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ON CONSTRUCTING A DISTRIBUTIVE LATTICE FROM A PARTIALLY ORDERED SET WITH DCC

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1. Introduction. It is well-known that an arbitrary partially ordered set (hereafter, poset) may be imbedded isomorphically in a distributive lattice, and this may be done in several different ways [3; 6]. The choice of a method of imbedding for a particular purpose will depend on such considerations as whether one is interested in completeness of the lattice, in efficiency (in the sense of not making the lattice unnecessarily large), in preserving some property of the poset (such as a chain condition), etc. In this note we are principally interested in imbeddings which preserve local finiteness. The reason for this interest is an application made in [7], where it is shown that an imbedding of one locally finite poset in another induces an imbedding of their incidence algebras, and incidence algebras of distributive lattices are the "nicest" kind.

The kind of imbedding best suited to our task (the "crown" construction) is the basis for Birkhoff's representation theory for distributive lattices with descending chain condition (DCC). Section 2 summarizes this theory, with references to [2] for some of the details. There is nothing essentially new here, except perhaps the presentation.

Section 3 deals with the locally finite case, and includes the principal theorem giving several equivalent sufficient conditions for being able to imbed a poset with DCC in a locally finite distributive lattice. The final section contains some remarks indicating the relationship (or lack of it) of the crown construction to other methods of imbedding posets in lattices.

2. Crowns and distributive lattices with DCC. Let P be an arbitrary poset, and let S be an arbitrary subset of P. We denote by S^* (respectively, S^+) the set of upper (respectively, lower) bounds of S in P. If S is a unit subset $\{x\}$, we will write x^* and x^+ for S^* and S^+, respectively. S is called M-closed if $x^+ \subseteq S$ for all $x \in S$. For any $x \in P$, x^+ is called the principal ideal generated by x. A subset J of P is called an (order) ideal [5] if $F^{*+} \subseteq J$ for every finite subset F of J. It is clear that a principal ideal x^+ is an ideal, and indeed is the smallest ideal containing x. It is also clear that an ideal is an M-closed subset.

A subset C of P is called a crown [2] if C is M-closed and has a finite set F of maximal elements such that every element of C is dominated by an element of F. If this is the case, F is clearly an incomparable subset of P and $C = \bigcup_{x \in F} x^+$. It is also clear that every finite incomparable subset F of P is the set of maximal elements of a unique crown, which we denote $\bar{F}$. Note that principal ideals are crowns, but a crown need not be an ideal.

We denote by $L(P)$ the set of all crowns of P , ordered by set inclusion. This poset is obviously isomorphic to the set of finite incomparable subsets of P , ordered by the relation: $F_1 \leq F_2$ if and only if for each $x \in F_1$ there exists $y \in F_2$ such that $x \leq y$. We will identify these isomorphic posets and use whichever representation of $L(P)$ is convenient.

<u>Lemma 1</u> [2,p.182]. If P satisfies DCC, then so does $L(P)$.

<u>Lemma 2</u>. $L(P)$ is an upper (join) semi-lattice, and if it is a lattice, it is distributive.

<u>Proof</u>. The union of the crowns $\bar{F}_1$ and $\bar{F}_2$ is the crown determined by the set of maximal elements in $F_1 \cup F_2$. The intersection of crowns (indeed of M-closed subsets) is clearly M-closed. Any candidate for $\bar{F}_1 \wedge \bar{F}_2$ must, of course, be contained in $\bar{F}_1 \cap \bar{F}_2$. We claim that if $F_1 \cap F_2$ is not a crown, then it contains no largest crown. It follows that if $L(P)$ is a lattice, it is a sublattice of the lattice of subsets of P , hence is distributive. To establish the claim, we observe that, if $\bar{F}$ is a crown properly contained in $\bar{F}_1 \cap \bar{F}_2$, and if $x \in (\bar{F}_1 \cap \bar{F}_2) - \bar{F}$, then the set of maximal elements in $F \cup \{x\}$ determines a crown properly containing $\bar{F}$.

In [1,p.142] (but not in the later edition [2]) it is asserted that if P has DCC , then $L(P)$ is a distributive lattice with DCC. The following example shows that this need not be so. Let $P = A \cup \{x,y\}$, where A is an infinite set, and the order in P is defined by well-ordering A and letting each of x and y be an upper bound for A. P clearly satisfies DCC. The only incomparable subsets are the one-element subsets and $\{x,y\}$. Thus $L(P)$ is obtained from P by adjoining a largest element, and hence is clearly not a lattice.

<u>Theorem 1</u>. The following are equivalent for an arbitrary poset P :

(a) $L(P)$ is a distributive lattice.

(b) The intersection of crowns in P is a crown.

(c) The intersection of principal ideals in P is a crown.

<u>Proof</u>. The equivalence of (a) and (b) follows from the proof of Lemma 2, and it is clear that (b) implies (c). Suppose P satisfies (c), and let $\bar{F}_1$ and $\bar{F}_2$ be arbitrary crowns in P. An element $z \in \bar{F}_1 \cap \bar{F}_2$ if and only if $z \in \{x,y\}^+ = x^+ \cap y^+$ for some $x \in F_1$, $y \in F_2$. Thus $\bar{F}_1 \cap \bar{F}_2 = \bigcup_{x \in F_1} (\bigcup_{y \in F_2} (x^+ \cap y^+))$. By assumption, $x^+ \cap y^+$ is a crown, and the finite union of crowns is a crown. Hence (c) implies (b).

For convenience, we will say a poset P is (<u>lower</u>) <u>crowned</u> if it satisfies any, hence all, of the conditions of Theorem 1. Condition (c) may be stated in terms of elements as follows: for arbitrary $x, y \in P$, $\{x,y\}^+$ has a finite set of maximal elements which dominate every element.

Henceforth, let L denote a distributive lattice satisfying DCC, and let $P(L)$ denote the poset of join-irreducible [2,p.58] elements of L. Clearly, $P(L)$ also satisfies DCC. It is known [2,pp.181-183] that every element of L has a unique representation as an irredundant join of a finite number of join-irreducibles. Thus the representation theory for distributive lattices with a chain condition is summarized by the following theorem. (Note, in particular, that $P(L)$ is a crowned poset.)

<u>Theorem 2</u> (Birkhoff). If L is a distributive lattice with DCC, then $L(P(L))$ is a distributive lattice isomorphic to L.

Conversely, if P is a crowned poset with DCC, Lemma 1 and Theorem 1 assert that $L(P)$ is a distributive lattice with DCC. P is clearly isomorphic to the subset of $L(P)$ consisting of one-element subsets of P (or of principal ideals), and these are clearly the join-irreducible elements of $L(P)$.

<u>Corollary</u>. There is a one-to-one correspondence between distributive lattices L with DCC and crowned posets P with DCC. Under this correspondence $P \cong P(L)$ and $L \cong L(P)$.

3. <u>Preserving local finiteness</u>. A poset is <u>locally finite</u> if every interval $[x,y] = \{z \mid x \leq z \leq y\}$ is finite. We observe that if P is locally finite (even with DCC), $L(P)$ may not be, whether it is a lattice or not. First suppose $P = A \cup \{x\}$, where A is an infinite

totally unordered set and x is an upper bound for A. Then P is obviously crowned, locally finite, and has DCC. $L(P)$ is just the lattice of all finite subsets of A with a unit element 1 ($= \{x\}$) adjoined, and the interval $[0,1] = L(P)$ is clearly not finite. Next suppose $P = A \cup \{x,y\}$, where A is as above, and x and y are unrelated upper bounds for A. Then P is locally finite and has DCC, but $\{x,y\}^+$ is not a crown. In this case $L(P)$ is neither locally finite nor a lattice.

Since P is always imbedded isomorphically in $L(P)$, an obviously necessary condition for $L(P)$ to be locally finite lattice with DCC is that any subset A of P with an upper bound x be finite; for, in $L(P)$, A would be a subset of the interval $[0,x]$. This condition is also sufficient.

<u>Theorem 3</u>. The following are equivalent for an arbitrary poset with DCC:

(a) $L(P)$ is a locally finite distributive lattice.

(b) Every subset of P bounded above is finite.

(c) Every principal ideal in P is finite.

(d) Every crown in P is finite.

(e) P is locally finite, and every incomparable subset of P bounded above is finite.

<u>Proof</u>. We have just observed that (a) implies (b). (b) implies (c) trivially. Since a crown is a finite union of principal ideals, (c) implies (d). An interval $[x,y]$ is a subset of y^+, and a subset A bounded above by z is a subset of z^+; since principal ideals are crowns,

(d) implies (e).

Now suppose (e) holds, and let A be an arbitrary subset which has an upper bound y in P. By DCC, every element of A dominates a minimal element of A. The set of minimal elements of A is an incomparable set bounded above by y, hence is a finite set $\{x_1, \dots, x_n\}$. Then $A \subseteq \bigcup_{i=1}^{n} [x_i, y]$, a union of finite intervals, so (b), (c), and (d) hold. It follows that P is crowned, since the intersection of principal ideals is finite and M-closed, therefore a crown. By Theorem 1, L(P) is a distributive lattice. To show that L(P) is locally finite, it suffices to show that every interval $[0,C]$ is finite, where C is an arbitrary crown. (The zero element of L(P) is the empty crown.) But this follows from (d): C has only a finite number of subsets, hence dominates only finitely many crowns. Thus (e) implies (a), and the proof is complete.

A poset P with DCC satisfying any, hence all, of the conditions of Theorem 3 will be called <u>strongly locally finite</u>. We remark that the equivalence of conditions (b), (c), and (d), and the fact that (e) follows from any one of these, make no use of DCC.

<u>Corollary 1</u>. There is a one-to-one correspondence between locally finite distributive lattices L with DCC and strongly localy finite posets P with DCC. Under the correspondence $L \longrightarrow P(L)$ and $P \longrightarrow L(P)$, we have $L \cong L(P(L))$ and $P \cong P(L(P))$.

Corollary 2. If P is a locally finite poset and is a dual crown (i.e. has finitely many minimal elements, and every element dominates one of these), then $L(P)$ is a locally finite distributive lattice with DCC.

Proof. The hypotheses clearly imply P has DCC. For any $x \in P$, x^+ is the union of finitely many finite intervals, hence is finite.

Corollary 3. If P is a locally finite poset with 0, then $L(P)$ is a locally finite distributive lattice with DCC.

The special cases of strongly locally finite posets in Corollaries 2,3,and 4 are not typical, for at the other extreme we have the example of an arbitrary totally unordered set P, for which $L(P)$ is the lattice of finite subsets of P, ordered by inclusion.

4. Some remarks concerning other imbeddings of partially ordered sets in distributive lattices.[1)] We have already remarked that there are many ways to imbed a poset P in a lattice, and perhaps we should comment on why the crown construction was used exclusively above. For example, P can be represented isomorphically as the set of principal ideals in the complete distributive lattice of all M-closed subsets of P [3], in the (smaller) comple-

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1) The author wishes to thank a referee for another journal who read an earlier version of this paper, and whose suggestions led to the addition of this section, as well as to some improvements in the preceding sections.

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te lattice of ideals [5] in the (still smaller) lattice of <u>normal ideals</u> (subsets S of P such that $S^{*+} = S$), and sometimes the sublattice of the normal ideals generated by the principal ideals will be smaller still. (See [5] for further remarks and references on other imbeddings of posets in lattices.)

Since we were principally interested in imbeddings in distributive lattices, none of the lattices just described would be generally appropriate except the lattice of M-closed subsets. (For example, let P be either 5-element non-distributive lattice. Then every ideal is principal, and the lattice of ideals (normal ideals, principal ideals) is P itself. In any finite lattice, the M-closed subsets coincide with the crowns.) Our main objective was an imbedding theorem which involved preservation of local finiteness, and for this purpose there would obviously be no point in considering complete lattices. Indeed, the essential difference between the crown construction and the M-closed subset construction is the "finitary" character of the former.

However, one may reasonably ask if there is an "infinitary" analogue to the results of section 2, perhaps for some class of posets and complete distributive lattices with DCC. The following facts suggest this possibility: (a) A distributive lattice with DDC is either complete or becomes so by adjoining a unit element. (b) In any complete lattice with DCC, every element is a (possibly infiniteú join of completely join irreducibles. (For a much more general statement, see [4, Corollary to Lemma 2.5].) (c) One may easily characterize those posets P with DCC for which the lattice of M-closed subsets also has DCC : this is the case if and only if every incomparable subset of P is finite.

The attempt to proceed any farther with an analogy to section 2 fails for the following reasons: (d) The principal ideals need not be completely join irreducible elements in the lattice of M-closed subsets. (e) Even in a complete distributive lattice with DCC, there need not be a unique irredundant representation of an element as a join of completely join irreducibles. (f) Non-isomorphic complete distributive lattices with DCC may have the same poset of completely join irreducibles.

All of these difficulties are illustrated by the following simple example. Let N denote the chain of non-negative integers and $\bar{N} = N \cup \{\infty\}$ its completion by adjoining a unit. Let L denote the complete distributive lattice with DCC obtained by adjoining a unit to $\bar{N} \times N$. L is the lattice of M-closed subsets of the poset $P = (\bar{N} \times \{0\}) \cup$ $\cup (\{0\} \times N)$ of its join irreducibles. But the element $(\infty, 0)$ is not completely join irreducible, and has infinitely many representations as a join of completely join irreducibles, none of which is irredundant. Furthermore, the completion L' of $N \times N$ is a complete distributive lattice with DCC having the same set of completely join irreducibles as L, but not isomorphic to L.

R e f e r e n c e s

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