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ON THE ROW SUM CONDITION AND THE CONVERGENCE OF ITERATION PROCESSES

(Preliminary communication)

Ivo MAREK, Praha

In his recent paper [10] W. Walter shows several sufficient conditions for the convergence of the Jacobi and Gauss-Seidel type iterations for linear algebraic systems. More precisely, he gives some conditions which lead to estimates of the spectral radius of the investigated matrix, and these are less than unity. One of Walter's requirements is the fulfilling of the so called weak "row sum condition" (in German "Zeilensummenkriterium", see [10]). This single condition is not sufficient for the convergence in general and Walter gives new supplementary conditions under which the convergence is guaranteed. Walter also applies his method to infinite systems of lower equations. Our aim is to extend the application of the weak row sum condition to infinitely dimensional Banach spaces with cones.

Let $\mathcal{Y}$ be a real Banach space, $\mathcal{X}$ its complexification, $\mathcal{Y}'$ and $\mathcal{X}'$ the corresponding dual spaces, $[\mathcal{Y}]$ and $[\mathcal{X}]$ the spaces of bounded linear operators mapping $\mathcal{Y}$ and $\mathcal{X}$ into $\mathcal{Y}$ and $\mathcal{X}$ respectively. All these spaces are assumed to be normed in the usual way, so that they are Banach spaces.

Let $K \subset \mathcal{Y}$ be a generating and normal cone (see [2]).

By K' we denote the adjoint cone to K ([2]). A subset $H' \subset K'$ is termed K-total, if the relations $\langle x, x' \rangle \geqq \geqq 0$ for all $x' \in H'$ imply $x \in K$, where $\langle x, x' \rangle$ denotes $x'(x)$.

If $y - x = z \in K$, we write $x \prec y$ or $y \succ x$. An element $x \in K$ is called non-support ([5]) or quasiinterior ([7]) x), if $\langle x, x' \rangle \neq 0$ for every linear form $x' \in K'$, $x' \neq \sigma$, where σ denotes the zero-element in all spaces $\mathcal{Y}$, $\mathcal{Y}'$, $\mathfrak{X}$, $\mathfrak{X}'$.

Let $u_0 \in K$, $\| u_0 \| = 1$. We say that $x \in K$ is u_0-positive, if there is a $\tau > 0$ such that $x \succ \tau u_0$.

If $T \in [\mathcal{Y}]$ and $TK \subset K$, T is called K-positive or, simply, positive ([2]), and we write $T \succ \theta$ or $\theta \prec T$, where θ denotes the zero-operator.

A K-positive operator $T \in [\mathcal{Y}]$ is called semi-non-support ([5]), if for every pair $x \in K$ and $x' \in K'$, $x \neq \sigma$, $x' \neq \sigma$, there is a positive integer $p = p(x, x')$ such that $\langle T^p x, x' \rangle \neq 0$. A K-positive operator $T \in [\mathcal{Y}]$ is called u_0-positive ([1,p.60]), if there is an element $u_0 \in K$, $\| u_0 \| = 1$ such that for every $x \in K$, $x \neq \sigma$, there exist positive numbers $\alpha = \alpha(x)$ and $\beta = \beta(x)$ and a positive integer $p = p(x)$ such that

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x) It can be shown that an element $x \in K$ is non-support if and only if the linear hull of the set $Q = \{ y \mid \sigma \prec y \prec x \}$ is the whole space $\mathcal{Y}$.

$$\alpha u_0 \prec T^n x \prec \beta u_0 .$$

A space $\mathcal{Y}$ will be termed a Riesz space ([6]), if for every pair x and $y \in \mathcal{Y}$ there exist $\sup(x, y)$ and $\inf(x, y)$, both belonging to $\mathcal{Y}$.

Let $T \in [\mathcal{Y}]$. We denote by $\hat{T}$ the complex extension of T, which is defined as follows: $\hat{T}z = Tx + iTy$, where $z = x + iy$, x and $y \in \mathcal{Y}$. Further, we define $\sigma(T) = \sigma(\hat{T})$, where $\sigma(A)$ is the spectrum of the operator $A \in [\mathfrak{X}]$ and $r(T) = r(\hat{T})$, where $r(A)$ is the spectral radius of $A \in [\mathfrak{X}]$, i.e. $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$. Evidently $T \in [\mathcal{Y}]$ implies $\hat{T} \in [\mathfrak{X}]$.

Let $T \in [\mathcal{Y}]$. If $\mu_0 \in \sigma(T)$ is an isolated singularity of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$, then $R(\lambda, T)$ can be developped into Laurent expansion ([8, p.305])

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k (\lambda - \mu_0)^k + \sum_{k=1}^{\infty} B_k (\lambda - \mu_0)^{-k} ,$$

where $A_k \in [\mathfrak{X}]$, $k = 0, 1, \dots$; $B_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T)\, d\lambda$,

where $C_0 = \{\lambda \mid |\lambda - \mu_0| = \rho_0\}$ is such that $K_0 \cap \sigma(T) = \{\mu_0\}$ for $K_0 = \{\lambda \mid |\lambda - \mu_0| \leqq \rho_0\}$. Further, it is known that $B_{k+1} = (T - \mu_0 I) B_k$, $k = 1, 2, \dots$. In case $B_q \neq \theta$, $B_{q+1} = \theta$, the singularity μ_0 is called a pole of $R(\lambda, T)$ and q its multiplicity.

An operator $T \in [\mathcal{Y}]$ or $T \in [\mathfrak{X}]$, by definition, has property (S), if the conditions $\lambda \in \sigma(T)$, $|\lambda| = r(T)$, imply λ is a pole of the resolvent $R(\lambda, T)$.

We shall assume the space Y and the operators in $[Y]$ to be such that every $T \in [Y]$ can be written as $T = T^{+} - T^{-}$, where T^{+} and $T^{-} \in [Y]$ and $T^{+} \geqq \theta$ and $T^{-} \geqq \theta$. Then we put $|T| = T^{+} + T^{-}$. Evidently, $T \neq \theta$ implies $|T| \neq \theta$.

Suppose $H' \subset K'$ is a K-total set. Let $H' = H'_{\alpha} \cup H'_{\beta}$, where $H'_{\alpha} \cap H'_{\beta} = \emptyset$. We say that the operator $T \in [Y]$ fulfils the row sum condition, simply, condition (RS), if there is a quasiinterior element $\hat{x} \in K$ such that the relations

$$(1) \quad \begin{cases} \langle |T| \hat{x}, x' \rangle \leqq \varrho \langle \hat{x}, x' \rangle, & x' \in H'_{\alpha}, \\ \langle |T| \hat{x}, x' \rangle = \langle \hat{x}, x' \rangle, & x' \in H'_{\beta} \end{cases}$$

are valid with some ϱ, $0 \leqq \varrho < 1$ and $H'_{\alpha} \neq \emptyset$.

In case $H'_{\beta} = \emptyset$ ($H'_{\beta} \neq \emptyset$) we say that the operator T strongly (weakly) fulfils the condition (RS) or that T fulfils the strong (weak) condition (RS).

<u>Theorem 1</u>. Let $\mathcal{Y}$ be a Riesz space when ordered by the cone K. Further, let the interior $\mathcal{I}nt\, K$ be non-empty.

The conditions

$$(2) \quad \begin{cases} \langle |T| \hat{x}, x' \rangle \leqq \varkappa \langle \hat{x}, x' \rangle, & x' \in H'_{\alpha}, \\ \langle |T| \hat{x}, x' \rangle = \varkappa \langle \hat{x}, x' \rangle, & x' \in H'_{\beta}, \end{cases}$$

where $\hat{x} \in K$ is a quasiinterior element and H' is a K-total set decomposed as in (1), imply the relations

$$(3) \qquad r(T) \leqq r(|T|) \leqq \varkappa .$$

Theorem 1 is well known (see [9, p. 21]).

Theorem 2. Suppose the operator $T \in [\mathcal{Y}]$ has property (S). Then relations (3) hold.

Very simple examples show that in the inequality $r(T) \leqq \varkappa$ the equality cannot be excluded in general. Hence some additional assumptions must be made to guarantee the strict inequality.

Theorem 3. Let the operator $|T|$, where $T \in [\mathcal{Y}]$, have property (S) and let

(a) $|T|$ be semi-non-support,

or

(b) $|T|$ be u_0-positive.

Further, let relations (2) be valid with $H'_\alpha \neq \emptyset$. Then

(4) $$r(T) \leqq r(|T|) < \varkappa .$$

Corollary. Under assumptions of Theorem 3 let the operator T weakly fulfil the condition (RS). Then we have

(5) $$r(T) \leqq r(|T|) < 1 .$$

Remark. It is easy to see that the assertions of Theorem 3 and its corollary are true if $T \in [\mathcal{Y}]$ is such that $r(|T|)$ is an eigenvalue of $|T|$ to which there corresponds a quasiinterior proper vector $x_0 \in K$. Hence, assuming T fulfils the weak (RS) condition, none of assumptions (a) and (b) is necessary for the fulfilment of the relations (4) and (5).

We shall say that the operator $T \in [\mathcal{Y}]$ fulfils the condition $[Z1]$, if T fulfils the condition (RS) with a non-empty H'_α and at least one of the conditions (a) and (b).

A weaker condition than the condition $[Z1]$ is Walter's condition $[Z2]$ (see[10]) which we shall generalize.

<u>Condition $[Z2]$</u>. An operator $T \in [\mathcal{Y}]$ fulfils condition $[Z2]$, if there is a quasiinterior element $\hat{x} \in K$ and a K-total set $H' \subset K'$ such that the following conditions take place:

(i) $$\langle |T| \hat{x}, x' \rangle \leqq q \langle \hat{x}, x' \rangle, \quad x' \in H'_\alpha ,$$

$$\langle |T| \hat{x}, x' \rangle = \langle \hat{x}, x' \rangle , \quad x' \in H'_\beta ,$$

where $H'_\alpha \cup H'_\beta = H'$, $H'_\alpha \neq \emptyset$, $H'_\alpha \cap H'_\beta = \emptyset$ and $0 \leqq q < 1$.

(ii) If $y \in K$, $y' \in H'_\alpha$, then there is a positive integer n such that $\langle |T|^n y, y' \rangle \neq 0$;

(iii) If $y \in K$, $\langle y, y' \rangle = 0$ for all $y' \in H'_\beta$, then $\langle |T| y, x' \rangle \leqq q' \langle y, x' \rangle$ for all $x' \in H'$, where $0 \leqq q' < 1$.

(iv) If x and $y \in K$, $\langle x, y' \rangle \leqq \langle y, y' \rangle$ for all $y' \in H'_\beta$, then $\langle |T| x, y' \rangle \leqq \langle |T| y, y' \rangle$ for all $y' \in H'_\beta$.

<u>Theorem 4</u>. Suppose the operator $|T|$, where $T \in [\mathcal{Y}]$, has property (S) and (5) fulfils condition $[Z2]$. Then we have (5).

<u>Theorem 5</u>. Let the operator $T \in [\mathcal{Y}]$ fulfil at least one of conditions $[Z1]$ and $[Z2]$. Further, let there be an element $x^+ \in K$ such that,

$$\langle x^+, x' \rangle \geqq \gamma' \langle \hat{x}, x' \rangle \quad \text{for} \quad x' \in H'_\beta ,$$

$$\langle |T| x^+, x' \rangle \leqq \gamma \langle \hat{x}, x' \rangle \text{ for } x' \in H' ,$$

where $q + \gamma < 1$, $\gamma' > \gamma > 0$. Under these assumptions the relations (5) are guaranteed.

As a consequence of Theorem 5 one can obtain an analogue of Walter's theorem from section 5 of [10] under assumptions [Z1] or [Z2].

Theorems 3,4 and 5 imply the convergence of the Jacobi type iterations to the solution of the equation

$$x = Tx + f , \quad f \in \mathcal{Y} .$$

It is interesting that conditions [Z1] or [Z2] also guarantee the fulfilment of the relation $r(H) < 1$, where $H = (I - L)^{-1} U$, and $L + U = T$, $r(L) < 1$, and hence the convergence of the Gauss-Seidel type iterations.

<u>Theorem 6</u>. Let $T \in [\mathcal{Y}]$ have $r(|T|) < 1$. Let us put $|T| = L_1 + U_1$, $L_1 \succcurlyeq \theta$, $U_1 \succcurlyeq \theta$ and let $r(H_1)$ be a proper vector of $H_1 = (I - L_1)^{-1} U_1$. Then we have

$$r(H) \leqq r(H_1) < 1 . \tag{6}$$

<u>Corollary</u>. If the operator $T \in [\mathcal{Y}]$ fulfils one of conditions [Z1] and [Z2], then there hold relations (6).

<u>Remark</u>. For $m \times m$ matrices the relation $r(H_1) < 1$ follows from the fact that H fulfils the strong (RS) condition as soon as the single matrix $L + U$ does fulfil one of [Z1] and (Z) (see [10]). But the fulfilment of the (RS) condition need not be verified to prove Theorem 6.

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