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PRODUCTS OF FILTERS

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Suppose that Z is a topological space, $\{ x_a \mid a \in A \}$ is a family of points of Z , $\mathcal{F}$ is a filter on A . If z is a point of Z and, for any neighborhood U of z , there exists a set $F \in \mathcal{F}$ such that $a \in F \Longrightarrow x_a \in U$, then z is called an $\mathcal{F}$-limit of $\{x_a\}$, written $x = \mathcal{F}\text{-}\lim x_a$.

These limits with respect to a filter may be useful, e.g., if Z is the space of mappings of a space X into a space Y . In this case, they make possible a classification of (discontinuous) mappings, roughly speaking, according to how complicated filters have to be used to obtain them from continuous mappings. The approach of the classical descriptive theory of functions is similar in the sense that discontinuous functions are obtained from continuous functions in a prescribed way; it is substantially different since the use of iterated limits is quite essential.

It may be shown, however, that iterated limits may be replaced, in a specified sense, by a single limit (of course, with respect to a more complicated filter) . Namely, if $\mathcal{F}$, $\mathcal{G}$ are filters, $x_a = \mathcal{G}\text{-}\lim x_{a,b}$ for every a , $x = \mathcal{F}\text{-}\lim x_a$, then $x = (\mathcal{F} \cdot \mathcal{G})\text{-}\lim x_{a,b}$, where $\mathcal{F} \cdot \mathcal{G}$ is the product of filters $\mathcal{F}$ and $\mathcal{G}$ (see 1.9 below).

Some results related to these questions are intended for publication elsewhere. Here they are mentioned only in order to show the motivation for the study of the product of filters, which is the main topic of the present note. It seems, however, that a thorough examination of filters and operations on them may be interesting and important in itself.

It must be stressed that only filters on countable sets are considered. The sometimes trivial, sometimes difficult problem of carrying the results over to filters on arbitrary sets is not considered here.

Some definitions, partly well known, and some auxiliary propositions are contained in § 1. In § 2 some easily proved assertions are given concerning non-commutativity of the product of filters. § 3 contains some definitions and simple assertions concerning sequences of positive numbers. In § 4 we show that in a rather general situation filters do not commute. Namely, if two filters $\mathcal{F}$ and $\mathcal{G}$ are "mutually singular" (see 4.6), then, for any filters $\mathcal{A}$, $\mathcal{B}$, the filters $\mathcal{A}.\mathcal{F}$ and $\mathcal{B}.\mathcal{G}$ are mutually singular and therefore non-equivalent; in particular, $\mathcal{F}.\mathcal{G}$ and $\mathcal{G}.\mathcal{F}$ are not equivalent.

§ 1.

1.1. The terminology and notation of [1] is used. On the whole, it does not differ from current terms and symbols, and only some points of difference should be mentioned. An (ordered) pair a,b is denoted by $\langle a,b \rangle$. A family of elements x_a with an indexing set A is denoted by

$\{x_\alpha \mid \alpha \in A\}$ or simply by $\{x_\alpha\}$, etc. If M is a set, then the collection of all subsets of M is denoted by exp M. If $\{X_\alpha \mid \alpha \in A\}$ is a family of sets, then $\sum\{X_\alpha \mid \alpha \in A\}$ (or $\sum_\alpha X_\alpha$, etc.) denotes the sum of $\{X_\alpha\}$, i.e. the set of all $\langle a,x \rangle$, $a \in A$, $x \in X_a$.

1.2. <u>Convention.</u> In what follows, a filter always means a free filter on a countable infinite set. Thus, e.g., the assertion that $\mathcal{F}$ is a filter (or an ultrafilter, etc.) on a set M, always implies that M is a countable infinite set.

1.3. <u>Definition.</u> Let $\mathcal{F}_i$ be a filter on M_i, $i = 1,2$. If there exists a bijective $\varphi : M_1 \to M_2$ such that $F \in \mathcal{F}_1 \Longleftrightarrow \varphi[F] \in \mathcal{F}_2$, we call $\mathcal{F}_1$ and $\mathcal{F}_2$ equivalent, and write $\mathcal{F}_1 \sim \mathcal{F}_2$.

Let τ be a fixed relation assigning to every filter $\mathcal{F}$ an element $\tau\mathcal{F}$ in such a way that $\tau\mathcal{F}_1 = \tau\mathcal{F}_2$ if and only if $\mathcal{F}_1 \sim \mathcal{F}_2$. We shall call $\tau\mathcal{F}$ the type of $\mathcal{F}$ and denote it by typ $\mathcal{F}$.

1.4. <u>Definition.</u> Let $\mathcal{F}_i$ be a filter on M_i, $i = 1,2$. A triple $\langle \mathcal{F}_1, \mathcal{F}_2, \varphi \rangle$, φ being a mapping of M_1 into M_2, will be called a morphism from $\mathcal{F}_1$ to $\mathcal{F}_2$ if $\varphi^{-1}[F] \in \mathcal{F}_1$ for every $F \in \mathcal{F}_2$; we often shall also call φ itself a morphism.

It is clear that filters as objects with morphisms just described form a category.

1.5. <u>Definition.</u> Let $\mathcal{F}_i$ be a filter on M_i, $i = 1,2$. An injective morphism φ from $\mathcal{F}_1$ to $\mathcal{F}_2$ is called an embedding if every $F_1 \in \mathcal{F}_1$ is of the form $\varphi^{-1}[F_2]$, $F_2 \in \mathcal{F}_2$. If such a morphism exists, we shall say that $\mathcal{F}_1$ can be embedded in $\mathcal{F}_2$.

Clearly, if $\mathcal{F}$ is an ultrafilter, then every filter embedded in $\mathcal{F}$ is equivalent to $\mathcal{F}$.

If τ_1 , τ_2 are types of filters, we shall say that τ_1 can be embedded in τ_2 if there are filters $\mathcal{F}_1$, $\mathcal{F}_2$ with $\tau_i = typ\, \mathcal{F}_i$ such that $\mathcal{F}_1$ can be embedded in $\mathcal{F}_2$.

1.6. <u>Definition.</u> If $\mathcal{F}_i$ is a filter on M_i , $i = 1,2$, and there exists a morphism from $\mathcal{F}_1$ to $\mathcal{F}_2$, we shall write $\mathcal{F}_1 \prec \mathcal{F}_2$. It is easy to see that the relation $\prec$ is transitive and reflexive; if $\mathcal{F}_i \sim \mathcal{F}_i'$, $i = 1, 2$, and $\mathcal{F}_1 \prec \mathcal{F}_2$, then $\mathcal{F}_1' \prec \mathcal{F}_2'$. If $\mathcal{F}_1 \prec \mathcal{F}_2$, $\tau_i = typ\, \mathcal{F}_i$, we put $\tau_1 \prec \tau_2$.

1.7. Most concepts defined for filters are immediately carried over to types of filters, as described e.g. in 1.5 and 1.6. Therefore, we shall use types of filters whenever necessary (as a matter of fact, only at few points) without further explanation.

1.8. <u>Definition.</u> Let $\mathcal{F}$ be a filter on A . For any $a \in A$, let $\mathcal{G}_a$ be a filter on B_a . Then the collection of all $\sum_{a \in F} G_a$, where $F \in \mathcal{F}$, $G_a \in \mathcal{G}_a$, is a base of a filter which will be denoted by $\sum_{\mathcal{F}} \{ \mathcal{G}_a \mid a \in A \}$ or by $\sum_{\mathcal{F}} \mathcal{G}_a$, and called the sum of the family $\{\mathcal{G}_a\}$ with respect to $\mathcal{F}$.

<u>Remarks.</u> - 1) If $\mathcal{F}$ and $\mathcal{F}'$ are equivalent filters on A , $\sum_{\mathcal{F}} \mathcal{G}_a$ and $\sum_{\mathcal{F}'} \mathcal{G}_a$ are not necessarily equivalent. Therefore, we may define the sum of a family of types of filters with respect to a given filter, whereas the sum of filters (or types) with respect to a type of a filter is meaningless except for special cases. - 2) For the case of

ultrafilters, the sum of filters with respect to a filter has been investigated by Z. Frolík; see, e.g.,[2]. For the general case (arbitrary filters on arbitrary sets) the sum of filters with respect to a filter has been considered by P. Vopěnka [5].

1.9. Definition. Let $\mathcal{F}$ be a filter on A , $\mathcal{G}$ a filter on B . If $\mathcal{G}_a = \mathcal{G}$ for every $a \in A$, then $\sum_{\mathcal{F}} \mathcal{G}_a$ is denoted by $\mathcal{F} \cdot \mathcal{G}$ and called the product of $\mathcal{F}$ and $\mathcal{G}$,

Remark. This product is different, of course, from the cartesian product $\mathcal{F}_1 \times \mathcal{F}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$, the base of which consists of all $F_1 \times F_2$, where $F_i \in \mathcal{F}_i$. Clearly, $\mathcal{F}_1 \times \mathcal{F}_2 \prec \mathcal{F}_1 \cdot \mathcal{F}_2$; I do not know whether $\mathcal{F}_1 \cdot \mathcal{F}_2$ can be equivalent to $\mathcal{F}_1 \times \mathcal{F}_2$ for some choice of $\mathcal{F}_1$ and $\mathcal{F}_2$.

1.10. Let $\mathcal{F}$ be a filter on A . For every $a \in A$, let $\mathcal{G}_a$ be a filter on B_a . For every $b \in B_a$ let $\mathcal{H}_{a,b}$ be a filter on $C_{a,b}$. Put $\mathcal{G} = \sum_{\mathcal{F}} \mathcal{G}_a$. Then it is easy to see that

$$\sum_{\mathcal{G}} \{\mathcal{H}_{a,b} \mid \langle a,b \rangle \in \sum_a B_a\} \sim \sum_{\mathcal{F}} \{ \sum_{\mathcal{G}_a} \{\mathcal{H}_{a,b} \mid b \in B_a\} \mid a \in A\} .$$

Consequently, we have $\mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}) \sim (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}$ for any filters $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

1.11. Let $\mathcal{F}$, $\mathcal{G}$ be filters. If there exist filters $\mathcal{G}_a$, $a \in \bigcup \mathcal{F}$, such that $\mathcal{G} \sim \sum_{\mathcal{F}} \mathcal{G}_a$, we write $\mathcal{G} \precapprox \mathcal{F}$.

Clearly, $\precapprox$ is transitive and $\mathcal{F}_1 \precapprox \mathcal{F}_2$ implies $\mathcal{F}_1 \prec \mathcal{F}_2$. However, the relation $\precapprox$ is not reflexive (since, by 2.6 below, if $\mathcal{F}$ is an ultrafilter, then $\mathcal{F} \precapprox \mathcal{F}$ does not hold; cf.,e.g., Z. Frolík [3]) . I do not know whether $\mathcal{F} \prec \mathcal{G}$ implies ($\mathcal{F} \precapprox \mathcal{G}$ or $\mathcal{F} \sim \mathcal{G}$)

whenever $\mathcal{F}$, $\mathcal{G}$ are ultrafilters.

It is to be noted that if τ is a type of filter, there exist at most $\exp \aleph_0$ types σ with $\tau \precsim \sigma$; cf. Z. Frolík [2] .

1.12. For any filters, $\mathcal{F}_1 \prec \mathcal{F}_2$, $\mathcal{G}_1 \prec \mathcal{G}_2$ implies $\mathcal{F}_1 \cdot \mathcal{G}_1 \prec \mathcal{F}_2 \cdot \mathcal{G}_2$. For any filters $\mathcal{F}$, $\mathcal{G}$ we have $\mathcal{F} \cdot \mathcal{G} \precsim \mathcal{F}$, $\mathcal{F} \cdot \mathcal{G} \prec \mathcal{G}$.

I do not know whether $\mathcal{F}_1 \precsim \mathcal{F}_2$, $\mathcal{G}_1 \precsim \mathcal{G}_2$ implies $\mathcal{F}_1 \cdot \mathcal{G}_1 \precsim \mathcal{F}_2 \cdot \mathcal{G}_2$, nor whether $\mathcal{F} \cdot \mathcal{G} \precsim \mathcal{G}$ for arbitrary filters $\mathcal{F}$ and $\mathcal{G}$.

1.13. Let $\mathcal{F}$ be an ultrafilter on A . Let $\mathcal{G}_a$, $a \in A$, be ultrafilters. Then $\sum_{\mathcal{F}} \mathcal{G}_a$ is an ultrafilter. - In particular: the product of two ultrafilters is an ultrafilter.

This is known; see P. Vopěnka [5], Z. Frolík [2] .

1.14. The following proposition is sometimes useful: if f is a mapping of a set X into a set Y , then there exist disjoint sets X_0 , X_1 , X_2 , X_3 such that (1) $X_0 \cup X_1 \cup X_2 \cup X_3 = X$, (2) $x \in X_0 \Longrightarrow fx = x$, (3) $f[X_i] \cap X_i = \emptyset$ for $i = 1,2,3$.

Remark. This proposition is contained in the author's note A theorem on mappings, Comment.Math.Univ.Carolinae 8 (1967), 431-434. As I have learned, it was found earlier by H. Kenyon and published in the form of a research problem [Amer.Math.Monthly 70 (1963),p.216; the solution appeared in vol.71 (1964),p.219].

1.15. Proposition. Let $\mathcal{F}_1$ be an ultrafilter, $\mathcal{F}_2$ a filter. Let $\mathcal{F}_1 \prec \mathcal{F}_2 \prec \mathcal{F}_1$. Then $\mathcal{F}_1$ can be embedded in $\mathcal{F}_2$; if $\mathcal{F}_2$ is an ultrafilter, then $\mathcal{F}_1 \sim \mathcal{F}_2$.

Proof. Let $f: M_1 \longrightarrow M_2$ be a morphism from $\mathcal{F}_1$ to $\mathcal{F}_2$, and let $g: M_2 \longrightarrow M_1$ be a morphism from $\mathcal{F}_2$ to $\mathcal{F}_1$. Put $h = g \circ f$. Choose sets X_0, X_1, X_2, X_3 possessing, with respect to h, properties from 1.14. If $i = 1,2,3$, then $h[X_i] \cap X_i = \emptyset$ and therefore, h being a morphism, X_i non $\in \mathcal{F}_i$. Since $\mathcal{F}_1$ is an ultrafilter, we get $X_0 \in \mathcal{F}_1$; and we have $x = hx$ whenever $x \in X_0$. Choose a set $X_0' \in \mathcal{F}_1$, $X_0' \subset X_0$, such that $X_0 - X_0'$ is infinite and choose a bijective $\psi : M_1 - X_0' \longrightarrow f[X_0 - X_0']$. For $x \in X_0'$ put $\varphi x = fx$; if $x \in M_1 - X_0$, put $\varphi x = \psi x$. It is easy to show that $\varphi : M_1 \longrightarrow M_2$ is an embedding of $\mathcal{F}_1$ into $\mathcal{F}_2$.

1.16. Convention. We denote by $\mathcal{N}$ the Fréchet filter on N, i.e. the filter consisting of all $X \subset N$ such that $N - X$ is finite.

§ 2.

2.1. Definition. We shall say that a filter $\mathcal{F}$ on A has property (α) if the following holds: if infinite sets $A_n \subset A$, $n \in N$, are disjoint, then there exists a set $F \in \mathcal{F}$ such that all $A_n - F$ are infinite.

It is easy to show that a filter $\mathcal{F}$ possesses property (α) if and only if neither $\mathcal{N}$ nor any filter of the form $\mathcal{G} \cdot \mathcal{N}$ can be embedded in $\mathcal{F}$.

2.2. Evidently, every ultrafilter possesses property (α).

2.3. Let Θ denote the class of all families $\{t_m \mid m \in M\}$ such that (1) M is a countable infinite set, (2) t_m are positive numbers, (3) for any $\varepsilon > 0$, there are only finitely many m with $t_m > \varepsilon$, (4) $\sum\{t_m \mid m \in M\} = \infty$. For any $t = \{t_m \mid m \in M\} \in \Theta$ we denote by $\mathcal{P}_t$ the collection of all $X \subset M$ such that $\sum\{t_m \mid m \in M - X\} < \infty$. It is easy to show that $\mathcal{P}_t$ is a filter possessing property (α).

2.4. <u>Theorem.</u> Let $\mathcal{F}$ be a filter on A. Let filters $\mathcal{G}_a$, $a \in A$, possess property (α). Then the filter $\sum_{\mathcal{F}} \mathcal{G}_a$ also possesses property (α).

<u>Proof.</u> We may suppose that $A = N$ and every $\mathcal{G}_a$ is a filter on N. Then $\mathcal{G} = \sum_{\mathcal{F}} \mathcal{G}_a$ is a filter on $N \times N$. Let $A_m \subset N \times N$, $m \in N$, be disjoint infinite sets. Put $B_n = (n) \times N$. For any $m \in N$, let $L(n)$ be the set of all $m \in N$ such that $A_m \cap B_n$ is infinite.

Let $n \in N$. If $L(n) = \emptyset$, put $H_n = N$. If $L(n) \neq \emptyset$, choose a set $H_n \in \mathcal{G}_n$ such that the set $A_m \cap B_n - (n) \times H_n$ is infinite for every $m \in L(n)$. Denote by H the set $\sum\{H_n \mid n \in N\}$. Clearly, $H \in \mathcal{G}$ and $A_m - H$ is infinite whenever $m \in \bigcup L(n)$.

Now put $K = N - \bigcup L(n)$. Then, for any $m \in K$, $n \in N$, the set $A_m \cap B_n$ is finite. For any $n \in N$ put $P_n = \bigcup\{A_m \cap B_n \mid m \in K, m \leq n\}$, $S = \bigcup\{B_n - P_n \mid n \in N\}$. Clearly, $S \in \mathcal{G}$ and $A_m - S$ is infinite whenever $m \in K$.

Put $G = H \cap S$. It is evident that $G \in \mathcal{G}$ and every $A_m - G$ is infinite.

2.5. <u>Corollaries.</u> - 1) If a filter $\mathcal{F}$ possesses property (α), then $\mathcal{F} \cdot \mathcal{N}$ and $\mathcal{N} \cdot \mathcal{F}$ are not equivalent. - 2) If a filter $\mathcal{F}$ commutes with $\mathcal{N}$, then $\mathcal{F}$ is of the form $\mathcal{G} \cdot \mathcal{N}$.

<u>Remark.</u> I do not know what conditions on a filter $\mathcal{F}$

are necessary and sufficient for the equivalence $\mathcal{F} \cdot \mathcal{N} \sim$ $\sim \mathcal{N} \cdot \mathcal{F}$.

2.6. Theorem. Let $\mathcal{F}$ be an ultrafilter on a set A . Let $\mathcal{G}_a$, $a \in A$, be filters. Then $\mathcal{F} \prec \sum_{\mathcal{F}} \mathcal{G}_a$ does not hold (hence, neither $\mathcal{F} \sim \sum_{\mathcal{F}} \mathcal{G}_a$ nor $\mathcal{F} \dot{\prec} \sum_{\mathcal{F}} \mathcal{G}_a$ holds).

Remark. For the assertion in parentheses see Z. Frolík [3].

Proof. Suppose that $\mathcal{F} \prec \sum_{\mathcal{F}} \mathcal{G}_a$. Put $B_a = \bigcup \mathcal{G}_a$, $B = \bigcup \{B_a \mid a \in A\}$, $\mathcal{G} = \sum_{\mathcal{F}} \mathcal{G}_a$; then $\mathcal{G}$ is a filter on B . Let $\varphi : A \rightarrow B$ be a morphism from $\mathcal{F}$ to $\mathcal{G}$. Denote by π the mapping which assigns a to $\langle a, b \rangle \in A \times B$ and put $\psi = \pi \circ \varphi$. Then ψ is a morphism from $\mathcal{F}$ to $\mathcal{F}$. From 1.14 it follows that there exists a set $M \in \mathcal{F}$ such that $\psi x = x$ whenever $x \in M$. Clearly, $\varphi[M]$ intersects every $(a) \times B_a$ at one point at most. Therefore we have $B - \varphi[M] \in \mathcal{G}$, hence $A - \varphi^{-1}[\varphi[M]] \in \mathcal{F}$, which is a contradiction since $M \in \mathcal{F}$.

2.7. Let $\mathcal{F}$, $\mathcal{G}$ be filters. If $\mathcal{F} \prec \mathcal{F} \cdot \mathcal{G}$, then $\mathcal{F}$ is not an ultrafilter. If $\mathcal{G} \prec \mathcal{F} \cdot \mathcal{G}$, then $\mathcal{G}$ is not an ultrafilter.

Proof. We are going to prove the second assertion; the first follows from 2.6.

Suppose that $\mathcal{F}$ is a filter on A , $\mathcal{G}$ is an ultrafilter and that $\varphi : B \rightarrow A \times B$ is such that $X \in \mathcal{F} \cdot \mathcal{G} \Rightarrow \varphi^{-1}[X] \in \mathcal{G}$. Denote by p the projection of $A \times B$ onto B . Then $\psi = p \circ \varphi$ is a morphism from $\mathcal{G}$ to $\mathcal{G}$. By 1.14, there exists a set $M \in \mathcal{G}$ such that $x \in M \Rightarrow \psi x = x$.

For every $a \in A$ denote by B_a the set of all points

$y \in M$ such that $\langle a, y \rangle = \varphi y$. It is easy to see that $B_a \cap$ $\cap B_{a'} = \emptyset$ if $a \neq a'$. Therefore B_a non $\in \mathcal{G}$ for every $a \in A$ with one exception at most. It follows that $(A \times B) - \sum_{a \in A} B_a$ belongs to $\mathcal{F} \cdot \mathcal{G}$. On the other hand, $\sum_{a \in A} B_a = \varphi[M]$. We obtain a contradiction.

2.8. Theorem. Let $\mathcal{F}$, $\mathcal{G}$ be ultrafilters. If $\mathcal{F} \cdot \mathcal{G} \sim$ $\sim \mathcal{G} \cdot \mathcal{F}$, then either $\mathcal{F} \sim \mathcal{G}$ or $\mathcal{F} \lessapprox \mathcal{G}$ or $\mathcal{G} \lessapprox \mathcal{F}$.

This proposition follows from Theorem 2 of Z. Frolík [4]. I give a different proof not using topological concepts. We may suppose that $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on N. Suppose $\mathcal{F} \cdot \mathcal{G} \sim \mathcal{G} \cdot \mathcal{F}$. For convenience, let $\mathcal{K}$ be a filter on K and let $f: N \times N \to K$, $g: N \times N \to K$ be bijective mappings such that $X \in \mathcal{F} \cdot \mathcal{G} \Longleftrightarrow f[X] \in \mathcal{K}$, $Y \in \mathcal{G} \cdot \mathcal{F} \Longleftrightarrow g[Y] \in \mathcal{K}$.

Let $C_{x,y}$, $A_{x,y}$, $B_{x,y}$ be defined for $\langle x, y \rangle \in N \times N$ as follows: $C_{x,y} = f[(x) \times N] \cap g[(y) \times N]$, $C_{x,y} = g[(y) \times A_{x,y}] = f[(x) \times B_{x,y}]$. For any $M \subset N \times N$ put $C(M) = \bigcup\{C_{x,y} \mid \langle x, y \rangle \in M\}$. Consider the following conditions: (0) $A_{x,y}$ non $\in \mathcal{F}$, $B_{x,y}$ non $\in \mathcal{G}$, (1) $A_{x,y} \in \mathcal{F}$, $B_{x,y} \in \mathcal{G}$, (2) $A_{x,y}$ non $\in \mathcal{F}$, $B_{x,y} \in \mathcal{G}$, (3) $A_{x,y} \in \mathcal{F}$, $B_{x,y}$ non $\in \mathcal{G}$. Denote by λ_i, $i = 0,1,2,3$, the set of all points $\langle x, y \rangle \in N \times N$ such that the condition (i) is satisfied. By 1.13, $\mathcal{K}$ is an ultrafilter; hence exactly one of the sets $C(\lambda_i)$ belongs to $\mathcal{K}$.

Now we show that $C(\lambda_0)$ non $\in \mathcal{K}$. Let λ'_0 consist of $\langle x, y \rangle \in \lambda_0$ such that $x < y$ and put $\lambda''_0 = \lambda_0 - \lambda'_0$. It is easy to show that $g^{-1}[C(\lambda'_0)]$ non $\in \mathcal{G} \cdot \mathcal{F}$, $f^{-1}[C(\lambda''_0)]$ non $\in \mathcal{F} \cdot \mathcal{G}$. This implies $C(\lambda_0)$ non $\in \mathcal{K}$.

We are going to consider the case $C(\lambda_2) \in \mathcal{K}$. The case $C(\lambda_3) \in \mathcal{K}$ is quite similar. If $C(\lambda_1) \in \mathcal{K}$, we obtain the equivalence $\mathcal{F} \sim \mathcal{G}$; the proof may be omitted.

Write φ instead of λ_2 . Thus φ consists of all $\langle x,y \rangle$ such that $A_{x,y}$ non $\in \mathcal{F}$, $B_{x,y} \in \mathcal{G}$. Clearly, φ is a single-valued relation. Denote by Q the set of all $y \in N$ such that $\bigcup\{A_{x,y} \mid \varphi x = y\} \in \mathcal{F}$; put $P = \varphi^{-1}[Q]$. It is easy to show that $Q \in \mathcal{G}$, $P \in \mathcal{F}$.

If $y \in Q$, define a filter $\mathcal{T}_y$ on $\varphi^{-1}[y]$ in the following way: $X \subset \varphi^{-1}[y]$ belongs to $\mathcal{T}_y$ if and only if $\bigcup\{A_{x,y} \mid x \in X\} \in \mathcal{F}$. Let $\mathcal{F}'$ consist of all $P \cap F$, $F \in \mathcal{F}$; let $\mathcal{G}'$ consist of all $Q \cap G$, $G \in \mathcal{G}$. For any $x \in P$ put $\psi x = \langle \varphi x, x \rangle$. It is easy to prove that $\psi : P \rightarrow \sum_{Q} \varphi^{-1}[y]$ is bijective and $X \in \mathcal{F}' \Longleftrightarrow \psi[X] \in \sum_{\mathcal{G}'} \mathcal{T}_y$. From this, $\mathcal{F} \preccurlyeq \mathcal{G}$ follows at once.

§ 3.

3.1.<u>Definition.</u> Let $u \in \Theta$, $v \in \Theta$ (see 2.3) ; let $u = \{u_k \mid k \in K\}$, $v = \{v_m \mid m \in M\}$. If, for any set H and any bijective mappings $\varphi : H \rightarrow K$, $\psi : H \rightarrow M$, we have $\sum\{\min(u_{\varphi h}, v_{\psi h}) \mid h \in H\} < \infty$, then we shall say that u and v are mutually singular (or that u is singular with respect to v , v is singular with respect to u) .

3.2. <u>Definition.</u> We denote by Θ^* the set of all sequences $t = \{t_n \mid n \in N\} \in \Theta$ such that $t_n \geq t_{n+1}$, $n = 0,1,2,\ldots$. If $u = \{u_k \mid k \in K\} \in \Theta$, $v = \{v_m \mid m \in M\} \in \Theta$ and there exists a bijective $\varphi : K \rightarrow M$ such that $v_{\varphi k} = u_k$ for any $k \in K$, we shall say that u and v are equivalent. It is easy to see that for any $u \in \Theta$ there exists exactly one sequence $t = \{t_n \mid n \in N\} \in \Theta^*$ which is equivalent to u . It will be denoted by $u^* = \{u_n^* \mid n \in N\}$.

3.3. Let $u = \{u_m \mid m \in M\} \in \Theta$. Let $p = \{p_n \mid n \in N\}$, $p_n \in N$, $p_n \geqq 1$. We denote by $p * u$ the sequence obtained from $u^* = \{u_n^*\}$ by repeating the n-th member p_n times (i.e., $p * u = \{z_n\}$, $z_n = u_k^*$ if $\sum_{i=0}^{k-1} p_i \leqq n < \sum_{i=0}^{k} p_i$).

3.4. Let $u = \{u_k \mid k \in K\}$, $v = \{v_m \mid m \in M\}$ belong to Θ . Let $\varphi : N \to K$, $\psi : N \to M$ be bijective mappings. Then $\sum_{i=0}^{n} \min(u_{\varphi i}, v_{\psi i}) \leqq \sum_{i=0}^{n} \min(u_i^*, v_i^*)$ for any $n \in N$.

The proof of this almost evident proposition may be omitted.

3.5. Proposition. Let $a_i > 0$, $b_i > 0$, $a_i > b_i > a_{i+1}$ for every $i \in N$; let $\lim a_n = 0$. Let $p_i \in N$, $q_i \in N$, $p_i > 0$, $\sum p_i a_i = \infty$, $\sum q_i b_i = \infty$, $\sum p_i b_i < \infty$, $\sum q_i a_{i+1} < \infty$. Put $a = \{a_i\}$, $b = \{b_i\}$, $p = \{p_i\}$, $q = \{q_i\}$, $s = p * a$, $t = q * b$. Then s and t are mutually singular.

Proof. By 3.4 it is sufficient to prove that $\sum\{\min(s_n, t_n) \mid n \in N\} < \infty$. Let K be the set of n such that $s_n \geqq t_n$, and put $L = N - K$. For $i \in N$ let K_i consist of all $n \in K$ such that $s_n = a_i$, and let L_i be the set of all $n \in L$ such that $t_n = b_i$. Clearly, K_i has p_i elements at most, L_i has q_i elements at most.

If $n \in K_i$, then $t_n \leqq s_n = a_i$, hence $t_n \leqq b$. Therefore, $\sum\{t_n \mid n \in K_i\} \leqq p_i b_i$ and $\sum\{\min(s_n, t_n) \mid n \in K\} \leqq \sum p_i b_i < \infty$. Similarly, if $n \in L_i$, then $s_n < t_n = b_i$, hence $s_n < a_{i+1}$. Therefore, $\sum\{t_n \mid n \in L_i\} < q_i a_{i+1}$ and $\sum\{\min(s_n, t_n) \mid n \in L\} \leqq \sum\{t_n \mid n \in L\} < \sum q_i a_{i+1} < \infty$.

3.6. Example. Let $r_n \in N$, $r_n \geqq 2$, $\sum r_n^{-1} < \infty$. Put $p_k = \prod_{i=0}^{2k} r_i$, $q_k = \prod_{i=0}^{2k+1} r_i$, $a_k = p_k^{-1}$, $b_k = q_k^{-1}$. Clearly, $a = \{a_k\}$, $b = \{b_k\}$, $p = \{p_k\}$, $q = \{q_k\}$ satisfy the conditions described in 3.5 and therefore $s = p * a$ and $t = q * b$ are mutually singular. In addition, if $m = \{m_k\}$, $m_k \in N$, $m_k \geqq 1$ and $\sum m_k r_k^{-1} < \infty$, then $m * s$ and $m * t$ are mutually singular.

3.7. Remark. If $u = \{u_n | n \in N\} \in \Theta^*$, $\liminf n u_n > 0$, then no $v \in \Theta$ is singular with respect to u.

Clearly, it is sufficient to consider the case $u = \{u_n\}$, $u_0 = 1$, $u_n = n^{-1}$ for $n = 1, 2, \ldots$. Suppose that $v = \{v_n | n \in N\} \in \Theta^*$, $\sum \min(u_n, v_n) < \infty$. Put $z_k = \sum \{\min(u_n, v_n) \mid 2^{k-1} \leqq n < 2^k\}$; then $\sum \{\min(u_n, v_n) | n \in N\} = \sum z_k$. Put $w_k = v_{2^k}$. Denote by K the set of all k such that $w_k \geqq 2^{-k}$. Then for $k \in K$ we have $z_k \geqq 2^{k-1} \cdot 2^{-k} = \frac{1}{2}$; therefore, K is finite. For $k \in N - K$ we have $z_k \geqq 2^{k-1} \cdot w_k$; hence $\sum 2^{k-1} \cdot z_k < \infty$. Clearly, for any $k \in N - K$, $\sum \{v_n | 2^k \leqq n < 2^{k+1}\} \leqq 2^k w_k$. It follows that $\sum_{k \in N-K} \sum \{v_n | 2^k \leqq n < 2^{k+1}\} < \infty$, hence $\sum \{v_n | n \in N\} < \infty$, which is a contradiction.

3.8. Let $u = \{u_n\} \in \Theta^*$, $v = \{v_n\} \in \Theta^*$ be mutually singular. Then there exists a sequence $p = \{p_n\}$ such that $p_n \in N$, $0 < p_0 \leqq p_1 \leqq p_2 \leqq \ldots$, $p_n \to \infty$, and the sequences $p * a$ and $p * b$ are mutually singular.

Proof. We have $\sum \min(u_n, v_n) < \infty$. Clearly, there exist positive integers p_n such that $p_0 \leqq p_1 \leqq \ldots$ and $\sum p_n \min(u_n, v_n) < \infty$.

§ 4.

4.1. Definition. Let $\mathcal{F}$ be a filter on M . Let t = $=\{t_k \mid k \in K\}$ belong to Θ . If, for some bijective φ: : $M \to K$, $X \subset M$ belongs to $\mathcal{F}$ whenever $\sum\{t_{\varphi m} \mid m \in M - X\} < \infty$, we shall say that $\mathcal{F}$ is strongly dominated by t .

Clearly, if $\mathcal{F}$ is strongly dominated by $t \in \Theta$, $\mathcal{G} \sim \mathcal{F}$ and $s \in \Theta$ is equivalent to t , then $\mathcal{G}$ is strongly dominated by s ; if $\mathcal{F}$ is strongly dominated by s = $=\{s_m \mid m \in M\} \in \Theta$ and $t = \{t_m \mid m \in M\} \in \Theta$, then it is strongly dominated by $\{\min(s_m, t_m) \mid m \in M\}$. It is easy to see that if $\mathcal{F}$ is strongly dominated by $t = \{t_m \mid m \in M\} \in \Theta$, then it is strongly dominated by any $\{t_m \mid m \in M - A\}$, A finite.

4.2. Definition. Denote by Ψ the set of all sequences $p = \{p_n\}$ such that (1) for any $n \in N$, $p_n \in N$, $p_n \geq 1$, $p_n \leq p_{n+1}$, (2) $p_n \to \infty$.

Let $\mathcal{F}$ be a filter on M . Let $t = \{t_k \mid k \in K\}$ belong to Θ . If, for any $p \in \Psi$, $\mathcal{F}$ is strongly dominated by $p * t$, we shall say that $\mathcal{F}$ is dominated by t .
The statements in 4.1 remain true if "strongly dominated" is replaced by "dominated".

4.3. Theorem. Let $\mathcal{F}$ be a filter on A . For any $a \in A$, let $\mathcal{G}_a$ be a filter. Let $t = \{t_k \mid k \in K\}$ belong to Θ . If every $\mathcal{G}_a$ is dominated by t , then $\sum_{\mathcal{F}} \mathcal{G}_a$ is dominated by t .

Proof. I. Suppose that every $\mathcal{G}_a$ is strongly dominated by t . Let $p = \{p_n\} \in \Psi$. We may suppose that $A = N$ and every $\mathcal{G}_a$ is a filter on N . Put $c_0 = 0$;

for $k = 1,2,\ldots$ let c_k be equal to the least n such that $p_n > k$. For $\langle m,n\rangle \in N \times N$ put $\varphi\langle m,n\rangle = c_m + n$. It is easy to see that, for any $k \in N$, the set $\varphi^{-1}[k]$ contains p_k elements. For any $\langle m,n\rangle \in N \times N$ put $u_{m,n} = t_{\varphi\langle m,n\rangle}$; put $u = \{u_{m,n} \mid m \in N,\ n \in N\}$. It can be easily shown that $u^* = p * t$.

We are going to prove that $\sum_{\mathcal{F}} \mathcal{G}_a$ is strongly dominated by u. Let $K \subset N \times N$ be such that $\sum\{u_{m,n} \mid \langle m,n\rangle \in K\} < \infty$. For every $m \in N$, the sequence $\{u_{m,n} \mid n \in N\} = \{t_{c_m+n} \mid n \in N\}$ strongly dominates $\mathcal{G}_m$, and therefore the filter $\mathcal{G}_m$ contains the set of all n such that $\langle m,n\rangle$ mon $\in K$. This proves that $N \times N - K$ belongs to $\sum_{\mathcal{F}} \mathcal{G}_m$. We have shown that u strongly dominates $\mathcal{F} \cdot \mathcal{G}$.

II. If every $\mathcal{G}_a$ is dominated by t, consider an arbitrary $p \in \Psi$. It is easy to see that there exist $q \in \Psi$, $n \in \Psi$ such that $u_n \geq v_n$ where $\{u_n\} = p * t$, $\{v_n\} = n * (q * t)$. Then every $\mathcal{G}_a$ is strongly dominated by $q * t$, hence, by the first part of the proof, $\sum_{\mathcal{F}} \mathcal{G}_a$ is strongly dominated by $n * (q * t)$, hence by $p * t$.

4.4. <u>Proposition.</u> For any ultrafilter $\mathcal{F}$ there are $\exp\exp\aleph_0$ types of ultrafilters of the form $\mathcal{A} \cdot \mathcal{F}$.

<u>Proof.</u> For any $x \in \beta N - N$ let $\mathcal{F}(x)$ denote the ultrafilter of all $U \cap N$, U a neighborhood of x in βN. Clearly, there exists a discrete countable infinite set $S \subset \beta N - N$ such that $\mathcal{F}(x) \sim \mathcal{F}$ for every $x \in S$. It is easy to see that for any $y \in \overline{S} - S$, we have $\mathcal{F}(y) \sim \sum_{\mathcal{A}}\{\mathcal{F}(x) \mid x \in S\}$ where $\mathcal{A}$ is the filter consisting of all $V \cap S$, V being a neighborhood of x. Since

$\mathcal{F}(x) \sim \mathcal{F}$, we have $\mathcal{F}(y) \sim \mathcal{A} \cdot \mathcal{F}$ for every $y \in$ $\in \overline{S} - S$. It is well known that $\overline{S}$ contains exp exp $\aleph_0$ points.

4.5. Proposition. For any $u \in \Theta$, the set of all types of ultrafilters strongly dominated by u is of cardinality exp exp $\aleph_0$.

Proof. Clearly there exist $v \in \Theta$ and $p \in \Psi$ such that, for any $n \in N$, $u_n \geq w_n$, where $\{w_n\} = p * v$. Evidently, every filter dominated by v is strongly dominated by u . By 4.3 and 4.4, there are exp exp $\aleph_0$ types of ultrafilters dominated by v .

4.6. Definition. If $\mathcal{F}$, $\mathcal{G}$ are filters and there exist mutually singular $u \in \Theta$, $v \in \Theta$ such that u dominates $\mathcal{F}$, v dominates $\mathcal{G}$, we shall say that $\mathcal{F}$ and $\mathcal{G}$ are mutually singular.

4.7. Theorem. If $\mathcal{F}$ and $\mathcal{G}$ are mutually singular filters, then, for any filters $\mathcal{A}$, $\mathcal{B}$, the filters $\mathcal{A} \cdot \mathcal{F}$ and $\mathcal{B} \cdot \mathcal{G}$ are mutually singular (hence non-equivalent). In particular, $\mathcal{F} \cdot \mathcal{G}$ and $\mathcal{G} \cdot \mathcal{F}$ are not equivalent.

This follows at once from 4.4.

Remark. By 3.5 and 3.6, there do exist mutually s gular sequences from Θ , hence also, by 4.5, mutually s gular filters (even ultrafilters).

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