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SOME REMARKS ON THE SPECTRA AND NUMERICAL RANGE

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If A is bounded linear operator on an Hilbert space H then the numerical range of A is by definition the set

$$W(A) = \{ \langle Ax, x \rangle , \quad \| x \| = 1 \}$$

and it is known that its closure $\overline{W(A)}$ is a convex set and contains the spectrum $\sigma(T)$ of T. The purpose of this Note is to present some new results about connection of spectra and numerical range and also new proofs for known results strongly related to these.

1. In this section we consider operators on an Hilbert space H.

Theorem 1.1. If T is any operator such that

1) $ST^{p} S^{-1} = T^{*p}$, p an integer ≥ 1,

2) $0 \notin \overline{W(S)}$ and $1 + (\frac{\bar{\lambda}}{\mu}) + \dots + (\frac{\bar{\lambda}}{\mu})^{p-1} \neq 0$ for $\lambda, \mu \in \sigma(T) - \{0\}$

then the spectrum of T is real.

Proof. It is clear that it will suffice to consider only boundary points. Let $\lambda_0 \in \partial\sigma(T)$, $\operatorname{Im} \lambda_0 \neq 0$. We find a sequence of unit vectors $\{x_n\}$ such that

$$T x_n - \lambda_o x_n \longrightarrow 0$$

and thus

$$T^p x_n - \lambda_o^p x \longrightarrow 0$$

or

$$S^{-1} ST^p x_n - \lambda_o^p S^{-1} S x_n \rightarrow 0$$

and thus

$$S^{-1} (T^{*p} S) x_n - \lambda_o^p S^{-1} S x_n \rightarrow 0$$

which implies that

$$(T^{*p} - \lambda_o^p) S x_n \longrightarrow 0$$

and condition 2^o and the relation

$$(T^* - \lambda_o)(T^{*p-1} + \dots + \lambda_o^{p-1}) S x_n \longrightarrow 0$$

leads to the fact

$$(T^* - \lambda_o) S x_n \longrightarrow 0 .$$

As in [9] we obtain that $\operatorname{Im} \lambda_o \neq 0$ produces a contradiction. For completeness we give a proof. We consider

$$(\lambda_o - \overline{\lambda}_o)\langle S x_n, x_n\rangle| = |\langle (T^* - \lambda_o) S x_n, x_n\rangle -$$

$$- \langle (T^* - \lambda_o) S x_n, x_n\rangle|$$

which clearly tends to zero. But $0 \overline{\in} W(S)$ and this completes the proof.

Remarks. 1) For $p = 1$ the theorem was proved by J. Williams [9].

2) For the case when T is a hyponormal operator it is in [2].

Corollary. If T is an operator such that the conditions of theorem 1 are true and $\overline{W(T)} \doteq$ convex closure $\sigma(T)$ then T is a selfadjoint operator.

The corollary contains results about selfadjointness of operators similar to their adjoints [5],[6].

2. In this section we give now proofs for some known results as an application of the following theorem [8] th. 1.

Theorem. If A and B are operators on the complex Hilbert space H, and if $0 \notin \overline{W(A)}$, then

$$\sigma(A^{-1}B) \subset \frac{\overline{W(B)}}{\overline{W(A)}} .$$

Theorem 2.1. If A is a bounded linear operator on a Hilbert space and W(A) a subset of real numbers then A is a selfadjoint operator.

Proof. We may suppose, without loss of generality, that A is an ivertible operator and thus A = UP where U is a cramped unitary operator and P is a strictly positive.

From the theorem

$$\sigma(P^{-1}T^*) = \sigma(U^{-1}) \subset \frac{\overline{W(T)}}{\overline{W(P)}} \in R^1$$

which implies that U is equal to I or - I.

Theorem 2.2. If S is a selfadjoint operator and $STS^{-1} = T^*$ then the spectrum of T is real.

Proof. From the theorem

$$\sigma(T^*) = \sigma(S^{-1}TS) \subset \frac{\overline{W(TS)}}{\overline{W(S)}} \subset R^1 .$$

Remark. Theorem 2.2 furnishes a proof of theorem 1.1 in the case p = 1 since from theorem 2 of [8] we can consider S to be a selfadjoint operator.

Theorem 2.3. If y is a selfadjoint operator, r is a strictly positive and ryr + y = 0 then necessarily y = 0.

Proof. It is enough to prove that (y) = 0.

Since $(ry)^2 = - y^2$ from a consequence of a theorem ([8]p.24) $\sigma(ry)$ is nonnegative and by right member of the above equality is nonpositive, it follows that only 0 is in the spectrum of y .

3. In this section we consider the case of elements of a Banach algebra with identity [7].

For an element x of a Banach algebra B the numerical range is the set defined by

$$W_o(x) = \{ f(x) , \quad f \in \beta \}$$

where

$$\beta = \{ f, f \in B^*, \; f(x) = 1 = \| f \| \} .$$

It is known [7] that W_o is a convex closed set which contains the spectrum $\sigma(x)$ of x .

The following represents a translation to this case of theorem in section 2.

__Theorem 3.1__. If a,b are elements of B and if o $W_o(a)$ then

$$\sigma(a^{-1} b) \subset \frac{W_o(b)}{W_o(a)} .$$

__Proof__. Exactly as for the theorem.

We do not insist upon further translation to this case or when B is an involutive Banach algebra of results presented in the section 1 and 2.

Finally, I am indebted to Professor J. Williams for the possibility to see his papers before publication.

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