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ON AN ANALOGICAL ITERATIVE METHOD WITH THE METHOD OF THE TANGENT HYPERBOLAS

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In this paper is given (by analogy with method of [1]) an iterative method using the difference quotients of the second order for the solving the operational equation

(1) $$P(x) = 0 ,$$

where P is a continuous operator defined in Banach space X with values in X.

In [2] is given such a method, but the recoursive formula has more terms than the corresponding relation of this paper and in [2] is assumed existence of solution; we prove existence and uniqueness of the solution.

In our paper we use some properties of the difference quotients and corresponding notations given in [3], [4], [5], [6].

The studied method is given by the formula

(2) $$x_{n+1} = x_n - \Gamma_n [I - P(x_n, x_{n-1}, x_{n-2}) \Gamma_{n-1} P(x_{n-1}) \Gamma_n]^{-1} P(x_n)$$

$$(n = 0, 1, \dots)$$

where

$$\Gamma_n = [P(x_n, x_{n-1})]^{-1} .$$

Theorem. If in domain of definition of the operator P there are the points x_{-2}, x_{-1}, x_0 so that:

1^0 Γ_{-1}, $\bar{\Gamma}_0$, Γ_0 exists (where $\bar{\Gamma}_0 = [P(x_0, x_{-2})]^{-1}$), in sphere $S(x_0, r_0)$ exists $\Gamma = [P(x', x'')]^{-1}$ for any two points x', x'' and $Max\{\|\Gamma_{-1}\|, \|\bar{\Gamma}_0\|, \|\Gamma_0\|, \|\Gamma\|\} = B < \infty$;

2^0 $\|P(x_{-2})\| \leq \eta_{-2}$, $\|P(x_{-1})\| \leq \eta_{-1}$, $\|P(x_0)\| \leq \eta_0$, $(\eta_0 \leq \eta_{-1} \leq \eta_{-2})$;

3^0 In sphere $S(x_0, r_0)$ take place the delimitations

$$\|P(x', x'', x''')\| \leq M, \ \|P(x', x'', x''', x^{IV})\| \leq N;$$

4^0 $G_{-2} h_{-2} < 1$, where $h_n = B^2 M \eta_n$ $(n = -2, -1, 0, 1, 2, \dots)$, $h_{-2} < \frac{1}{2}$

and
$$G_n^2 = \frac{(1 + 2h_{n-1})(1 + 2h_n)}{(1 - h_{n+1})^2 (1 - 2h_{n+1})} \left(1 + \frac{N}{BM^2}\right)$$

$(n = -2, -1, 0, 1, 2, \dots)$ then in sphere $S(x_0, r_0)$ where

$$r_0 = \frac{2B\eta_{-2}}{1 - (G_{-2} h_{-2})^2}$$

the equation (1) has a solution x^*, which is unique in this sphere. The solution x^* is the limit of the sequence (x_n) given by (2) and the rapidity of convergence is given by the inequality

$$(3) \quad \|x^* - x_n\| \leq r_0 (G_{-2} h_{-2})^{2 \sum_{i=1}^{n-1} t_i}, \quad \text{where}$$

$$t_i = t_{i-1} + t_{i-2} + t_{i-3}, \quad t_{-1} = -1, \ t_0 = 1, \ t_1 = 1.$$

Remark 1. The condition 4^0 is verified if, for example, $h_{-2} < \frac{1}{4}$ and $\frac{N}{BM^2} \leq 1$.

Remark 2. We can easily verify (see [6]) that x_{-1}, x_{-2} are in $S(x_0, r_0)$.

Remark 3. The theorem is valid if in the place of condition of the boundedness in norm of the difference quotient of the third order, we assume that

$$\| P(x', x'', x''') - P(x'', x''', x^{IV}) \| \leq N \| x' - x^{IV} \| .$$

Proof. From conditions of the theorem, it results evidently that x_1 may be calculated by formula (2) and x_1 is in $S(x_0, r_0)$.

We prove that the conditions $1^0 - 4^0$ are verified too, for the points x_{-1}, x_0, x_1.

1^0 From preceding considerations it results evidently that this condition is verified.

2^0 Let be the auxiliary operator

$$F_n(x) = x - \Gamma_n [I - P(x_n, x_{n-1}, x_{n-2}) \Gamma_{n-1} P(x_{n-1}) \Gamma_n]^{-1} \cdot$$
$$\cdot P(x) + \Gamma_n [I - P(x_n, x_{n-1}, x_{n-2}) \Gamma_{n-1} P(x_{n-1}) \Gamma_n]^{-1} \cdot$$
$$\cdot P(x_n, x_{n-1}, x_{n-2}) \Gamma_{n-1} P(x)(x - x_n)$$

with properties

$$F_n(x_n) = F_n(x_{n-1}) = F_n(x_{n-2}) = x_{n+1} \quad ,$$

$$F_n(x_n, x_{n-1}) = 0, \quad F_n(x_n; x_{n-1}, x_{n-2}) = 0 \ .$$

Applying the Newton's formula to the operator F_0 in the point $x = x_1$, we obtain

$$\| P(x_1) \| \leq G_{-2}^2 h_{-1} h_{-2} \eta_0 \leq (G_{-2} h_{-2})^2 \eta_0 = \eta_1 < \eta_0 \ .$$

3^0 The conditions are evidently satisfied.

4^0 Evidently $h_{-1} \leq h_{-2}$, $G_{-1} \leq G_{-2}$, hence $h_{-1} G_{-1} < 1$.

By induction, it is easily to prove that x_n may be constructed by (2), $x_n \in S(x_0, r_0)$ and the conditions $1^0 - 4^0$ of the theorem take place for any n . Also

$$(5) \quad \eta_{n+1} = (h_{n-2} G_{n-2})^2 \eta_n , \qquad (n = 0, 1, \dots) ,$$

$$(6) \quad h_{n+1} = (h_{n-2} G_{n-2})^2 h_n \leq \frac{(h_{n-2} G_{n-2})^3}{G_{n-2}} ,$$

where $\| P(x_{n+1}) \| \leq \eta_{n+1}$.

From relations (5) and (6) we obtain

$$(7) \quad \eta_{n+1} \leq [(G_{-2} h_{-2})^2]^{\sum_{i=2}^{n} t_i} \cdot \eta_0$$

whence

$$(8) \quad \begin{aligned} \| x_{n+p} - x_n \| &\leq \sum_{k=1}^{p} \| x_{n+k} - x_{n+k-1} \| \leq \frac{B\eta_0}{1-(G_{-2} h_{-2})^{2n} h_{-2}} \cdot \\ &\cdot \sum_{k=1}^{p} \eta_{n+k-1} \leq \frac{B\eta_0}{1-h_{-2}} \sum_{k=1}^{p} \eta_{n+k-1} < \frac{B\eta_0}{1-h_{-2}} \frac{(G_{-2} h_{-2})^{2\sum_{i=2}^{n-1} t_i}}{1-(G_{-2} h_{-2})^6} < \\ &< r_0 (G_{-2} h_{-2})^{2\sum_{i=2}^{n-1} t_i} \end{aligned}$$

Hence the sequence obtained by (2) is fundamental and has an limit x^* . From (8) it results that $x^* \in S(x_0, r_0)$ and the delimitation (3) takes place.

From (7) it results that

$$\| P(x_n) \| \leq (G_{-2} h_{-2})^{2\sum_{i=2}^{n-1} t_i} \eta_0 \quad .$$

Hence, using the continuity of P we have

$$P(x^*) = 0 \; .$$

For proving the uniqueness of the solution, let be $\tilde{x} \in S(x_0, r_0)$, $\tilde{x} \neq x^*$ and $P(\tilde{x}) = 0$. From condition 1^0 it results the existence of the $[P(\tilde{x}, x_n)]^{-1}$ where x_n is constructed by formula (2)) and $\| [P(\tilde{x}, x_n)]^{-1} \| \leq B$. We have

$$\tilde{x} - x_n = [P(\tilde{x}, x_n)]^{-1}[P(\tilde{x}, x_n)](\tilde{x} - x_n) = -[P(\tilde{x}, x_n)]^{-1} P(x_n),$$

whence

$$\| \tilde{x} - x_n \| \leq B \eta_n \leq B[(G_{-2} h_{-2})^2]^{\sum_{i=2}^{n-1} t_i} \eta_0 \quad .$$

Hence

$$\lim_{n \to \infty} x_n = \tilde{x}$$

and using the uniqueness of the limit

$$\tilde{x} = x^* .$$

R e f e r e n c e s

[1] M.A. MERTVETZOVA: Analogical process with process of tangent hyperbolas for solving the generalized functional equations(Russian).DAN SSSR,88,4,611-614(1953).

[2] S.I. UL´M: Iterative methods with difference quotients of the second order(Russian).DAN SSSR 158,1,56-58(1961).

[3] A.S. SERGEJEFF: On method of chord(Russian).Sibirskii Math.Journ.II,2,282-289(1961).

[4] S.I. UL'M : On generalized difference quotients I (Russian).Izv.Akad.Nauk Est.SSR.XVI,fiz.-mat.,1,13-26(1967).

[5] G. GOLDNER and M. BALÁSZ: On method of chord and on its modification for solving the nonlinear operational equation (Roumanian).Being printed in Studii şi Cerc.Mat.Bucuresti (1968).

[6] M. BALÁSZ and G. GOLDNER: On an iterative method with difference quotients of the second order. To appear.

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