Commentationes Mathematicae Universitatis Carolinae
9,1 (1968)

MAPPINGS INTO SPACES OF OPERATORS

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In a recent paper [1] B.E.Johnson proved the fact that every strictly irreducible representation of a Banach algebra is continuous. It is the purpose of this note to show that a similar argument may be used to prove a more general result which is concerned with algebraic homomorphisms of Banach spaces into spaces of linear operators. In this preliminary report we only give the proof of this main result; it has a number of consequences which will be published in the full version of the note.

Theorem. Let (Y,q) and (X,ω) be two normed spaces. Let (A,p) be a Banach space and T an algebraic homomorphism of A into $L((Y,q),(X,\omega))$. Suppose that the following two conditions are satisfied.

1^o Given $y_1,\dots,y_n \in Y$ and $x_1,\dots,x_n \in X$ such that the y_i are linearly independent then there exists an $a \in A$ such that $T_a y_i = x_i$;

2^o for each $y \in Y$ the set $N(y) = \{a \in A ; T_a y = 0\}$ is closed in (A,p) .

Then either Y is finite-dimensional or the mapping T is continuous.

Proof. The proof will be divided into four steps.

For the sake of brevity, we shall write ay for $T_a y$.

I. Let us prove first the following assertion.

(F) There exists a finite sequence $y_1, \dots, y_n \in Y$ such that every $y \in Y$ is continuous on $N(y_1) \cap \dots \cap N(y_n)$.

Suppose that (F) is not true. Take a discontinuous $y_1 \in Y$ with $q(y_1) = 1$. Since y_1 is discontinuous on (A, p) there exists an $a_1 \in A$ with $p(a_1) \leq 1$ and $\omega(a_1 y_1) \geq 2$.

Since (F) is not true there exists a $y_2 \in Y$ which is discontinuous on $N(y_1)$ and $q(y_2) = 1$. Hence there exists an $a_2 \in N(y_1)$ for which $p(a_2) \leq 1$ and $\omega(a_2 y_2) \geq 2^2(2 + \frac{1}{2}\omega(a_1 y_2))$. Since there exist discontinuous y's on $N(y_1) \cap N(y_2)$ there exists a $y_3 \in Y$, $q(y_3) = 1$ and an $a_3 \in N(y_1) \cap N(y_2)$ with $p(a_3) \leq 1$ and $\omega(a_3 y_3) \geq 2^3(3 + \frac{1}{2}\omega(a_1 y_3) + (\frac{1}{2})^2\omega(a_2 y_3))$.

Proceeding by induction, we construct two sequences $a_i \in A$, $y_j \in Y$ such that $p(a_i) \leq 1$, $q(y_j) \leq 1$ and

$$a_i \in N(y_1) \cap \dots \cap N(y_{i-1}) \quad ,$$

$$\omega(a_n y_n) \geq 2^n\Big(n + \sum_{j \leq n-1} (\tfrac{1}{2})^j \omega(a_j y_n)\Big) \quad .$$

Define now $a \in A$ as

$a = \sum_{j \geq 1} (\frac{1}{2})^j a_j$ so that $p(a) \leq 1$. Given a natural number n, we have

$$a y_n = \sum_{j \leq n} (\tfrac{1}{2})^j a_j y_n + v y_n$$

where $v = \sum_{j \geq n+1} (\frac{1}{2})^j a_j$. For $j \geq n+1$ we have $a_j \in N(y_n)$ so that, $N(y_n)$ being closed, the vector v belongs to $N(y_n)$ as well. It follows that

$$\omega(a y_n) = \omega\Big((\tfrac{1}{2})^n a_n y_n + \sum_{j \leq n-1} (\tfrac{1}{2})^j a_j y_n \Big) \geq$$
$$\geq (\tfrac{1}{2})^n \omega(a_n y_n) - \sum_{j \leq n-1} (\tfrac{1}{2})^j \omega(a_j y_n) \geq n$$

which contradicts the continuity of a in the second variable. The proof of (F) is thus complete.

II. Let us show now that every $y \in Y$ which does not belong to the subspace generated by $y_1, \dots, y_n$ is already continuous. Take an arbitrary $y \in Y$ which is linearly independent of $y_1, \dots, y_n$. We may clearly assume that $y_1, \dots, y_n$ are linearly independent. We begin by proving that $A = N(y_1) \cap \dots \cap N(y_n) + N(y)$.

Indeed, let $a \in A$ be given. Since the $n+1$ elements $y_1, \dots, y_n, y$ are linearly independent, there exists, by assumption 1^o, a $c \in A$ with $cy_1 = \dots = cy_n = 0$ and $cy = ay$. If we write $a = c + (a - c)$, we have $c \in N(y_1) \cap \dots \cap N(y_n)$ and $(a - c) \in N(y)$.

Since (A,p) is complete and both $N(y_1) \cap \dots \cap N(y_n)$ and $N(y)$ are closed, there exists a $\sigma > 0$ such that every $a \in A$ may be written in the form $a = u + v$, $u \in N(y_1) \cap \dots N(y_n)$, $v \in N(y)$ and $p(u) + p(v) \leq \sigma p(a)$.

Now y is continuous on $N(y_1) \cap \dots \cap N(y_n)$ so that there exists a $\beta > 0$ such that $x \in N(y_1) \cap \dots \cap N(y_n)$ implies $\omega(xy) \leq \beta p(x)$.

If $a \in A$, we have

$$\omega(ay) = \omega(uy + vy) = \omega(uy) \leq \beta p(u) \leq \beta \sigma p(a)$$

and the proof is complete.

III. We have shown that there exists a finite-dimensional subspace of Y which contains all those $y \in Y$ which are not continuous on (A,p). Denote by Y_0 the smallest subspace of this property and let us show that either $Y_0 = Y$ or $Y_0 = 0$. Indeed, consider the case $Y_0 \neq Y$. Choose a $y^* \in Y$ outside Y_0. We intend to show that every $y \in Y$ is continuous. According to the preceding part of the proof, this is immediate if y lies outside Y_0.

If $y \in Y_0$ then both $y + y^*$ and $y - y^*$ belong to the complement of Y_0. Hence both $y + y^*$ and $y - y^*$ are continuous and so is $y = \frac{1}{2}(y+y^*) + \frac{1}{2}(y - y^*)$. It follows that $Y_0 = 0$.

IV. The proof is concluded by a standard category argument. Denote by B the unit ball of (Y,q). If Y is infinite-dimensional the space Y_0 has to be the zero space so that Y may be considered as a subspace of $L((A,p),(X,\omega))$. Let us show now that the set $B \subset L((A,p),(X,\omega))$ is pointwise bounded on A. This, however, is a consequence of the fact that T is a mapping into $L((Y,q),(X,\omega))$. Indeed, if $a \in A$ is fixed and if $y \in B$, we have

$$\omega(ay) \leq |T_a| q(y) \leq |T_a| .$$

It follows that the set B is bounded in $L((A,p),(X,\omega))$; hence there exists a $\sigma > 0$ such that

$$\omega(ay) \leq \sigma p(a) q(y)$$

which proves the theorem.

R e f e r e n c e s

[1] B.E. JOHNSON: The uniqueness of the complete norm topology, Bull.Amer.Math.Soc. 73(1967), 537-539.

(Received January 5,1968)