ with $n_{i_1} \geq 2$ and put $n_{i_1} = k$. If $w_1, w_2 \in W_\Delta$, then put $w_1 \cdot w_2 = f_{i_1}(w_1, w_2, \dots, w_2)$. If $w_1, \dots, w_n \in W_\Delta$, then the product $w_1 \dots w_n$ is defined in this way: if $n = 1$, it is equal to $w_1$; if $n > 1$, then $w_1 \dots w_n = (w_1 \dots w_{n-1}) \cdot w_n$. If $w_1 = \dots = w_n = w$, we write $w^n$ instead of $w_1 \dots w_n$.

Put $e = \langle u, v \rangle$, so that $u \neq v$. Let $N$ be the set of all integers $n \geq 2$. Let us fix a variable $x$. For each $n \in N$ put $e_n = \langle u \cdot x^n, v \cdot x^n \rangle$. For each $M \subseteq N$ let $E_M$ be the set of all $e_m$ with $m \in M$. Put $E = E_N$.

We have evidently $e \vdash E$.

Let a set $M \subseteq N$ be given. A finite sequence $e^{(1)}, \dots, e^{(h)}$ of $\Delta$-equations is called proof (with respect to $M$) if for each $j = 1, \dots, h$ one of the following cases takes place:

i) $e^{(j)}$ is trivial;

ii) there exists an $m \in M$ and an endomorphism $\varphi$ of $W_\Delta$ such that either $e^{(j)} = \langle \varphi(u \cdot x^m), \varphi(v \cdot x^m) \rangle$ or $e^{(j)} = \langle \varphi(v \cdot x^m), \varphi(u \cdot x^m) \rangle$;

iii) there exists an $i \in I$ and a sequence $\langle w_1, \overline{w_1} \rangle, \dots \langle w_{n_i}, \overline{w_{n_i}} \rangle$ of $\Delta$-equations such that all these equations occur among $e^{(1)}, \dots, e^{(j-1)}$ and $e^{(j)} = \langle f_i(w_1, \dots, w_{n_i}), f_i(\overline{w_1}, \dots, \overline{w_{n_i}}) \rangle$;

iv) there exist two equations $\langle w_1, w_2 \rangle$ and $\langle w_2, w_3 \rangle$ among $e^{(1)}, \dots, e^{(j-1)}$ such that $e^{(j)} = \langle w_1, w_3 \rangle$.

Let $R_M$ be the set of all those $\Delta$-equations that occur as the last member of a proof (with respect to $M$). It is easy to see that $R_M$ is a FI-congruence relation of $W_\Delta$, so that evidently

$R_M = Cn(E_M)$.

To prove the Lemma, it is evidently enough to prove that if $n \in N - M$, then $e_n \notin Cn(E_M)$. Suppose on the contrary that $e_n \in R_M$, so that $e_n$ is the last member of a proof (with respect to $M$ )` $e^{(1)}, \ldots, e^{(h)}$. We may suppose $\kappa(u) \leq \kappa(v)$; in the contrary case the proof would be analogous. We can not receive $e^{(h)}$ applying only the rules i) and iv); hence, there exists a $j \leq h$ and a $w \in W_\Delta$ such that $w \neq u \cdot x^n$ and $e^{(j)} = \langle u \cdot x^n, w \rangle$ and such that $e^{(j)}$ can be got applying ii) or iii). (In the case $\kappa(u) \geq \kappa(v)$ we would seek $e^{(j)}$ in the form $\langle w, v \cdot x^n \rangle$.) Suppose that $e^{(j)}$ can be got by iii). There exist elements $w_1, w_2, \ldots, w_k \in W_\Delta$ such that $w = f_{i_1}(w_1, w_2, \ldots, w_k)$ and
$\langle u, w_1 \rangle \in R_M$, $\langle x^n, w_2 \rangle \in R_M$, ...
..., $\langle x^n, w_k \rangle \in R_M$.

Let us call an element $t \in W_\Delta$ special if it has a subword $f_{i_1}(t_1, \ldots, t_k)$ where $t_k$ is not a variable. Evidently, we get a FI-congruence relation if we take the set of all those $\Delta$-equations $\langle t, \bar{t} \rangle$ such that either $t = \bar{t}$ or $t$ and $\bar{t}$ are both special. As $E_M$ and hence $R_M = Cn(E_M)$ is contained in this FI-congruence relation and as

$x^n = f_{i_1}(x^{n-1}, x, \ldots, x)$, $x^{n-1} =$

$= f_{i_1}(x^{n-2}, x, \ldots, x), \ldots,$

$x^2 = f_{i_1}(x, x, \ldots, x)$,

we get $x^n = w_2 = \ldots = w_k$. As $\kappa(u) \leq \kappa(v)$,

$\kappa(u \cdot x^m) > \kappa(u)$ for all $m \in M$ and $\langle u, w_1 \rangle \in R_M$, we get $u = w_1$ easily by Lemma 1. We get $w = u \cdot x^n$, a contradiction. Hence, $e^{(j)}$ is as in ii). We have either $u \cdot x^n = \varphi(u \cdot x^m) =$

$= \varphi(u) \cdot (\varphi(x))^m$ or

$u \cdot x^n = \varphi(v \cdot x^m) = \varphi(v) \cdot (\varphi(x))^m$;

in both these cases $x^n = (\varphi(x))^m$. As $n$, $m \geq 2$, we have $f_{i_1}(x^{n-1}, x, \ldots, x) =$

$= f_{i_1}((\varphi(x))^{m-1}, \varphi(x), \ldots, \varphi(x))$

and hence $x = \varphi(x)$ . From $x^n = x^m$ we get evidently $n = m \in M$ , a contradiction.

The Theorem is thus proved.

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