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A remark on selective functors

A REMARK ON SELECTIVE FUNCTORS [x)]

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In the present remark, we show that the definition of se-
lectivity from [1] may be simplified. Namely, we prove the
sufficiency of $\emptyset$ -selectivity for selectivity (i.e. Δ -se-
lectivity for any type Δ).

We use the terminology and notation of [1] (set functor,
I , Q_M , P^- etc.) with the following alterations and addi-
tions: We describe types by $(\alpha_\beta)_{\beta < \gamma}$ instead of the un-
precise $\{ \alpha_\beta \mid \beta < \gamma \}$. Similarly, relational systems are
denoted by $(\varkappa_\beta)_{\beta < \gamma}$ etc. We write $\mathcal{R}(\Delta)$ instead
of $\mathcal{T}(I, \Delta)$ (as was originally done in [2]). For the void
type, $\mathcal{R}(\emptyset)$ is identical with the category $\mathcal{T}$ of all
sets and all their mappings. Without fear of confusion we wri-
te again $\square$ for the natural forgetful functors of various
$\mathcal{R}(\Delta)$. (Thus, in the case of $\Delta = \emptyset$, $\square = I$.) If
$\tau : F \to G$ is a transformation of functors, we say that it
is a monotransformation (an epitransformation) if every τ^X
is a monomorphism (an epimorphism).

In [1], a one-to-one set functor F is said to be Δ -

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selective if there is a type Δ and a one-to-one functor Φ mapping $\mathscr{R}(\Delta)$ onto a full subcategory of $\mathscr{R}(\Delta')$ such that the diagram

$$\mathscr{R}(\Delta) \xrightarrow{\ \Phi\ } \mathscr{R}(\Delta')$$

$$\downarrow \square \qquad\qquad \downarrow \square$$

$$\gamma \xrightarrow{\ F\ } \gamma$$

commutes.

The aim of this note is to prove

Theorem: A set functor is selective if and only if it is $\emptyset$-selective.I.e.,a one-to-one set functorFis selective if and only if there is a type Δ and a one-to-one functor, mapping γ onto a full subcategory of $\mathscr{R}(\Delta)$ such that the diagram

$$\gamma \overset{\Phi}{\nearrow} \overset{\mathscr{R}(\Delta)}{\underset{\searrow_{F}}{\Big\downarrow \square}}$$

$$\gamma$$

commutes.

First,we formulate and prove some lemmas.

Lemma 1: Let F be a covariant one-to-one set functor. Then there exists a monotransformation $\mu : I \to F$.

Let F be a contravariant one-to-one set functor. Then there exist transformations $\mu : P^{-} \to F$ and $\varepsilon : F \to P^{-}$ such that $\varepsilon \cdot \mu$ is the identity transformation of P^{-} (consequently, μ is a mono- and ε an epitransformation).

Proof: We use the notation $1 = \{0\}$, $2 = \{0,1\}$ etc.

For every set X and every $x \in X$ define $\xi_x^X : 1 \to X$ by $\xi_x^X(0) = x$. We have obviously, for every $f : X \to Y$,

$$f \cdot \xi_x^X = \xi_{f(x)}^Y .$$

Let F be a one-to-one covariant set functor. Thus, there is an $a \in F(1)$ such that

$$F(\xi_0^2)(a) \neq F(\xi_1^2)(a) .$$

Put $\mu^X(x) = F(\xi_x^X)(a)$. For $f : X \to Y$ we obtain $F(f)\mu^X(x) = F(f \cdot \xi_x^X)(a) = F(\xi_{f(x)}^Y)(a) = \mu^Y(f(x))$ and, hence, μ is a transformation. If $x, y \in X$, $x \neq y$, there exists an $f : X \to 2$ such that $f(x) = 0$, $f(y) = 1$. Thus, $F(f)\mu^X(x) = \mu^Y(0) = F(\xi_0^2)(a) \neq F(\xi_1^2)(a) = F(f)\mu^X(y)$ and hence $\mu^X(x) \neq \mu^X(y)$.

Now, let F be a contravariant one-to-one set functor. Instead of for P^-, we shall prove the assertion for P_2 defined by:

$$P_2(X) = \{\alpha \mid \alpha \text{ a mapping of } X \text{ into } 2\} ,$$
$$P_2(f)(\alpha) = \alpha \cdot f ,$$

which is evidently naturally equivalent with P^-.

There is a $b \in F(2)$ such that

$$a_0 = F(\xi_0^2)(b) \neq F(\xi_1^2)(b) = a_1 .$$

Define $\mu : P_2 \to F$ and $\varepsilon : F \to P_2$ as follows:

for $\alpha : X \to 2$, $\mu^X(\alpha) = F(\alpha)(b)$,

for $c \in F(X)$, $\varepsilon^X(c)(x) = 1 \Longleftrightarrow F(\xi_x^X)(c) = a_1$.

If $f : X \to Y$, we have $F(f)\mu^Y(\alpha) = F(f)F(\alpha)(b) = F(\alpha \cdot f)(b) = \mu^X(\alpha \cdot f) = \mu^X P_2(f)(\alpha)$,

$$(P_2(f)\,\varepsilon^Y(c))(x) = \varepsilon^Y(c)(f(x)) = 1 \Longleftrightarrow a_1 = F(\xi^Y_{f(x)})(c) =$$

$$= F(f \cdot \xi^X_x)(c) = F(\xi^X_x)(F(f)(c)) \Longleftrightarrow (\varepsilon^X F(f)(c))(x) = 1.$$

For any $\alpha : X \to 2$, $\varepsilon^X(\mu^X(\alpha))(x) = 1 \Longleftrightarrow a_1 =$

$$= F(\xi^X_x)(\mu^X(\alpha)) = F(\alpha \cdot \xi^X_x)(b) \Longleftrightarrow \alpha \cdot \xi^X_x = \xi^2_1 \Longleftrightarrow \alpha(x) = 1 \text{ and hence}$$

$\varepsilon^X \mu^X(\alpha) = \alpha$.

Lemma 2 (Hedrlín): For $A \subset Q_M(X)$ define $q A \subset$

$\subset Q_M P^-(X)$ by

$$\varphi \in q A \Longleftrightarrow \forall \alpha : M \to X \; ((\forall m \; \alpha(m) \in \varphi(m)) \Longrightarrow \alpha \in A).$$

Let $A \subset Q_M(X)$, $B \subset Q_M(Y)$, $f : X \to Y$. Then

$$Q_M(f)(A) \subset B \Longleftrightarrow Q_M P^-(f)(q B) \subset q A .$$

Proof: Let $Q_M(f)(A) \subset B$, $\varphi \in q B$. Let $\alpha(m) \in$

$\in Q_M P^-(f)(\varphi)(m) = (P^-(f) \cdot \varphi)(m) = f^{-1}(\varphi(m))$ and hence

$f \cdot \alpha(m) \in \varphi(m)$ for any $m \in M$. Then $f \cdot \alpha \notin B$ and
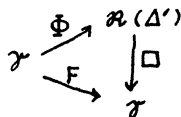
hence $\alpha \notin A$.

Let $Q_M P^-(f)(q B) \subset q A$. Let $\alpha : M \to X$.

If $f \cdot \alpha \notin B$, then $\varphi : M \to P^-(Y)$, defined by $\varphi(m) = \{f \cdot \alpha(m)\}$,

is an element of $q B$. Thus, $Q_M(P^-(f))(\varphi) = P^-(f) \cdot \varphi \in$

$\in q A$. Since $\alpha(m) \in f^{-1}\{f(\alpha(m))\} = P^-(f)(\varphi(m))$, $\alpha \notin A$.

Lemma 3. If q is a one-to-one mapping, then

$$g(f(A)) \subset g(B) \Longleftrightarrow f(A) \subset B .$$

Proof is trivial.

Proof of Theorem: Let F be $\emptyset$ -selective. Thus, there is a $\Delta' = (\alpha'_\beta)_{\beta < \gamma'}$ and a commutative diagram

with one-to-one functor Φ , mapping $\mathcal{Y}$ onto a full sub-category of $\mathcal{R}(\Delta')$.

 I. Let F be a covariant. Let $\Delta = (\alpha_\beta)_{\beta < \gamma}$. Put $\bar{\alpha}_\beta = \alpha'_\beta$ for $\beta < \gamma'$, $\bar{\alpha}_{\gamma'+\iota} = \alpha_\iota$, $\bar{\Delta} = (\alpha_\beta)_{\beta < \gamma'+\gamma}$. We have $\Phi(X) = (F(X), (\kappa^X_\beta)_{\beta < \gamma'})$. Now, we see easily that the following prescription:

 $\Psi(X, (\kappa_\beta)_{\beta<\gamma}) = (F(X), (\bar{\kappa}_\beta)_{\beta < \gamma'+\gamma})$, where $\bar{\kappa}_\beta = \kappa^X_\beta$ for $\beta < \gamma'$, $\bar{\kappa}_{\gamma'+\beta} = (Q_{\alpha_\beta} \mu^X)(\kappa_\beta)$ (for objects $(X, (\kappa_\beta)_{\beta < \gamma})$ of $\mathcal{R}(\Delta)$; μ from the first part of lemma 1),

$\square \cdot \Psi(f) = F \cdot \square(f)$ (for morphisms),

defines a one-to-one functor $\Psi : \mathcal{R}(\Delta) \to \mathcal{R}(\bar{\Delta})$. Let

$$\varphi : (F(X), (\bar{\kappa}_\beta)_{\beta < \gamma'+\gamma}) \to (F(Y), (\bar{\delta}_\beta)_{\beta < \gamma'+\gamma})$$

be a morphism in $\mathcal{R}(\bar{\Delta})$. Hence, according to $\bar{\kappa}_\beta$, $\bar{\delta}_\beta$ for $\beta < \gamma'$, $\square\varphi = F(f)$ for some $f : X \to Y$. For every $\beta < \gamma$ we obtain

$$Q_{\alpha_\beta} \mu^Y(Q_{\alpha_\beta}(f)(\kappa_\beta)) = Q_{\alpha_\beta} F(f)(Q_{\alpha_\beta} \mu^X)(\kappa_\beta) =$$

$$= Q_{\bar{\alpha}_{\gamma'+\beta}}(\square\varphi)(\bar{\kappa}_{\gamma'+\beta}) \subset \bar{\delta}_{\gamma'+\beta} = Q_{\alpha_\beta} \mu^Y(\delta_\beta)$$

and hence, by lemma 3, $Q_{\alpha_\beta}(f)(\kappa_\beta) \subset \delta_\beta$.

 II. Let F be contravariant. Take Δ , $\bar{\Delta}$ as in I, denote again $\Phi(X) = (F(X), (\kappa^X_\beta)_{\beta < \gamma'})$. For an object $(X, (\kappa_\beta)_{\beta < \gamma})$ of $\mathcal{R}(\Delta)$, put $\Psi(X, (\kappa_\beta)_{\beta < \gamma}) = (F(X), (\bar{\kappa}_\beta)_{\beta < \gamma'+\gamma})$, where $\bar{\kappa}_\beta = \kappa^X_\beta$ for $\beta < \gamma'$, $\bar{\kappa}_{\gamma'+\beta} = (Q_{\alpha_\beta} \mu^X)(q \kappa_\beta)$ (μ from the second part of lemma 1, q from lemma 2), for morphisms define Ψ by $\square \cdot \Psi(\varphi) = F \cdot \square(\varphi)$. Again, it is easy to see that this defines a one-to-one functor $\Psi : \mathcal{R}(\Delta) \to \mathcal{R}(\bar{\Delta})$. Let $\varphi : (F(Y), (\bar{\delta}_\beta)_{\beta<\gamma'+\gamma}) \to (F(X), (\bar{\kappa}_\beta)_{\beta < \gamma'+\gamma})$ be a

morphism in $\mathcal{R}(\bar{\Delta})$. According to $\bar{\rho}_\beta$, $\bar{\pi}_\beta$ with $\beta <$
$< \gamma'$ we obtain $\square \varphi = F(f)$ for some $f : X \to Y$. We
have, for $\beta < \gamma$,

$$Q_{\alpha_\beta} \mu^x (Q_{\alpha_\beta} P^-(f)(q \, \rho_\beta)) = Q_{\bar{\alpha}_{\gamma'+\beta}} F(f)(Q_{\alpha_\beta} \mu^x(q \, \rho_\beta)) =$$

$$= Q_{\bar{\alpha}_{\gamma'+\beta}} (\square \varphi)(\bar{\rho}_{\gamma'+\beta}) \subset \bar{\pi}_{\gamma'+\beta} = Q_{\alpha_\beta} \mu^x(q \, \pi_\beta) \, .$$

Thus, by lemma 3, $Q_{\alpha_\beta} P^-(f)(q \, \rho_\beta) \subset q \, \pi_\beta$ and hence
finally by lemma 2 $Q_{\alpha_\beta}(f)(\pi_\beta) \subset \rho_\beta$.

<div align="center">R e f e r e n c e s</div>

[1] Z. HEDRLÍN, A. PULTR: On categorial embeddings of topolo-
        gical structures into algebraic,Comment.Math.
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[2] Z. HEDRLÍN, A.PULTR: On full embeddings of categories of
        algebras, Illinois J.of Math.,10,3(1966),392-
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