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ON FULL EMBEDDINGS OF CONCRETE CATEGORIES WITH RESPECT TO FORGETFUL FUNCTORS x)

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Introduction.

Every semigroup with unity S^1 is isomorphic to the semigroup of all the mappings of a set X into itself preserving a suitable binary relation R. On the other hand, if we have a semigroup S^1 of mappings of X into itself, there is rarely a binary relation R on X such that the R-preserving mappings are exactly the elements of S^1. Thus, the problem of a representation of semigroups of mappings by binary relations without full omitting the concrete forms of the semigroups gives rise naturally to the following question:

(1) Let S^1 be a semigroup of mappings of X into itself. Does there exist a $Y \supset X$ and a binary relation R on Y such that the semigroup of all the mappings of Y into itself preserving R consists exactly of (uniquely determined) extensions of the elements of S^1?

(Similarly, instead of binary relations, we may consider algebraic structures of a given type etc.)

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More generally, investigating full embeddings of concrete categories (by a concrete category $(\mathfrak{K}, \square)$ we mean a category $\mathfrak{K}$ together with a firmly given forgetful functor $\square$), we see that the condition of preserving the carrying sets of objects and the carrying mappings of morphisms (i.e. that the full embedding is a realisation, see [6]) may be, even for simplest cases, rarely satisfied. In the present paper we investigate two kinds of embeddings (pseudorealisation and strong embedding, see Definition 1.1) under which the carrying sets are only augmented by additional elements and the carrying mappings are extensions of the original ones. By the strong embedding, we require moreover, roughly speaking, that this expansion depends on the carrying sets only, not on the objects themselves.

In the first paragraph, the definitions are given and some fundamental properties are proved. It is shown, that we may often obtain a pseudorealisation by means of a construction from a full embedding. In particular, we obtain a positive answer to the question (1) (see 1.12.3)) for binary relations and other structures.

Paragraph 2 contains a construction which is, later on, used to prove that the category of graphs $\mathcal{R}$ may be strongly imbedded into the category of undirected graphs $\mathcal{R}_{s}$ (the full embedding of $\mathcal{R}$ into $\mathcal{R}_{s}$ constructed in [2] is not a strong embedding). Lemma 2.5 is formulated substantially stronger than necessary for the present paper; this formulation will allow some other applications which shall appear elsewhere.

In paragraph 3, first of all, the strong embeddability of $\mathcal{R}(\Delta)$ (see Notation) into $\mathcal{R}_b$ is proved. As a consequence (by means of lemma 3.3, which is, in fact, a reformulation of Theorem 5 from [3]) we obtain strong embeddings of several other categories into $\mathcal{R}_b$. In particular, we formulate a necessary and sufficient condition for a strong embeddability of small concrete categories into $\mathcal{R}$. Finally, in the last paragraph 4, we show that the existence of a strong embedding into categories of quasialgebras is equivalent with the existence of a strong embedding into $\mathcal{R}$ (while, see 3.5, the existence of a strong embedding of a concrete category into a category of algebras is a substantially stronger property).

Notation: An ordinal, in particular a natural number, is always considered as the set of all smaller ordinals. We use the Gödel-Bernays set theory. For some statements, we require, moreover, that in the theory the following assumption holds:

(M) There exists a cardinal σ such that every σ-additive two-valued measure is γ-additive for any cardinal γ.

(I.e., roughly speaking, there are not too many measurable cardinals.)

A functor, mapping the category $\mathcal{S}$ of all sets and all mappings into itself is called a set functor. A definition of the TB-functor may be found in [7]. A transformation $\mu : F \to G$ where F,G are functors from a category into $\mathcal{S}$ is said to be a monotransformation if all the mappings μ^a are one-to-one.

A graph is a couple (X,R), where X is a set and R is a binary relation on X. An undirected graph is a graph (X,R) such that R is symmetrical. If (X,R) and (Y,S) are graphs and f a mapping of X into Y, we say that f is compatible (more exactly, RS-compatible) if it preserves the relations, i.e., if $(f(x), f(y)) \in S$ whenever $(x, y) \in R$. More generally, if r is an A-relation on X (i.e. a set of mappings of A into X) and s is an A-relation on Y, $f : X \to Y$ is said to be rs-compatible if $f \circ \alpha \in s$ whenever $\alpha \in r$. A graph (X,R) is said to be rigid, if there is no non-identical RR-compatible mapping.

A type $\Delta = (\alpha_\beta)_{\beta < \gamma}$ is a sequence of ordinals indexed by ordinals, $\sum \Delta$ is the usual ordinal sum of the sequence. A relational system r of the type Δ on X is a sequence $(r_\beta)_{\beta < \gamma}$, where r_β is an α_β-relation on X. If r, s are relational systems of the type Δ, a mapping f is said to be rs-compatible, if it is $r_\beta s_\beta$-compatible for every $\beta < \gamma$. The category of all sets with relational systems and their compatible mappings is denoted by $\mathcal{R}(\Delta)$. The symbol $\mathcal{O}(\Delta)$ ($\mathcal{Q}(\Delta)$ resp.) designates the category of all algebras (quasialgebras, resp.) of a type Δ and all their homomorphisms. (For a more detailed description of $\mathcal{R}(\Delta)$, $\mathcal{O}(\Delta)$, $\mathcal{Q}(\Delta)$ see e.g. [1]).

$S(F)$, where F is a set functor, is a category, the objects of which are couples (X,r) with $r \subset F(X)$ and morphisms from (X,r) into (Y,s) the mappings $f : X \to Y$

for which $F(f)(\kappa) \subset \lambda$ if F is covariant, $F(f)(\lambda) \subset \kappa$ if F is contravariant. (See also e.g. [6]).

The categories $\mathcal{R}$, $\mathcal{R}_{\lambda}$, $\mathcal{R}(\Delta)$, $\mathcal{Q}(\Delta)$, $\mathcal{O}\!\mathcal{t}(\Delta)$, $S(F)$ are always treated as concrete categories, endowed by the obvious forgetful functor (this is, as a rule, denoted by $\square$).

A faithful set functor F is said to be selective, if for every type Δ there are a type Δ' and a one-to-one functor Φ mapping $\mathcal{R}(\Delta)$ onto a full subcategory of $\mathcal{R}(\Delta')$ so that $\square \circ \Phi = F \circ \square$. (See [3]).

If X,Y are sets, we write $X \vee Y = X \times \{0\} \cup \cup\, Y \times \{1\}$ (the disjoint union of X and Y). If there is no danger of confusion, we write, for the elements of $X \vee Y$, simply x instead of (x,0), y instead of (y,1). $\langle X, Y \rangle$ is the set of all mappings of X into Y.

§ 1. Generalities

1.1. Definition: Let $(\mathfrak{K}, \square)$, $(\mathfrak{K}', \square')$ be concrete categories. A full embedding $\Phi : \mathfrak{K} \to \mathfrak{K}'$ is said to be a pseudorealization if for every object a of $\mathfrak{K}$ there is a set Z(a) such that

$$\square'\Phi(a) = \square a \cup Z(a), \quad Z(a) \cap \square a = \emptyset ,$$

and if for every morphism $\varphi : a \to b$ and every $x \in \square a$

$$\square'\Phi(\varphi)(x) = \square\varphi(x) .$$

A full embedding $\Phi : \mathfrak{K} \to \mathfrak{K}'$ is said to be a strong embedding if there is a faithful set functor F such that the diagram

$$\begin{array}{ccc} \mathfrak{K} & \xrightarrow{\ \Phi\ } & \mathfrak{K}' \\ \downarrow{\scriptstyle \square} & & \downarrow{\scriptstyle \square'} \\ \mathcal{S} & \xrightarrow{\ F\ } & \mathcal{S} \end{array}$$

commutes.

1.2. Remarks: 1) The realization (see [6]) is a particular case of both pseudorealization ($Z(a) = \emptyset$ for any a) and strong embedding (F = I) .

2) A composition of pseudorealizations (strong embeddings) is obviously a pseudorealization (a strong embedding).

1.3. Definition: We say that a concrete category ($\mathfrak{K}$, $\square$) has the property of transfer (shortly, (T)) if for every object a of $\mathfrak{K}$ and for every one-to-one mapping f of $\square a$ onto an arbitrary set X there is an isomorphism φ in $\mathfrak{K}$ such that $\square\varphi = f$.

We say that a concrete category ($\mathfrak{K}$, $\square$) has the property of unicity (shortly, (U)) , if every isomorphism φ of $\mathfrak{K}$ such that $\square\varphi = id$ is an identity.

1.4. Remarks: 1) Obviously, S(F) , $\mathfrak{X}(\Delta)$, $\mathcal{G}(\Delta)$, $\mathcal{A}(\Delta)$ have both (T) and (U) .

2) If ($\mathfrak{K}$, $\square$) has (T) and ($\mathfrak{K}'$, $\square'$) is a concrete subcategory of ($\mathfrak{K}$, $\square$) containing with any object all the isomorphic objects, then ($\mathfrak{K}'$, $\square'$) has (T) .

3) If ($\mathfrak{K}$, $\square$) has (U) and if there exists a strong embedding $\Phi : (\mathfrak{K}', \square') \longrightarrow (\mathfrak{K}, \square)$ then ($\mathfrak{K}'$, $\square'$) has (U) . Really, denote by F the associated set functor. Let φ be an isomorphism in $\mathfrak{K}$, $\square\varphi = id$. Then $\Phi(\varphi)$ is an isomorphism and $\square\Phi(\varphi) = F\square\varphi = id$,

so that $\Phi(\varphi)$ is an identity. Thus, φ is an identity.

1.5. Lemma: Let $(\mathfrak{K}, \Box)$, $(\mathfrak{K}', \Box')$ be concrete categories. Let $(\mathfrak{K}, \Box)$ have (U) and $(\mathfrak{K}', \Box')$ have (T). Let $\Phi : \mathfrak{K} \to \mathfrak{K}'$ be a faithful functor such that for every $\varphi : \Phi(a) \to \Phi(b)$ there is a $\psi : a \to b$ with $\varphi = \Phi(\psi)$. Let there exist a monotransformation $\mu : \Box \to \Box' \cdot \Phi$.

Then there exists a pseudorealization $\Psi : (\mathfrak{K}, \Box) \to \to (\mathfrak{K}', \Box')$. If always $\mu^a(\Box a) \neq \Box' \cdot \Phi(a)$, the assumption of (U) may be left out.

Proof: Put $Z(a) = (\Box'\Phi(a) - \mu^a(\Box a)) \times \{(a, \Box a)\}$ and define $\varkappa_a : \Box' \cdot \Phi(a) \to \Box a \cup Z(a)$ by $\varkappa_a(\mu^a(x)) = x$, $\varkappa_a(y) = (y, a, \Box a)$ otherwise. Evidently, $\varkappa_a$ is one-to-one.

Since $(\mathfrak{K}', \Box')$ has (T), there are isomorphisms $\alpha_a : \Phi(a) \to \Psi(a)$ with $\Box'\alpha_a = \varkappa_a$ (and, hence, $\Box'\Psi(a) = \Box a \cup Z(a)$).

Put, for every morphism $\varphi : a \to b$ in $\mathfrak{K}$,

$$\Psi(\varphi) = \alpha_b \cdot \Phi(\varphi) \cdot \alpha_a^{-1} .$$

Thus, we defined a functor $\Psi : \mathfrak{K} \to \mathfrak{K}'$. For $\varphi : a \to b$ and $x \in \Box a$ we have $\Box'\Psi(\varphi)(x) = \varkappa_b \cdot \Box'\Phi(\varphi) \cdot \varkappa_a^{-1}(x) = \varkappa_b \Box'\Phi(\varphi) \cdot \mu^a(x) = \varkappa_b \mu^b \Box\varphi(x)$.

If $\psi : \Psi(a) \to \Psi(b)$ is a morphism, we obtain $\alpha_b^{-1} \cdot \psi \cdot \alpha_a : \Phi(a) \to \Phi(b)$ and hence $\alpha_b^{-1} \psi \alpha_a = \Phi(\varphi)$ for some $\varphi : a \to b$. Thus, it remains to prove that Ψ is one-to-one.

Ψ is evidently faithful. If $\Psi(a) = \Psi(b)$, we have $\Box a \cup Z(a) = \Box'\Psi(a) = \Box b \cup Z(b)$. If both Z(a) and

Z(b) are non-void, we have, for some x, $(x, a, \Box a) \in$ $\in \Box b \cup Z(b)$ and, for some y, $(y, b, \Box b) \in$ $\in \Box a \cup Z(a)$. If $a \neq b$, we obtain $(x, a, \Box a) \in \Box b$, $(y, b, \Box b) \in \Box a$ in a contradiction with the set theory.

Anyway, if $\Psi(a) = \Psi(b)$, there are (see above) $\alpha :$ $: a \to b$ and $\beta : b \to a$ with $\Psi(\alpha) = \Psi(\beta) = id_{\Psi(a)}$. Thus, $\Psi(\alpha\beta) = \Psi(\beta\alpha) = id_{\Psi(a)}$ and, since Ψ is faithful, $\beta\alpha = id_a$, $\alpha\beta = id_b$. Consequently, by the isomorphism α, we see easily that in the case of $Z(a) = \emptyset$ we also have $Z(b) = \emptyset$ and hence $\Box a =$ $= \Box b$. If $x \in \Box a$, $\Box\alpha(x) = \Box' \Psi(\alpha)(x) = x$. Thus, $\Box\alpha = id_a$ and hence, finally (by (U)), a = b.

1.6. <u>Remarks</u>: 1) If there exists a pseudorealization Φ' of $(\mathfrak{K}', \Box')$ in itself with non-void Z(a) for every object a, we may leave out the assumption of (U) in 1,5. It suffices to use $\Phi'\Phi$ instead of Φ.

The existence of such Φ' for $\mathcal{O}\!\iota(\Delta)$, $\mathcal{G}(\Delta)$ and $\mathcal{R}$ follows by the constructions in [1], for $\mathcal{R}_s$ by the composition of the trivial embedding of $\mathcal{R}_s$ into $\mathcal{R}$ and the construction in [2] (or, the construction which will be described in §§ 2,3).

Thus, applying 1.5 and its immediate corollaries for the mentioned categories in the role of $(\mathfrak{K}', \Box')$, we need not assume (U) for $(\mathfrak{K}, \Box)$.

2) By lemma 1.5, to prove the existence of a pseudorealization, it suffices to find a faithful functor with the required property. Similarly, for the strong embedding we have

Theorem: Let $(\mathfrak{K}, \square)$ have (U), $(\mathfrak{K}', \square')$ have (T). Let there be a faithful functor $\Phi : \mathfrak{K} \to \mathfrak{K}'$ such that for every two objects a,b of $\mathfrak{K}$ and for every morphism $\varphi : \Phi(a) \to \Phi(b)$ there is a $\psi : a \to b$ with $\varphi = \Phi(\psi)$. Let $\square' \cdot \Phi = F \cdot \square$ for some faithful set functor F. Then there exists a strong embedding of $(\mathfrak{K}, \square)$ into $(\mathfrak{K}', \square')$.

Proof: First, define a set functor G by $G(X) = F(X) \times \{X\}$ for sets X, $G(f)(x, X) = (F(f)(x), Y)$ for mappings $f : X \to Y$. Obviously, G is faihtul. Define $k_X : F(X) \to G(X)$ by $k_X(x) = (x, X)$. By (T), for every object a of $\mathfrak{K}$ there is an isomorphism $\alpha_a : \Phi(a) \to \Psi(a)$ such that $k_{\square a} = \square' \alpha_a$.

For a morphism $\varphi : a \to b$ put $\Psi(\varphi) = \alpha_b \cdot \Phi(\varphi) \cdot \alpha_a^{-1}$. Thus, we obtain a faithful functor $\Psi : \mathfrak{K} \to \mathfrak{K}'$.

Similarly as in the proof of 1.5 we see that for every $\varphi : \Psi(a) \to \Psi(b)$ there is a $\psi : a \to b$ with $\varphi = \Psi(\psi)$. Since we have always $\square'(\alpha_b \cdot \Phi(\varphi) \cdot \alpha_a^{-1}) = k_{\square b} \cdot F(\square \varphi) \cdot k_{\square a}^{-1} = G(\square \varphi)$, it remains to show that Ψ is one-to-one.

Let $\Psi(a) = \Psi(b)$. Thus, $F(\square a) \times \{\square a\} = F(\square b) \times \{\square b\}$ and hence $\square a = \square b$. For $id : \Psi(a) \to \Psi(b)$ there are $\alpha : a \to b$ and $\beta : b \to a$ with $id = \Psi(\alpha) = \Psi(\beta)$. Consequently, $\Psi(\alpha\beta) = \Psi(\beta\alpha) = id$, and, since Ψ is faithful, $\beta\alpha = id_a$, $\alpha\beta = id_b$, so that α is an isomorphism. We have $G(\square\alpha) = \square'\Psi(\alpha) = id$. Since G is faithful, $\square\alpha = id_a$. Thus, by (U), $\alpha = id_a$ and hence a = b.

1.7. Theorem: Let there exist a strong embedding of $(\mathfrak{K}, \square)$ into $(\mathfrak{K}', \square')$. Let $(\mathfrak{K}, \square)$ have (U) and $(\mathfrak{K}', \square')$ have (T). Then there exists a pseudorealization of $(\mathfrak{K}, \square)$ into $(\mathfrak{K}', \square')$.

Proof: Let $\Phi : \mathfrak{K} \to \mathfrak{K}'$ be a strong embedding, F the faithful set functor with $F \cdot \square = \square' \cdot \Phi$. Since F is faithful, there is a monotransformation $\nu : I \to F$ (define $\xi^X_x : 1 \to X$ by $\xi^X_x(0) = x$, take an $a \in F(1)$ with $\xi^2_0(a) \neq \xi^2_1(a)$, and put $\nu^X(x) = F(\xi^X_x)(a)$). Now, it suffices to put $\mu = \nu \square$ and use lemma 1.5.

1.8. Theorem: Let there be objects Z and K in $(\mathfrak{K}, \square)$ such that

1) $\square Z = \{z\}$ and for every object a of $\mathfrak{K}$ and every $x \in \square a$ there is a morphism $\alpha^a_x : Z \to a$ with $\square \alpha^a_x(z) = x$.

2) $\square K$ contains distinct elements u, v and for every object a of $\mathfrak{K}$ and distinct x, $y \in \square a$ there is a $\beta^a_{xy} : a \to K$ with $\square\beta^a_{xy}(x) = u$, $\square\beta^a_{xy}(y) = v$.

Let $(\mathfrak{K}, \square)$ have (U) and let there exist a full embedding of $\mathfrak{K}$ into $\mathfrak{K}'$. Then there exists a pseudorealization of $(\mathfrak{K}, \square)$ into any $(\mathfrak{K}', \square')$ with (T).

Proof: First, since $\square$ is faithful, we see easily that there is, for every $x \in \square a$, exactly one required α^a_x. Thus, since $\square(\varphi\alpha^a_x)(z) = \square\varphi(x)$ for any $\varphi : a \to b$, we obtain

$$\varphi \cdot \alpha^a_x = \alpha^b_{\square\varphi(x)} .$$

Now, let $\Phi : \mathfrak{K} \to \mathfrak{K}'$ be a full embedding. For an object a of $\mathfrak{K}$ define $\mu^a : \square a \to \square'\Phi(a)$ by

$$\mu^a(x) = \Box'\Phi(\alpha_x^a)(x_0),$$

where $x_0 \in \Box'\Phi(Z)$ is such that $\Box'\Phi(\alpha_u^K)(x_0) \neq \Box'\Phi(\alpha_v^K)(x_0)$. This defines a monotransformation $\mu : \Box \to \Box' \cdot \Phi$. Really, we have, for $\varphi : a \to b$,

$$\Box'\Phi(\varphi)\mu^a(x) = \Box'\Phi(\varphi\alpha_x^a)(x_0) = \Box'\Phi(\alpha^b_{\Box\varphi(x)})(x_0) =$$

$$= \mu^b\Box\varphi(x); \text{ if } x, y \in \Box a, x \neq y, \Box'\Phi(\beta^a_{xy})(\mu^a(x)) =$$

$$= \Box'\Phi(\alpha_u^K)(x_0) \neq \Box'\Phi(\alpha_v^K)(x_0) = \Box'\Phi(\beta^a_{xy})(\mu^a(y))$$

and hence $\mu^a(x) \neq \mu^a(y)$.

1.9. Corollary: Under the assumption (M) on set theory, if $(\mathfrak{K}, \Box)$ is realizable in some S(F) with a TB-functor F, then there are pseudorealizations of $(\mathfrak{K}, \Box)$ in $\mathcal{R}$ and in any $\mathcal{O}\mathcal{L}(\Box)$ with $\Sigma\Delta \geq 2$. In particular, there exist pseudorealizations of $\mathcal{R}$ in any $\mathcal{O}\mathcal{L}(\Delta)$ with $\Sigma\Delta \geq 2$.

Proof: According to [7] (particularly theorems 4.2 and 4.4),[1] and Theorem 1.8, it suffices to find objects Z and K in S(F). Take $Z = (1, \emptyset)$, $K = (2, F(2))$.

1.10. Definition: Let $(\mathfrak{K}, \Box)$ be a concrete category. We describe a concrete category $((\mathfrak{K}, \Box)^+, \Box^+)$:

The objects of $(\mathfrak{K}, \Box)^+$ are all the couples (a,0), where a is an object of $\mathfrak{K}$, (0,1) and (1,1). Morphisms between (a,0) and (b,0) are all the couples $(\varphi, 0)$ where $\varphi : a \to b$ in $\mathfrak{K}'$, morphisms between (0,1) and (a,0) are $((x, a), 1)$, where $x \in \Box a$, morphisms between (a,0) and (1,1) are $((u, a), 2)$, where $u \subset \Box a$ and, finally, morphisms between (0,1) and (1,1) are (0,2) and (1,2) .

The morphisms are composed by the following rules:

$$(\alpha, 0)\cdot(\beta, 0) = (\alpha\beta, 0),$$

for $\alpha : a \to b$, $(\alpha, 0)\cdot((x,a),1) = ((\square\alpha(x), b), 1)$,

for $\beta : b \to a$, $((u,a),2)\cdot(\beta,0) = (((\square\beta)^{-1}(u), b), 2)$,

$$((u,a),2)\cdot((x,a),1) = \begin{cases} (0,2) & \text{for } x \notin u, \\ (1,2) & \text{for } x \in u. \end{cases}$$

The forgetful functor $\square^+$ is defined by:

$\square^+(a,0) = \square a$, $\square^+(\varphi, 0) = \square\varphi$, $\square^+(0,1) = 1$, $\square^+(1,1) = 2$,

$\square^+((x,a),1)(0) = x$, $\square^+(i,2)(0) = i$,

$$\square^+((u,a),1)(x) = \begin{cases} 0 & \text{for } x \notin u, \\ 1 & \text{for } x \in u. \end{cases}$$

1.11. __Theorem__: If $(\mathfrak{K}, \square)$ has (U) and $(\mathfrak{K}', \square')$ has (T) , and if there exists a full embedding of $(\mathfrak{K}, \square)^+$ into $\mathfrak{K}'$, then there exists a pseudorealization of $(\mathfrak{K}, \square)$ into $(\mathfrak{K}, \square')$.

__Proof__: follows immediately from Definition 1.10 and 1.8.

1.12. __Remarks__: 1) By 1.11 and 1.6.1 we may express the following contribution to the unsolved problem of the existence of a non-boundable category (see [4],[5]; other term: non-algebraic):

Every concretisable category is boundable if and only if every concrete category is pseudorealizable in $\mathfrak{R}$.

2) If $\mathfrak{K}$ is a small category, $(\mathfrak{K}, \square)^+$ is a small category. Thus, since every small category may be

fully embedded into $\mathcal{R}$ ($\mathcal{R}_s$, $\mathcal{O}\iota(\Delta)$ with $\sum \Delta > 2$ etc., see [1],[2]), every small concrete category ($\mathfrak{K}$, $\square$) is pseudorealizable in $\mathcal{R}$ ($\mathcal{R}_s$, $\mathcal{O}\iota(\Delta)$ with $\sum \Delta \geq 2$ etc.).

3) In particular, if S is a semigroup of mapping of a set X into itself (containing the identity mapping), there is a $Y \supset X$ and a binary relation (binary symmetrical relation, binary operation, a couple of unary operations etc.), such that the semigroup of all the mappings of Y into itself preserving the relation (all the endomorphisms, resp.) consists exactly of (uniquely determined) extensions of the elements of S .

§ 2. A construction

This paragraph contains a construction and a lemma concerning this, which will be used in the following paragraph for embeddings into $\mathcal{R}_s$.

2.1. Conventions: Let (X,R) be an undirected graph. The distance $\rho(x, y)$ of two distinct points $x, y \in X$ is the least n such that there are $x_0, x_1, \ldots, x_n$ with

$$x_{i-1} R x_i \quad \text{for } i = 1, \ldots, n, \quad x = x_0, \quad y = x_n$$

(if such an n exists).

A triangle in (X,R) is every $\{x_1, x_2, x_3\} \subset X$ such that $x_i R x_j$ for all distinct i, j. A graph (X,R) is said to be t-connected if for any two distinct $x, y \in X$ there are triangles $t_1, t_2, \ldots, t_n$ such that $x \in t_1$, $y \in t_n$ and $t_i \cap t_{i+1} \neq \emptyset$ for i = 1, 2, ..., n - 1 . A subset $Y \subset X$ is said to be a t-connected subset of (X,R), if $(Y, R \cap Y \times Y)$ is t-connected.

The following is evident:

2.2. <u>Lemma</u>: Let $f: (X, R) \to (Y, S)$ be a compatible mapping, Z a t-connected subset in (X,R). Then f(Z) is t-connected.

2.3. <u>Construction</u>: A system $((A,T), (a_\iota)_{\iota<\alpha}, (b_i)_{i\in J})$, where (A,T) is a t-connected undirected graph without loops, α an ordinal, J a set and a_ι, b_i elements of A such that

$$(\rho) \qquad \begin{array}{l} \text{for } \iota \neq 0 \;\; \rho(a_0, a_\iota) \geq 4 \text{, in general, for} \\ \iota \neq \varkappa,\; \rho(a, a_\varkappa) \geq 2, \\ \text{for } i \neq j \;\; \rho(b_i, b_j) \geq 2 \;\; \text{always } \rho(a_\iota, b_i) \geq 2, \end{array}$$

is said to be a H-system.

For every couple α, β of cardinals with $\beta \leq \alpha$ choose once for ever a mapping $p_{\alpha\beta}$ of α onto β. If there is no danger of confusion, we shall write simply p.

Let $\mathcal{A} = ((A,T), (a_\iota)_{\iota<\alpha}, (b_i)_{i\in J})$ be a H-system, X a set, r_i (for $i \in J$) α_i-relations on X. Let $\alpha \geq \sup\{\alpha_i \mid i \in J\}$.

The undirected graph $\mathcal{H}(\mathcal{A}, (X, (r_i)_{i\in J})) = (X \times \alpha \vee \langle \alpha, X \rangle \vee A, R)$ is defined as follows:

(1) For $a, b \in A$, $aRb \Leftrightarrow aTb$

(2) For $a = A$, $\varphi: \alpha \to X$,

$aR\varphi \Leftrightarrow \varphi Ra \Leftrightarrow \exists i\, ((a = b_i) \,\&\, \exists \psi \in r_i,\ \varphi = \psi \cdot p))$

(3) For $a \in A$, $(x,\iota) \in X \times \alpha$, $aR(x,\iota) \Leftrightarrow (x,\iota)Ra \Leftrightarrow a = a_\iota$

(4) For $\varphi, \psi: \alpha \to X$ there is never $\varphi R \psi$

(5) For $\varphi: \alpha \to X$, $(x,\iota) \in X \times \alpha$,

$\varphi R(x,\iota) \Longleftrightarrow (x,\iota) R \varphi \Longleftrightarrow \varphi(\iota) = x$

(6) For $(x,\iota), (y,\varkappa) \in X \times \alpha$,

$(x,\iota) R (y,\varkappa) \Longleftrightarrow x = y$ and exactly one of $\iota, \varkappa$ is zero.

2.4. Lemma: Put $K_x = \{x\} \times \alpha \cup \{\varphi \mid \varphi(0) = x\}$. Then

1) Every t-connected subset of $\mathcal{K}(\mathcal{A}, (X, (\varkappa_i)_{i \in J}))$ is either a subset of A or a subset of some K_x.

2) Every K_x is 3-coloured.

Proof: 1) Since $K_x \cap K_y = \emptyset$ for $x \neq y$ and $K_x \cap A = \emptyset$, it suffices to show that every triangle is contained either in A or in some K_x. Let a triangle t not be contained in A. Then, by Construction (see condition $(\wp)$), $|t \cap A| < 2$. If $|t \cap A| = 1$, the single point of $t \cap A$ is either some a_ι, or some b_i. No of these points, however, is joined with two joined points outside of A.

Thus, according to (4) in 2.3, $t = \{(x,\iota), (y,\varkappa), \varphi\}$. By (5) and (6) we obtain $x = y$, ι or $\varkappa$ equal to zero and $\varphi(0) = x$.

2) Put $\chi(x,0) = 0$, $\chi(x,\iota) = 1$ for $\iota \neq 0$, $\chi(\varphi) = 2$ for $\varphi : \alpha \to X$.

2.5. Lemma: Let $\mathcal{A} = ((A,T), (a_\iota)_{\iota < \alpha}, (b_i)_{i \in J})$, $\mathcal{A}' = ((A',T'), (a'_\iota)_{\iota < \alpha}, (b'_i)_{i \in J'})$ be H-systems such that $\emptyset \neq J \subset J'$, that there is exactly one compatible $h : (A,T) \to (A',T')$ and that there holds

$h(a_\iota) = a'_\iota$ for $\iota < \alpha$, $h(b_i) = b'_i$ for

$i \in J$.

Let r_i for $i \in J$ (r_i' for $i \in J'$) be α_i-relations on X (on X') , let $\alpha \geq \max(2, \sup\{\alpha_i \mid i \in J'\})$. Then for every compatible

$$g : \mathcal{K}(\mathcal{Q},(X,(\kappa_i)_{i\in J})) \rightarrow \mathcal{K}(\mathcal{Q}',(X',(\kappa_i')_{i\in J'}))$$

there is an $f : X \rightarrow X'$ which is $r_i r_i'$ -compatible for every $i \in J$, such that, for $a \in A$, $g(a) = h(a)$ for $(x,\iota) \in X$ $g(x,\iota) = (f(x),\iota)$ and for $\varphi : \alpha \rightarrow X$ $g(\varphi) = f \cdot \varphi$.

Proof: Since $\alpha \neq 0$, $J \neq \emptyset$, there are points a_0, b_i with $\rho(a_0, b_i) \geq 2$ in A'. Since (A', T') is t-connected, there is a triangle t containing a_0 and, of course, not containing b_i. Thus, there is no compatible mapping of (A,T) into a 3-coloured graph - in that case there were possible to map (A,T) into t in a contradiction with the properties of h.

Thus, by lemmas 2.2 and 2.4, $g(A) = A'$ and, by (1) in Construction, $g(a) = h(a)$ for every $a \in A$. In particular, we obtain $g(a_\iota) = a_\iota'$. Take $(x, 0)$, $(x,\iota) \in X \times \alpha$, $\iota \neq 0$. If $g(x,0) \notin X' \times \{0\}$, we have necessarily $g(x, 0) \in A'$ and, by (φ), also $g(x,\iota) \in A'$, so that

$$a_0' R' g(x,0) R' g(x,\iota) R' a_\iota'$$

in a contradiction with (φ). Thus, $g(X \times \{0\}) \subset X' \times \{0\}$ and analogously $g(X \times \{\iota\}) \subset X' \times \{\iota\}$.

Define $f : X \rightarrow X'$ by $(f(x), 0) = g(x, 0)$. We obtain immediately $g(x,\iota) = (f(x),\iota)$ by the conditions $g(x,\iota) \in X' \times \{\iota\}$ and $g(x,\iota)R'(f(x),0)$. Now, let $\varphi: \alpha \to X$. For every $\iota, \varphi R(\varphi(\iota),\iota)$ and hence $g(\varphi)R'(f\varphi(\iota),\iota)$, so that, first, $g(\varphi): : \alpha \to X'$ (there are no other elements y with both $yR(u,0), yR(v,1)$) and, further, by (5), $g(\varphi) = = f\cdot\varphi$. If $\varphi \in \kappa_i$, we have $\varphi\cdot p R b_i$ and hence $f\cdot\varphi\cdot p R' b_i'$. Thus, there is a $\psi \in \kappa_i'$ with $f\cdot\varphi\cdot \cdot p = \psi\cdot p$. Since p is a mapping onto, we obtain $f\cdot\varphi = \psi \in \kappa_i'$. Thus, f is $r_i r_i'$-compatible.

§ 3. Strong embeddings into $\mathcal{R}_\wp$ and related categories

3.1. Lemma: Let α be an ordinal, J a set. Then there exists a H-system $\mathcal{A} = ((A,T),(a_\iota)_{\iota<\alpha},(b_i)_{i\in J})$ such that (A,T) is rigid.

Proof: In this proof, we shall use the methods and results of [2]. Thus, we preserve the terminology and, in some extent, also the notation of that paper.

Define $(\bar{A},\bar{T})$ as follows:

$\bar{A} = \{0,1,2,3,4,5,6,7,3',4',5',5'',6'',8,8',8''\}$,

$\bar{T}$ is the binary symmetrical relation generated by the couples

(1,2),(2,3),(3,4),(4,5),(5,6),(6,7),(7,1),
(1,0),(0,3'),(3',4'),(4',5'),(5',6),(4,5"),(5",6"),
(6",0)
(8,1),(8,2),(8,3),(8,4),(8,5),(8,6),(8,7),
(8',1),(8',0),(8',3'),(8',4'),(8',5'),(8',6),(8',7),
(8",0),(8",1),(8",2),(8",3).(8",4),(8",5"),(8",6").

(Thus, (A,T) is obtained from (Z,T) described in [2] by adding points 8,8',8" and joining each of them with the points of one of the 7-cycles.)

First, we see that the only elements $x \in \bar{A}$ such that there is a carrier of an odd cycle in $\{y \mid (x, y) \in \bar{T}\}$ are 8,8', 8". Considering this, we may prove that $(\bar{A}, \bar{T}, 5', 5")$ is strongly rigid in a way quite analogous to the proof of the strong rigidity of $(Z, T, 5', 5'')$ in [2].

Put $(A, T) = (\bar{A}, \bar{T}, 5', 5'') * (D, R)$, where (D,R) is the rigid graph constructed in [9]. By theorem 1 in [2], (A,T) is rigid. We have $\rho(5', 5'') = 4$ in $(\bar{A}, \bar{T})$. Now, it follows easily from the construction in [9] that (A,T) contains sufficiently many sufficiently distant elements, if (D,R) is taken large enough.

3.2. Theorem: For any type Δ there exists a strong embedding of $\mathcal{R}(\Delta)$ into $\mathcal{R}_{\mathcal{S}}$.

Proof: If $\Delta = (\alpha_\beta)_{\beta < \gamma}$, put $\alpha = \max(2, \sup\{\alpha_\beta \mid \beta < \gamma\})$ and take (see 3.1) some H-system $\mathcal{A} = ((A, T), (a_\iota)_{\iota < \alpha}, (b_\iota)_{\iota < \gamma})$ with rigid (A,T). For an object $(X, (r_\beta)_{\beta < \gamma})$ of $\mathcal{R}(\Delta)$ define $\Phi(X, (r_\beta)_{\beta < \gamma}) = \mathcal{H}(\mathcal{A}, (X, (r_\beta)_{\beta < \gamma}))$. Put $F = V_A \cdot (K_\alpha \vee Q_\alpha)$ (see Construction 2.3 and [6]). We see easily that for any morphism $\varphi : (X, (r_\beta)) \to (Y, (s_\beta))$ there is a unique $\Phi(\varphi) : \Phi(X, (r_\beta)) \to \Phi(Y, (s_\beta))$ with $\square \cdot \Phi(\varphi) = F \cdot \square(\varphi)$ and that the functor Φ thus described is one-to-one. By 2.5, Φ is a full embedding.

3.3. Lemma: Let F be a covariant selective functor. Then there is a strong embedding of S(F) into some $\mathcal{R}(\Delta)$.

Proof: Take a $\Delta' = (\alpha_\beta)_{\beta < \gamma}$ such that there is a full embedding $\Phi : \gamma \longrightarrow \mathcal{R}(\Delta')$ such that $\square \cdot \Phi = F$. Put $\Delta = (\alpha_\beta)_{\beta < \gamma + 1}$, where $\alpha_\gamma = 1$. If (X,r) is an object of S(F), put $\Psi(X, r) = (F(X), (r_\beta)_{\beta < \gamma+1})$ where $(F(X), (r_\beta)_{\beta<\gamma}) = \Phi(X)$, $r_\gamma = r$.

If $\Psi(Y, s) = (F(Y), (s_\beta)_{\beta < \gamma + 1})$ and $f : X \to Y$ is rs-compatible, F(f) is evidently $r_\beta s_\beta$-compatible for every $\beta < \gamma + 1$. Thus, Ψ may be extended to a functor by the prescription $\square \Psi(\varphi) = F(\square \varphi)$. Ψ is evidently one-to-one. If $g : F(X) \longrightarrow F(Y)$ is $(r_\beta)_{\gamma+1}$ $(s_\beta)_{\gamma+1}$ -compatible, it is $(r_\beta)_\gamma$ $(s_\beta)_\gamma$ - compatible and hence $g = F(f)$ for some $f : X \to Y$. Since $g(r_\gamma) \subset s_\gamma$, f is rs-compatible.

3.4. Theorem: Under the assumption (M) on the set theory, the following two statements are equivalent:

(1) There is a strong embedding of $(\mathfrak{K}, \square)$ into some S(F) with a TB-functor F,

(2) There is a strong embedding of $(\mathfrak{K}, \square)$ into $\mathcal{R}_s$.

Proof: Trivially, (2) $\Longrightarrow$ (1). Let (1) hold. By [7] (theorems 4.2 and 4.3) there is a strong embedding of $(\mathfrak{K}, \square)$ into an S(G) with covariant selective G. (2) follows by 3.2 and 3.3.

3.5. Remark: Thus, e.g., every $\mathcal{O}\ell(\Delta)$ is strongly embeddable into $\mathcal{R}_s$. We saw in 1.9 that $\mathcal{R}_s$ is pseudorealisable in $\mathfrak{A}(\Delta)$ with $\Sigma \Delta \geq 2$. On the other hand, we have

<u>Proposition</u>: Let there exist a strong embedding of $(\mathfrak{K}, \square)$ into an $\mathfrak{A}(\Delta)$. Then a morphism φ of $\mathfrak{K}$ is an isomorphism if and only if $\square \varphi$ is a one-to-one mapping onto.

<u>Proof</u>: Let $\varphi : a \to b$ be a morphism such that $\square \varphi$ is a one-to-one mapping of $\square a$ onto $\square b$. Let Φ be a strong embedding of $(\mathfrak{K}, \square)$ into $\mathfrak{A}(\Delta)$, let F be the set functor with $F \cdot \square = \square \cdot \Phi$. Thus, $\square \Phi(\varphi) = F(\square \varphi)$ is one-to-one onto and hence $\Phi(\varphi)$ is an isomorphism. Φ is a full embedding and hence there is a $\psi : b \to a$ with $\Phi(\psi) = (\Phi(\varphi))^{-1}$. Thus, $\Phi(\varphi\psi) = id$, $\Phi(\psi\varphi) = id$. Since Φ is one-to-one, $\varphi\psi = id$ and $\psi\varphi = id$.

Thus, $\mathcal{R}_{\flat}$ is strongly embeddable in <u>no</u> $\mathfrak{A}(\Delta)$, see e.g. the identity imbedding of $(X, \emptyset)$ into $(X, X \times X)$. In [8] is proved that any $\mathfrak{A}(\Delta)$ is strongly embeddable into every $\mathfrak{A}(\Delta')$ with $\Sigma \Delta' \geq 2$. Recently, V. Trnková proved that e.g. the category of Hausdorff compact spaces is strongly embeddable into the categories of algebras.

3.6. <u>Lemma</u>: Let $(\mathfrak{K}, \square)$ be a small concrete category with (U). Then there exists a type Δ and a realization of $(\mathfrak{K}, \square)$ in $\mathcal{R}(\Delta)$.

<u>Proof</u>: Let a be a one-to-one mapping of an ordinal γ onto the set of objects of $\mathfrak{K}$. Put $\alpha_\beta = card\, \square a(\beta)$, $\Delta = (\alpha_\beta)_{\beta < \gamma}$. For every $\beta < \gamma$ choose a one-to-

-one mapping m_β of α_β onto $\Box a(\beta)$.

Let b be an object of $\mathfrak{K}$. Define a relational system $\kappa^b = (\kappa^b_\beta)_{\beta<\gamma}$ of the type Δ on $\Box b$ as follows:

$f \in \kappa^b_\beta \Longleftrightarrow f: \alpha_\beta \rightarrow \Box a \;\&\; \exists \varphi: a(\beta) \rightarrow b$ with $f = \Box\varphi \cdot m_\beta$.

Let $\varphi: a(\iota) \rightarrow a(\varkappa)$ be a morphism. Let $f \in \kappa^{a(\iota)}_\beta$. Thus, $f = \Box\psi \cdot m_\beta$ for some $\psi: a(\beta) \rightarrow a(\iota)$, so that $\Box\varphi \cdot f = \Box(\varphi\psi) \cdot m_\beta$ and hence $\Box\varphi \cdot f \in \kappa^{a(\varkappa)}_\beta$. Thus, $\Box\varphi$ is $\kappa^{a(\iota)}\,\kappa^{a(\varkappa)}$-compatible.

Let $g: \Box a(\iota) \rightarrow \Box a(\varkappa)$ be $\kappa^{a(\iota)}\,\kappa^{a(\varkappa)}$-compatible. We have $m_\iota = \Box id_{a(\iota)} \cdot m_\iota \in \kappa^{a(\iota)}_\iota$ and hence $g \cdot m_\iota \in \kappa^{a(\varkappa)}_\iota$. Thus, there is a $\varphi: a(\iota) \rightarrow a(\iota)$ with $g \cdot m_\iota = \Box\varphi \cdot m_\iota$, so that $g = \Box\varphi$.

It remains to show that $(\Box b, \kappa^b) \neq (\Box c, \kappa^c)$ whenever $b \neq c$. Let $(\Box a(\iota), \kappa^{a(\iota)}) = (\Box a(\varkappa), \kappa^{a(\varkappa)})$. We have $m_\iota = \Box id_{a(\iota)} \cdot m_\iota \in \kappa^{a(\iota)}$ and hence $m_\iota \in \kappa^{a(\varkappa)}_\iota$. Thus, there is a $\varphi: a(\iota) \rightarrow a(\varkappa)$ with $m_\iota = \Box\varphi \cdot m_\iota$; consequently, $\Box\varphi = id$. Similarly we obtain a $\psi: a(\varkappa) \rightarrow a(\iota)$ with $\Box\psi = id$. Thus, $\Box\psi\varphi = \Box\varphi\psi = id$. Since $\Box$ is faithful, φ is an isomorphism. Hence, by (U), $a(\iota) = a(\varkappa)$.

3.7. <u>Theorem</u>: A small concrete category $(\mathfrak{K}, \Box)$ is strongly embeddable into $\mathcal{R}_n$ if and only if it has (U).

<u>Proof</u>: (U) is necessary by 1.4.3. It is sufficient by 3.6, 3.2 and 1.2.

§ 4. Strong embeddings into categories of quasi-algebras

4.1. Lemma: $\mathcal{R}$ is realisable in $\mathcal{Q}(2)$ and strongly embeddable into $\mathcal{Q}(2,0)$.

Proof: To prove the first statement, it suffices to define $\Phi(X,R) = (X,\omega)$, where $\omega(x,y)$ is defined if and only if $(x,y) \in R$ and equals x. Now, we obtain easily a strong embedding of $\mathcal{R}$ into $\mathcal{Q}(2,0)$ combining this construction with the construction of the strong embedding of $\mathcal{R}$ into $\mathcal{R}_{\diamond}$ by 3.2. Any point of A may be taken for the required nullary operation.

4.2. Lemma: $\mathcal{Q}(2)$ is strongly embeddable into $\mathcal{Q}(1,1)$ and into $\mathcal{Q}(1,1,0)$.

Proof: Put $F(X) = X \times X \times 3 \vee 1$, $F(f)(x,y,i) = (f(x), f(y), i)$, $F(f)(0) = 0$. For an object (X,ω) of $\mathcal{Q}(2)$ put

$$\Psi(X,\omega) = (F(X), \varphi, \psi) \quad (\ldots = (F(X), \varphi, \psi, 0) \text{ resp.}),$$

where $\varphi(x,y,i) = (x,y,i+1)$ for $i = 0,1$, $\varphi(x,y,2) = (y,x,0)$, $\varphi(0) = 0$; $\psi(x,y,0) = (x,y,0)$, $\psi(x,y,1) = (x,x,2)$, $\psi(x,y,2)$ defined as equal $(\omega(x,y), \omega(x,y), 1)$ if and only if $\omega(x,y)$ is defined; $\psi(0)$ is not defined. Further, define $\Psi(f)$ for morphisms by $\Box\Psi(f) = F(\Box f)$. Evidently, Ψ is a one-to-one functor mapping $\mathcal{Q}(2)$ into $(1,1)$ $(\mathcal{Q}(1,1,0)$ resp.).

Let $g: (F(X), \varphi, \psi) \to (F(X'), \varphi', \psi')$ be a homomorphism. Since 0 is the only fixed point of φ, we have $g(0) = 0$. Similarly, considering ψ, $g(X \times X \times \{0\}) \subset$

$\subset X' \times X' \times \{0\}$. Define $f, f' : X \to X'$ by $g(x,x,0) = (f(x), f'(x), 0)$. Put $g(x,y,0) = (x', y', 0)$. We have $(x',x',0) = \varphi\psi\varphi g(x,y,0) = g(x,x,0) = (f(x), f'(x), 0)$ and hence $x' = f(x) = f'(x)$ and similarly, by $\varphi\psi\varphi^4$, $y' = f(y)$.

Thus, for i = 0,1,2, $g(x,y,i) = g\varphi^i(x,y,0) = \varphi^i(f(x), f(y),0) = (f(x), f(y), i)$ and hence $g = F(f)$. If $\omega(x,y)$ is defined, we have $(f\omega(x,y), f\omega(x,y), 1) = g\psi(x,y,2) = \psi(f(x), f(y), 2)$ and hence $\omega'(f(x), f(y))$ is defined and equal $f\omega(x,y)$.

4.3. Lemma: Let $\Delta_1 = (\varkappa_\alpha)_{\alpha<\beta}$, $\Delta_2 = (\lambda_\gamma)_{\gamma<\delta}$ and let there exist a one-to-one mapping $\varphi : \beta \to \delta$ such that $\varkappa_\alpha \leq \lambda_{\varphi(\alpha)}$ for every $\alpha < \beta$. Let at least one of the following two conditions be satisfied:

(1) there is an $\alpha < \beta$ with $\varkappa_\alpha = 0$,

(2) $\lambda_\gamma \neq 0$ for $\gamma \in \delta - \varphi(\beta)$.

Then $\mathcal{G}(\Delta_1)$ is realizable in $\mathcal{G}(\Delta_2)$.

Proof is quite analogous to the proof of similar Lemma 1 in [1] concerning $\mathcal{O}(\Delta_1)$ and $\mathcal{O}(\Delta_2)$.

4.4. Theorem: $\mathcal{R}$ is strongly embeddable into any $\mathcal{G}(\Delta)$ with $\Sigma\Delta \geq 2$.

Proof: If $\Sigma\Delta \geq 2$, at least one of $\mathcal{G}(1,1)$, (2), $(1,1,0)$, $(2,0)$ is realizable in $\mathcal{G}(\Delta)$ by 4.3. Thus, the statement follows by 4.1 and 4.2.

4.5. Corollary: The statements (1) and (2) in Theorem 3.4 are equivalent with the following ones:

(3) There is a strong embedding of $(\mathcal{R}, \square)$ into some $\mathcal{G}(\Delta)$ with $\Sigma\Delta \geq 2$,

(4) There are strong embeddings of $(\mathfrak{R}, \square)$ into any $\mathfrak{A}(\Delta)$ with $\Sigma \Delta \geq 2$.

R e f e r e n c e s

[1] Z. HEDRLÍN, A. PULTR: On full embeddings of categories of algebras,Illinois J.of Math.10,3(1966),392-406.

[2] Z. HEDRLÍN, A. PULTR: Symmetric Relations(Undirected Graphs)with Given Semigroups,Monatshefte f. Math.69,4(1965),318-322.

[3] Z. HEDRLÍN, A. PULTR: On categorial embeddings of topological structures into algebraic,Comment.Math. Univ.Carolinae 7,3(1966),377-400.

[4] J.R. ISBELL: Adequate subcategories,Illinois J.of Math. 4(1960),541-552.

[5] J.R. ISBELL: Subobjects,adequacy,completeness and categories of algebras,Rozprawy matematyczne XXXVI, Warszawa 1964.

[6] A. PULTR: On selecting of morphisms among all mappings between underlying sets of objects in concrete categories and realisations of these,Comment. Math.Univ.Carolinae 8,1(1967),53-83.

[7] A. PULTR: Limits of functors and realisations of categories,Comment.Math.Univ.Carolinae 8,4(1967), 663-682.

[8] A. PULTR: Eine Bemerkung über volle Einbettungen von Kategorien von Algebren,submitted to Math.Annalen.

[9] P. VOPĚNKA, A. PULTR, Z. HEDRLÍN: A rigid relation exists on any set, Comment.Math.Univ.Carolinae 6(1965),149-155.

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