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ON THE DIFFERENTIABILITY OF OPERATORS AND CONVEX FUNCTIONALS

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Introduction. This paper is a continuation of our considerations [1 - 4] concerning the differentiability of operators and convex functionals.

Theorem 1 establishes sufficient conditions under which the Gâteaux derivative $F'(0)$ of a mapping F at 0 is the Fréchet derivative. This result can be useful for instance in branching theory. It is shown (Th.2) that for convex subadditive functional f (under some further assumptions) the existence of the Fréchet differential $df(0,h)$ at 0 and the Gâteaux differential $Vf(x,h)$ in some open convex neighbourhood $U(0)$ of 0 imply the existence of the Fréchet derivative $f'(x)$ on $U(0)$. Theorem 3 concerns with so-called weak one-sided Lipschitz condition, while Theorem 4 gives some sufficient conditions for continuity of a linear functional f by means of properties of a convex functional g. For the recent results in these topics see the bibliography cited in [1 - 4].

1. Notations and definitions. Let X,Y be real linear normed spaces, X^*, Y^* their duals, $F: X \to Y$ a

mapping of X into Y. We shall use the symbols "$\longrightarrow$", "$\xrightarrow{w}$" to denote the strong and weak convergence in X, Y. Then

a) F is said to be strongly continuous at x_0 if $x_n \xrightarrow{w} x_0$ implies $F(x_n) \longrightarrow F(x_0)$.

b) a functional f is said to be weakly continuous at x_0 if $x_n \xrightarrow{w} x_0$ implies $f(x_n) \longrightarrow f(x_0)$.

c) $F: X \longrightarrow Y$ is called compact on a set $M \subseteq X$ if for every bounded subset $N \subset M$ the set $F(N)$ is compact in Y.

d) A functional f defined on a convex open subset $M \subseteq X$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for each $x, y \in M$ and $\lambda \in \langle 0, 1 \rangle$.

For the Gâteaux and Fréchet differentials and derivatives we shall use the notions and notations given in [5, chapt.I]. By $V_+ f(x_0, h)$ we mean the one-sided Gâteaux differential of a real function f at x_0. Through this paper we shall assume that functionals f, $V_+ f(x, h)$ are finite. $D(0, R)$ denotes the closed ball with the radius $R > 0$ and the center 0.

2. We shall prove the following

Theorem 1. Let X,Y be linear normed spaces, X reflexive, $F: X \longrightarrow Y$ a mapping of X into Y having at 0 the Gâteaux derivative $F'(0)$. Assume that $F'(0)$ is compact. If either a) F is strongly continuous on $D(0,1)$ and for each $u, v \in D(0, 1)$ and real λ

$$\| F(\lambda u) - F(\lambda v) \| = |\lambda| \; \| F(u) - F(v) \|$$

or b) F is bounded on $D(0, 1)$ and for each u, $v \in D(0, 1)$ and real λ

$$\| F(\lambda u) - F(\lambda v) \| = |\lambda|^p \; \| F(u) - F(v) \|$$

with $p > 1$, then F possesses the Fréchet derivative $F'(0)$ at 0.

Proof. Let h be an arbitrary (but fixed) element of X. By our hypothesis for given $\varepsilon > 0$ there exists a number $\delta_1(\varepsilon, h) > 0$ such that

$$(1) \qquad \left\| \frac{1}{t} \omega(0, th) \right\| < \varepsilon$$

whenever $0 < |t| < \delta_1$, where

$$\omega(0, th) = F(th) - F(0) - F'(0)th .$$

To prove our theorem we need to show that the numbers $\delta_1(\varepsilon, h)$ have a positive lower bound $\delta(\varepsilon)$ for any $h \in X$ with $\| h \| = 1$ and that (1) is valid for these h. Suppose contrary, there exist a positive number ε_0 and sequences $\{h_n\} \in X$ with $\| h_n \| = 1$ $(n = 1, 2, \dots)$, $\{t_n\}$ with $0 < |t_n| < \frac{1}{n}$ such that

$$(2) \qquad \left\| \frac{1}{t_n} \omega(0, t_n h_n) \right\| > \varepsilon_0 .$$

Since X is reflexive and $\{h_n\}$ is bounded, passing to a subsequence $\{h_{n_k}\}$ we have that $h_{n_k} \xrightarrow{w} h_0$. Being $D(0, 1)$ weakly closed, $h_0 \in D(0, 1)$. For given ε_0, $h_0 \in X$ there exists a positive constant $\delta_2(\varepsilon_0, h_0)$ such that if $0 < |t| < \delta_2$, then

(3) $$\left\| \frac{1}{t} \omega(0, t h_0) \right\| < \frac{\varepsilon_0}{3} .$$

Since $\{h_{n_k}\}$ is a subsequence of $\{h_n\}$ then there exists t_{n_k} with $0 < |t_{n_k}| \leq \frac{1}{n_k}$ such that

(4) $$\left\| \frac{1}{t_{n_k}} \omega(0, t_{n_k} h_{n_k}) \right\| > \varepsilon_0 .$$

We shall show that this conclusion leads to a contradiction. By our hypothesis

(5) $$F(t_{n_k} h_{n_k}) - F(0) = F'(0) t_{n_k} h_{n_k} + \omega(0, t_{n_k} h_{n_k}) ,$$

$$F(t_{n_k} h_0) - F(0) = F'(0) t_{n_k} h_0 + \omega(0, t_{n_k} h_0) .$$

Hence

(6) $$\omega(0, t_{n_k} h_{n_k}) = F(t_{n_k} h_{n_k}) - F(t_{n_k} h_0) + \\ + t_{n_k} F'(0)(h_0 - h_{n_k}) + \omega(0, t_{n_k} h_0) .$$

Assuming a) we have that

(7) $$\left\| \frac{1}{t_{n_k}} \omega(0, t_{n_k} h_{n_k}) \right\| \leq \| F(h_{n_k}) - F(h_0) \| + \\ + \| F'(0)(h_0 - h_{n_k}) \| + \left\| \frac{1}{t_{n_k}} \omega(0, t_{n_k} h_0) \right\| .$$

Since $h_{n_k} \xrightarrow{w} h_0$ as $k \to \infty$, $h_{n_k}, h_0 \in D(0,1)$ and F is strongly continuous on $D(0,1)$, $F(h_{n_k}) \to \to F(h_0)$ as $k \to \infty$. Furthermore, $F'(0)$ as a linear continuous operator from X into Y is weakly continuous, i.e. $F'(0) h_{n_k} \xrightarrow{w} F'(0) h_0$. But $F'(0) D(0,1)$ is compact set in Y and weak convergence in compact set gives a strong one (see [5], Lemma 4.1, p.68). Hence $F'(0)(h_0 - h_{n_k}) \to 0$ as $k \to \infty$. The third term on the right side of (7) tends to zero for $t_{n_k} \to 0$ as $k \to \infty$ and F has the Gâteaux derivative $F'(0)$ at 0 . Hence

$$\| \frac{1}{t_{n_k}} \omega(0, t_{n_k} h_{n_k}) \| \to 0$$

as $k \to \infty$ and this is a contradiction with (4). Assuming b), according to (6) it is sufficient to show that

$$\frac{1}{|t_{n_k}|} \| F(t_{n_k} h_{n_k}) - F(t_{n_k} h_0) \| \to 0$$

whenever $k \to \infty$. But the desired conclusion follows at once from the following relations:

$$|t_{n_k}|^{-1} \| F(t_{n_k} h_{n_k}) - F(t_{n_k} h_0) \| \leqq$$

$$\leqq |t_{n_k}|^{\alpha} (\| F(h_{n_k}) \| + \| F(h_0) \|) \leqq 2C |t_{n_k}|^{\alpha} \to 0$$

as $k \to \infty$ for $t_{n_k} \to 0$ as $k \to \infty$, $\alpha > 0$ and C is a constant from the boundedness of F on $D(0,1)$. Now proceeding as above, we obtain a contradiction with (4). This concludes the proof.

Corollary 1. Let X be a reflexive linear normed space, f a functional on X having at 0 the Gâteaux derivative $f'(0)$. If either a) f is weakly continuous on $D(0,1)$ and for each $h \in D(0,1)$ and real λ $f(\lambda h) = |\lambda| f(h)$, or b), f is bounded on $D(0,1)$ and for real λ $f(\lambda h) = |\lambda|^p f(h)$ with $p > 1$, then f possesses at 0 the Fréchet derivative $f'(0)$.

Corollary 1 follows immediately from Theorem 1 if we aware that the Gâteaux derivative $f'(0)$ as an element of X^* is weakly continuous. Theorem 1 can be useful for instance in branching theory. It is well-known [5] that the points of bifurcation of completely continuous operator F (under further special conditions on F) may be only the eigenvalues of the Fréchet derivative $F'(0)$ of F at 0.

Let X,Y be linear normed spaces, $F: X \to Y$ a mapping of X into Y. The following result is due to M.M. Vajnberg [5,Th.3.3]: If there exists the Gâteaux derivative $F'(x)$ of F in some neighbourhood $U(x_0)$ of $x_0 \in X$ and this derivative is continuous at x_0 in the norm of the space $(X \to Y)$ of all linear continuous operations from X into Y, then F possesses the Fréchet derivative $F'(x_0)$ at x_0.

Now we shall prove that for convex subadditive functional f (with some further properties) the existence of the Gâteaux differential $Vf(x, h)$ in some neighbourhood $U(0)$ of 0 and the Fréchet differential $df(0, h)$ at 0 imply the existence of the Fréchet derivative $f'(x)$ on $U(0)$. More exactly we have the following

Theorem 2. Let X be a reflexive linear normed space, f a convex subadditive functional on X such that f is upper-bounded on some convex open subset $M \neq \emptyset$ of X and $f(0) = 0$. Assume f possesses the Gâteaux differential $Vf(x, h)$ for each x, $x \neq 0$ of some open convex neighbourhood $U(0)$ of 0 and that there exists the Fréchet differential $df(0, h)$ of f at 0. Then f possesses the Fréchet derivative $f'(x)$ on $U(0)$.

Proof. Continuity of f follows at once from Theorem 2 [6,II,§ 5]. Convexity of f implies that $Vf(x, h) = Df(x, h)$ for each $x \in U(0)$ and every $h \in X$.

According to Proposition 6 [7] $\mathcal{D}f(x,h) = f'(x,h)$ for each $x \in U(0)$ and every $h \in X$, where $f'(x)$ denotes the Gâteaux derivative of f at x. By our hypothesis, $df(0,h)$ exists and hence f possesses the Fréchet derivative $f'(0)$ at 0. Suppose there does not exist the Fréchet derivative $f'(x)$ at some $x \in U(0)$, $x \neq 0$. We proceed as in the proof of Theorem 1. In relations (1),(2),(3),(4) write x for 0, f for F and the remainder in (1) replace by

$$\omega(x,th) = f(x+th) - f(x) - f'(x)th .$$

Since the one-sided Gâteaux derivative $V_+ f(x,h)$ is equal to $f'(x)h$ and f is convex, we deal here only with a sequence $\{t_n\}$ of positive numbers. The elements h_0, $\{h_n\}_{n=1}^{\infty}$ and the sequence $\{t_n\}$ have there the same meaning as in proof of Theorem 1. Instead (5) we have

$$(8)\ f(x+t_{n_k}h_{n_k}) - f(x) = f'(x)t_{n_k}h_{n_k} + \omega(x,t_{n_k}h_{n_k}),$$
$$f(x+t_{n_k}h_0) - f(x) = f'(x)t_{n_k}h_0 + \omega(x,t_{n_k}h_0).$$

By convexity of f and in view of Lemma 2 [3]

$$(9)\qquad \omega(x,t_{n_k}h_{n_k}) \geqq 0,\quad \omega(x,t_{n_k}h_0) \geqq 0$$

for each k $(k = 1,2,\dots)$. Again in view of subadditivity and convexity of f we have that

$$(10)\qquad f(x+t_{n_k}h_{n_k}) - f(x) \leqq f(t_{n_k}h_{n_k})$$

and

$$(11)\qquad f(x) - f(x+t_{n_k}h_0) \leqq f(x-t_{n_k}h_0) - f(x) \leqq$$
$$\leqq f(-t_{n_k}h_0).$$

Hence from (8),(9),(10),(11) one obtains that

(12) $0 \leq \omega(x, t_{n_k} h_{n_k}) \leq f(t_{n_k} h_{n_k}) + f(-t_{n_k} h_0) +$
$$+ f'(x) t_{n_k}(h_0 - h_{n_k}) + \omega(x, t_{n_k} h_0) .$$

Since $f(0) = 0$ and f is Fréchet-differentiable at 0,

(13) $f(t_{n_k} h_{n_k}) = f'(0) t_{n_k} h_{n_k} + \omega(0, t_{n_k} h_{n_k})$,
$$f(-t_{n_k} h_0) = -f'(0) t_{n_k} h_0 + \omega(0, -t_{n_k} h_0) .$$

From (12) and (13) it follows that
$$0 \leq \frac{1}{t_{n_k}} \omega(x, t_{n_k} h_{n_k}) \leq f'(0)(h_{n_k} - h_0) +$$
$$+ f'(x)(h_0 - h_{n_k}) + \frac{1}{t_{n_k}} \omega(x, t_{n_k} h_0) + \frac{1}{t_{n_k}} \omega(0, t_{n_k} h_{n_k}) +$$
$$+ \frac{1}{t_{n_k}} \omega(0, -t_{n_k} h_0) .$$

Since $h_{n_k} \xrightarrow{w} h_0$ and $f'(0)$, $f'(x)$ are weakly continuous ($f'(0)$, $f'(x)$ belong to X^*), $f'(0)(h_{n_k} - h_0) \to 0$, $f'(x)(h_0 - h_{n_k}) \to 0$ as $k \to \infty$. By our hypothesis f has the Gâteaux derivative $f'(x)$ on $U(0)$ (see the first part of this proof) and thus
$$\frac{1}{t_{n_k}} \omega(x, t_{n_k} h_0) \to 0, \quad \frac{1}{t_{n_k}} \omega(0, -t_{n_k} h_0) \to 0$$
whenever $k \to \infty$, for $t_{n_k} \to 0$. The term $\frac{1}{t_{n_k}} \omega(0, t_{n_k} h_{n_k})$ tends to zero as $k \to \infty$ in view of the existence of the Fréchet derivative $f'(0)$ of f at 0 and the fact that $t_{n_k} \to 0$ as $k \to \infty$ and $\| h_{n_k} \| = 1$. Hence
$$\frac{1}{t_{n_k}} \omega(x, t_{n_k} h_{n_k}) \to 0$$
as $k \to \infty$. We have obtained a contradiction. Thus f possesses the Fréchet derivative $f'(x)$ on $U(0)$.

This concludes the proof.

Corollary 2. Let X be a reflexive linear normed space, f a subadditive positive homogeneous (i.e. $f(\lambda x) = \lambda f(x)$ for any $\lambda \geq 0$ and $x \in X$) functional on X such that f is upper bounded on some open convex subset $M \neq \emptyset$ of X. Moreover, suppose f possesses the Gâteaux differential $Vf(x, h)$ for each x, $x \neq 0$ of some open convex neighbourhood $U(0)$ of 0 and the Fréchet differential $df(0, h)$ at 0. Then f has the Fréchet derivative $f'(x)$ on $U(0)$.

Remark 1. If a functional f defined on a Banach space X is either a) upper-semicontinuous at some point $x_0 \in X$ or b) lower-semicontinuous on X, then there exists an open ball D and a constant N such that f is upper bounded on D by the number N. The assertion a) follows at once from definition of upper-semicontinuity of f at x_0, while b) follows immediately from Theorem [8, p. 31]. Recall that a reflexive linear normed space is a Banach (reflexive) space.

Now we shall deal with so-called weak one sided Lipschitz condition (compare [5], chapt. I). We make first

Definition. We shall say that a convex functional f defined on a linear normed space X satisfies the condition (A) at $x_0 \in X$ if for each $h \in X$ with $\|h\| = 1$ there exists a number $\sigma(h) > 0$ such that

$$f(x_0 + th) + f(x_0 - th) - 2f(x_0) \leq Ct\|h\|$$

whenever $0 < t < \sigma(h)$, where the constant C does not depend on $h \in X$ ($\|h\| = 1$).

A functional f is said to satisfy a weak one-sided Lipschitz condition at $x_0 \in X$ if for each $h \in X$ with $\|h\| = 1$ there exists a number $\delta(h) > 0$ such that if $0 < t < \delta(h)$ there is

$$|f(x_0 + th) - f(x_0)| \leq N t \|h\| ,$$

where the constant $N > 0$ does not depend on $h \in X$ $(\|h\| = 1)$.

Theorem 3. Let X be a linear normed space, f a convex functional on X satisfying the condition (A) at $x_0 \in X$. Let one of the following three conditions be fulfilled: a) f is continuous at x_0 ; b) $V_+ f(x_0, h)$ is upper bounded on some open convex subset $M \neq \emptyset$ of X ; c) X is complete and $V_+ f(x_0, h)$ is lower-semicontinuous on X . Then f satisfies a weak one-sided Lipschitz condition at x_0 .

Proof. Since f is convex, $V_+ f(x_0, h)$ is subadditive and positive homogeneous [9] and hence convex on X . Assuming b) and using Theorem 2 [6,II,§ 5] we see that $V_+ f(x_0, h)$ is continuous on X . But continuity of this mapping implies the boundedness of $V_+ f(x_0, h)$ in some neighbourhood of 0 . Now the positive homogeneity of $V_+ f(x_0, h)$ implies that there exists a constant $C_1 > 0$ such that

$$|V_+ f(x_0, h)| \leq C_1 \|h\| . \tag{14}$$

The case c) we transfer to b), see remark 1. Assume a), $V_+ f(x_0, h)$ satisfies (14) by Theorem 8a) [3]. Set

$$\varphi(x_0, t, h) = f(x_0 + th) + f(x_0 - th) - 2f(x_0)$$

for $t > 0$ and $h \in X$. Then

(15) $f(x_0+th)-f(x_0)=\varphi(x_0,t,h)+f(x_0)-f(x_0-th)$.

By our hypothesis for each $h \in X$ with $\|h\|=1$ there exists a number $\sigma(h)>0$ such that if $0<$ $<t<\sigma(h)$, then

(16) $$\varphi(x_0,t,h) \leqq C t \|h\| .$$

By (15),(16) and (14) and according to lemma 2 [3]

$$f(x_0+th)-f(x_0) \leqq C t \|h\| + |V_+ f(x_0,th)| \leqq$$

$$\leqq N t \|h\| \; ; \quad N = C + C_1$$

if $0<t<\sigma(h)$ and h is an arbitrary (but fixed) element of X with $\|h\|=1$. On the other hand, by lemma 2 [3] and (14)

$$f(x_0+th)-f(x_0) \geqq V_+ f(x_0,th) \geqq -C_1 t \|h\| .$$

Hence

$$|f(x_0+th)-f(x_0)| \leqq N t \|h\|$$

whenever $0<t<\sigma(h)$ and $\|h\|=1$. This concludes the proof.

Remark 2. We shall say that a functional f has one-sided symmetric differential $V_+^s f(x_0,h)$ at $x_0 \in X$ if there exists for arbitrary (but fixed) $h \in X$ the limit

$$\lim_{t \to 0_+} \frac{1}{t}\left(f(x_0+th)-f(x_0-th)\right) = V_+^s f(x_0,h) .$$

For convex functional f the one-sided symmetric differential $V_+^s f(x,h)$ always exists for every $x \in X$. Moreover, if $V_+^s f(x_0,h) = V_+ f(x_0,h)$ for every $h \in X$, where f is a convex functional, then f possesses a linear Gâteaux differential $Df(x_0,h)$ at x_0. Thus, if $V_+^s f(x_0,h) = V_+ f(x_0,h)$ for every $h \in X$ and f is for instance continuous at x_0, then f possesses the Gâteaux derivative $f'(x_0)$ at x_0.

Theorem 4. Let X be a linear normed space, f a linear functional on X. Suppose there exists a convex functional g such that for some $x_0 \in X$ $f(x_0) = g(x_0)$ and $f(x) \leq g(x)$ for every $x \in X$. Then f is continuous on X if one of the following three conditions is fulfilled: a) g is continuous at x_0 ; b) $V_+ g(x_0, h)$ is upper bounded on some convex open subset $M \neq \emptyset$ of X ; c) X is complete and $V_+ g(x_0, h)$ is lower-semicontinuous on X.

Proof. Let $h \in X$ and $t > 0$. Then

$$g(x_0)+tf(h)=f(x_0)+tf(h)=f(x_0+th) \leq g(x_0+th).$$

Hence

$$f(h) \leq V_+ g(x_0, h), \quad h \in X. \tag{17}$$

Furthermore,

$$f(h) = -f(-h) \geq -V_+ g(x_0, -h) \tag{18}$$

for every $h \in X$. The inequalities (17),(18) and lemma 2 [3] give

$$g(x_0) - g(x_0-h) \leq -V_+ g(x_0,-h) \leq f(h) \leq$$

$$\leq V_+ g(x_0, h) \leq g(x_0+h) - g(x_0)$$

for every $h \in X$. Assuming a) the continuity of g at x_0 implies continuity of f at $h = 0$. Being f linear, f is continuous on X. For the cases b),c) we proceed as in the beginning of the proof of Theorem 3. This completes the proof.

Remark 3. From the assumptions of Theorem 4 [7] it follows that f is continuous everywhere in X (and not only on the open ball B_R). The same assertion follows at once from the conclusion of Corollary 1 [4]. The result of Proposition 1 [4] one may rewrite as follows: if f is a convex functional on a linear normed space X , then f possesses a linear Gâteaux differential $Df(x_0, h)$ at $x_0 \in X$ if and only if f is directionally smooth at x_0 (see [4]). Hence Theorems 2,3 [4] and the result of Ivanov [10] imply the following assertions:

(a) If X is a linear separable normed space, f a convex functional on X such that f is upper bounded on some open convex subset $M \neq \emptyset$ of X , then the set P of all $x \in X$ where f is directionally smooth is a $F_{\sigma\sigma}$ -set. The same conclusion is valid if X is a separable Banach space and f a convex lower-semicontinuous functional on X .

(b) If f is convex and Lipschitzian in a separable Banach space, then the set P of all $x \in X$ where f is directionally smooth is a $F_{\sigma\sigma}$ -set of the second category in X .

(c) Let X be a linear normed space with $\dim X < \infty$, f a convex functional on X such that f is directionally smooth at $x_0 \in X$ and Lipschitzian in some convex neighbourhood of x_0 Then f has the Fréchet derivative $f'(x_0)$ at x_0 .

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