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Commentationes Mathematicae Universitatis Carolinae, Vol. 9 (1968), No. 1, 157--159

Persistent URL: http://dml.cz/dmlcz/105165

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9,1 (1968)

A PARTIAL GENERALIZATION OF A THEOREM OF HURSCH

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This paper is based on part V of the author's thesis, Symmetric generalized uniform and proximity spaces, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the graduate school of Arts and Sciences of the University of Connecticut. The author wishes to acknowledge his indebtedness to Professor E.S. Wolk, under whose direction the thesis was written.

Let $\mathcal{P}$ be a symmetric generalized proximity space (c.f. [1]) with proximity class $\Pi(\mathcal{P})$.

Theorem 1. There exists a symmetric generalized uniform space (cf. [2]), $\mathcal{U}(\mathcal{P})_1$, such that $\mathcal{P}(\mathcal{U}(\mathcal{P})_1) = \mathcal{P}$.

Proof. (for notation c.f. [2]) Let X be a set with power set $P(X)$. For every A, B in $P(X)$ let $U_{A,B}$ equal $(X \times X) - ((A \times B) \cup (B \times A))$. Let $\mathcal{V} = \{ U_{A,B} \mid (A, B) \notin \mathcal{P} \}$. Clearly, $\mathcal{V}$ satisfies M_2. Suppose $A \bar{\mathcal{P}} B$. Then $U_{A,B}[A] \cap B = \Phi$.

Conversely, suppose there exists C, D such that $C \bar{\mathcal{P}} D$ and $U_{C,D}[A] \cap B = \Phi$. Then it is easily shown that $(A \subseteq C$ and $B \subseteq D)$ or $(A \subseteq D$ and $B \subseteq C)$. Hence $A \bar{\mathcal{P}} B$. So that (by theorem 1 in [2]) we have that $\mathcal{V}$ satisfies M_1, M_3, and M_4. Let $\mathcal{U}(\mathcal{P})_1$ equal $\{ U \mid U = U^{-1}$ and $U \supseteq V$ for some V in $\mathcal{V}\}$. $\mathcal{U}(\mathcal{P})_1$ (by theorem 5 in [2]) is a symmetric generalized uniform space on X. It is easy to show that $\mathcal{P}(\mathcal{U}(\mathcal{P})_1) = \mathcal{P}$.

Theorem 2. $\mathcal{U}(\mathcal{P})_1$ is the least element of $\Pi(\mathcal{P})$.

Proof. Let $U_{A,B}$ be in $\mathcal{U}(\mathcal{P})_1$. Then $A \bar{\mathcal{P}} B$. But it is easily shown that there exists V in $\mathcal{U}$ such that $(A \times B) \cap V = \Phi$. But since $V = V^{-1}$ we have that $(B \times A) \cap V = \Phi$., Hence $U_{A,B} \supseteq V$.

Theorem 3. If $\mathcal{P}$ is the usual proximity for the reals X, then $\mathcal{U}(\mathcal{P})_1$ is properly contained in $\mathcal{U}(\mathcal{P})^*$, the Alfsen-Fenstad uniformity in $\Pi(\mathcal{P})$ (c.f. [3]).

Proof. We know by the previous theorem that $\mathcal{U}(\mathcal{P})_1 \subseteq \mathcal{U}(\mathcal{P})^*$. Let $A = [1,2]$; $B = [2,3]$; $A_1 = [3,4]$; $B_1 = [4,5]$. Clearly, $A \bar{\mathcal{P}} A_1$ and $B \bar{\mathcal{P}} B_1$. Suppose there exists P, Q such that $P \bar{\mathcal{P}} Q$ and $U_{P,Q} \subseteq U_{A,A_1} \cap U_{B,B_1}$; then $P \times Q \supseteq (A \times A_1) \cup (B \times B_1) = E_1$ and $Q \times P \supseteq (A_1 \times A) \cup (B_1 \times B) = E_2$. But $(3,5) \in (B \times B_1)$ implies $(3,5) \in E_1$ implies $(3,5) \in P \times Q$; implies $3 \in P$; and $(3,1) \in (A_1 \times A)$ implies $(3,1) \in E_2$ implies $(3,5) \in Q \times P$ implies $3 \in Q$. Hence $P \cap Q \neq \Phi$ implies $P \mathcal{P} Q$ which is a contradiction. Hence there does not exist P, Q such that $P \bar{\mathcal{P}} Q$ and $U_{P,Q} \subseteq U_{A,A_1} \cap U_{B,B_1}$.

Theorem 4. (Hursch). Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces, and let $\mathcal{V}$ be totally bounded. If $f: (X, \mathcal{P}(\mathcal{U}))$ into $(Y, \mathcal{P}(\mathcal{V}))$ is p-continuous, then it is uniformly continuous from $(X, \mathcal{U})$ into $(Y, \mathcal{V})$. (c.f. [4] p.202).

We obtain a partial generalization of the Hursch theorem with the following

Theorem 5. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be symmetric generalized uniform spaces, and let $\mathcal{V}$ be equal to $\mathcal{U}(\mathcal{P})_1$

for some symmetric generalized proximity space $\mathcal{P}$ .If f: $:(X, \mathcal{P}(\mathcal{U}))$ into $(Y, \mathcal{P}(\mathcal{V}))$ is p-continuous, then it is uniformly continuous from $(X, \mathcal{U})$ into $(Y, \mathcal{V})$.

<u>Lemma</u>: $f:(X, \mathcal{U}(\mathcal{P})_1)$ into $(Y, \mathcal{V}(\mathcal{P})_1)$ is uniformly continuous.

<u>Proof of Lemma</u>: Suppose $V \in \mathcal{V}(\mathcal{P})_1$. There exists C, D such that $C\ \overline{\mathcal{P}(\mathcal{V})}\ D$ and $V \supseteq U_{C,D}$. But since f is p-continuous, $f^{-1}(C)\ \overline{\mathcal{P}(\mathcal{U})}\ f^{-1}(D)$. Let U = $= U_{f^{-1}(C), f^{-1}(D)}$. It is easily shown that $(x,y) \in$ $\in$ U implies $(f(x), f(y)) \in V$. The proof of theorem 5 is an immediate consequence of the lemma and theorem 2.

$\mathcal{U}(\mathcal{P})_1$ is easily shown to be totally bounded; hence, by theorem 2 and theorem 3 we have that a complete generalization of the Hursch theorem for symmetric generalized uniform spaces is not possible.

R e f e r e n c e s

[1] M.W. LODATO: On topologically induced generalized proximity relations II,Pacific Journal of Mathematics,17,No.1(1966),131-135.

[2] C.J. MOZZOCHI: A correct system of axioms for a symmetric generalized uniform space(to appear), Mathematica Scandinavica.

[3] C.J. MOZZOCHI: A generalization of a theorem of Alfsen and Fenstad(to appear), Pacific Journal of Mathematics.

[4] W.J. THRON: Topological Structures, Holt,Rinehart,and Winston,New York 1966.

(Received January 11,1968)