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REMARKS ON NONLINEAR FUNCTIONALS

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Introduction. Weakly lower-semicontinous functionals play an important role in the theory of variational methods. The following well-known result [1,th.9.1] is basic in the theory of the extrema: Let X be a reflexive linear normed space, f a weakly lower-semicontinuous finite functional on a bounded weakly closed subset E of X. Then f is bounded below and attains its lower bound on E. M.M.Vajnberg [1,chapt.III] has introduced so-called m-property of weakly lower-semicontinuous functionals as follows: A weakly lower-semicontinuous functional f is said to have the m-property in X if there exist a bounded weakly closed subset $E \subset X$ and interior point x_0 of E such that $f(x) > f(x_0)$ for each x on the boundary of E. In such spaces X, a G-differentiable (i.e. a G-derivative $f'(x)$ exists at every $x \in X$) functional with the m-property has at least one critical point. Some recent investigations in these topics have been obtained by M.M.Vajnberg [2], R.I.Kačurovskij [3],[4], B.T.Poljak [5],[6], E.S.Levitin-B.T.Poljak [7], M.Z.Nashed [8] and others.

Section 2 of this note contains a theorem concerning the global extrema of weakly lower-semicontinuous functionals defined on the whole space X. Thus this theorem permits to investigate the extrema of such functionals defined not only on bounded subsets of X, but on the whole X. Further some basic properties of weakly lower-semicontinuous functionals are described. Section 3 contains some remarks close related to [9] concerning the G-differentiability and boundedness of convex functionals.

1. Notation and definitions. Let X be a real linear normed space, X^* its dual, E_1 the set of all real numbers, (x, e^*) a pairing between $e^* \in X^*$ and $x \in X$. A functional f defined on a convex set $M \subset X$ is called convex (quasi-convex - see for instance [8]) if $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ $(f(\lambda x+(1-\lambda)y) \leq \max(f(x), f(y)))$ for each $x, y \in M$ and $\lambda \in (0, 1)$. A functional f is said to be strictly quasi-convex [8] if $f(x) < f(y)$ implies $f(\lambda x + (1-\lambda)y) < f(y)$. We shall use the symbols "$\longrightarrow$", "$\xrightarrow{w}$" to denote the strong and weak convergence in X. A functional $f: X \longrightarrow E_1$ is said to be weakly lower-semicontinuous (weakly continuous) at $x_0 \in X$ if $x_n \xrightarrow{w} x_0$ implies $f(x_0) \leq \varliminf_{n\to\infty} f(x_n)$ $(f(x_n) \to f(x_0))$.

We shall use the notions and notations by M.M.Vajnberg [1, chapt.I] for differentials and derivatives of mappings in abstract spaces. Recall that a reflexive linear normed space is a Banach space.

2. Weakly lower-semicontinuous functionals.

In next we shall use the following

Proposition 1. Suppose that $f: X \to E_1$ is weakly lower-semicontinuous on X. Then for each real constant c the set $E(c) = \{x \in X : f(x) \leqq c\}$ is weakly closed in X. Conversely if $f: D \to X$, where $D \subseteqq X$ and $E^*(c) = \{x \in D : f(x) \leqq c\}$ is weakly closed in X for each real constant c, then f is weakly lower-semicontinuous on D.

Theorem 1. Let X be a reflexive linear normed space, $f: X \to E_1$ a weakly lower-semicontinuous functional on X. Suppose that for some real number a the set $E(a) = \{x \in X : f(x) \leqq a\}$ is bounded in X and $f(x) \neq -\infty$ for each $x \in E(a)$. Then f is bounded below on X. Furthermore, if $E(a) \neq \emptyset$, then there exists $u_0 \in X$ such that $f(u_0) = \inf_{x \in X} f(x)$ and $u_0 \in E(a)$.

Proof. If $E(a) = \emptyset$, then the first assertion is obvious. Suppose that $E(a) \neq \emptyset$ and is bounded for some a. Assume f is not bounded below on X. Then there exist $x_n \in X$ such that $f(x_n) < -n$ $(n = 1, 2, \dots)$. Hence there exists an index n_0 such that for $n \geqq n_0$ we have $f(x_n) \leqq a$. Thus $x_n \in E(a)$ for each n, $n \geqq n_0$. According to our assumption, $\{x_n\}$ is bounded in X. Since X is a reflexive Banach space, passing a subsequence $\{x_{n_k}\}$, we obtain that $x_{n_k} \xrightarrow{w} x_0$. Hence $x_{n_k} \in E(a)$ for each $k \geqq k_0$. Since f is weakly lower-semicontinuous on X, using Proposition 1, we see that $E(a)$ is weakly closed in X. Hence $x_0 \in E(a)$. In view of weak lower-semicontinuity of f, $f(x_0) \leqq \underline{\lim}_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(x_{k_n}) = -\infty$,

which is a contradiction with the fact that f does not attain the value $-\infty$ on $E(a)$. Set $d = \inf_{x \in X} f(x)$. Then $d \leqq a$. If $d = a$, then $f(x) = a$ for each $x \in E(a)$, and f attains its lower bound on $E(a)$. If $d < a$, we choose a positive number ε such that $d + \varepsilon < a$. There exists a sequence $\{u_n\} \in X$ such that $f(u_n) \to d$ and hence there exists an index n_1 such that for each $n \geqq n_1$ we have $f(u_n) \leqq d + \varepsilon$. Therefore $u_n \in E(a)$ for each $n \geqq n_1$, and $\{u_n\}$ is bounded in X. Again, in view of reflexivity of X, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \xrightarrow{w} u_0$ and $u_0 \in X$. Since $u_{n_k} \in E(a)$ for each $k \geqq k_1$, $u_0 \in E(a)$. But

$$f(u_0) \leqq \underline{\lim}_{k \to \infty} f(u_{n_k}) = \lim_{k \to \infty} f(u_{n_k}) = d .$$

On the other hand $f(u_0) \geqq d$. Thus $f(u_0) = d$ and this completes the proof.

Corollary 1. Let X be a reflexive linear normed space, $f: X \to E_1$ a quasi-convex lower-semicontinuous functional on X. Suppose that for some real number a the set $E(a) = \{x \in X : f(x) \leqq a\}$ is bounded in X and $f(x) \neq -\infty$ for each $x \in E(a)$. Then f is bounded below on X. If $E(a) \neq \emptyset$, then there exists $u_0 \in X$ such that $f(u_0) = \inf_{x \in X} f(x)$ and $u_0 \in E(a)$. Moreover, if f is strictly quasi-convex, then u_0 is unique.

A quasi-convex lower-semicontinuous functional on X is weakly lower-semicontinuous. This fact has been observed by B.T.Poljak [5] and M.Z.Nashed [8]. Since for each real constant c the set $E(c) = \{x \in X : f(x) \leqq c\}$ is convex and closed and hence weakly closed in X, this fact follows also at once from Proposition 1.

Analysing the proof of Theorem 1 it is easy to see that the assertion of Th.1 one may rewrite as follows:

Theorem 2. Let X be a reflexive linear normed space, $f: D \to E_1$ a functional defined on $D \subseteq X$ and such that for some real number a the set $E^*(a) = \{x \in D : f(x) \leqq a\}$ is bounded and weakly closed in X. If f is weakly lower-semicontinuous on $E^*(a)$ and $f(x) \neq -\infty$ for each $x \in$ $\in E^*(a)$, then f is bounded below on X. Moreover, if $E^*(a) \neq \emptyset$, then there exists $u_0 \in X$ such that $f(u_0) =$ $= \inf_{x \in X} f(x)$ and $u_0 \in E^*(a)$.

Proposition 2. Let X be a linear normed space, G a non-empty set of weakly lower-semicontinuous functionals on X. If $f(x) = \sup\{\varphi(x); \varphi \in G\}$ for every $x \in X$, then f is weakly lower-semicontinuous on X. In particular, if $\{f_n\}$ is a sequence of weakly lower-semicontinuous functionals and $f_n \nearrow f$ on X, then f is weakly lower-semicontinuous on X.

Proof. Let c be any real number. Set $E(c) = \{x \in X :$ $: f(x) \leqq c\}$. We shall prove that $E(c)$ is weakly closed in X. Let $x_n \in E(c)$ and $x_n \xrightarrow{w} x_0$ in X. Then $f(x_n) \leqq c$. Since $f(x) = \sup\{\varphi(x) : \varphi \in G\}$, $\varphi(x_n) \leqq f(x_n) \leqq c$ for an arbitrary $\varphi \in G$. Since $\varphi \in G$ are weakly lower-semicontinuous, $\varphi(x_0) \leqq \underline{\lim}_{n \to \infty} \varphi(x_n) \leqq c$ for any $\varphi \in G$. Hence $\sup_{\varphi \in G} \varphi(x_0) \leqq c$ and therefore $f(x_0) \leqq$ $\leqq c$. Thus $x_0 \in E(c)$ and $E(c)$ is weakly closed in X. According to Proposition 1 f is weakly lower-semicontinuous on X. If $f_n \nearrow f$, then we set $f(x) = \sup_n f_n(x)$ for every $x \in X$. Using the first part of our theorem to $f(x)$

we obtain at once the second assertion. This concludes the proof.

Corollary 2. Let $\{f_n\}$ be a monotone increasing (decreasing) sequence of functionals $f_n : X \to E_1$ $(n = 1, 2, \dots)$. If f_n $(n = 1, 2, \dots)$ are weakly continuous on X, then $f = \lim_{n\to\infty} f_n$ is weakly lower-semicontinuous (weakly upper-semicontinuous) on X.

Proposition 3. Let X be a linear normed space, $f: X \to E_1$ a functional weakly lower-semicontinuous at $x_0 \in X$. Then for each number A, $A < f(x_0)$ there exist a number $\delta(A, x_0) > 0$ and $e^*_{A,x_0} \in X^*$ such that if $|(x - x_0, e^*_{A,x_0})| < \delta$, then $f(x) > A$.

Proof. Suppose f is weakly lower-semicontinuous at $x_0 \in X$. Assume on the contrary that there exists $A_0 < f(x_0)$ such that for every $\delta_n = \frac{1}{n}$ and $e^* \in X^*$ there exists x_n $(n = 1, 2, \dots)$ such that

(1) $|(x_n - x_0, e^*)| < \frac{1}{n}$ imply $f(x_n) < A_0$

$(n = 1, 2, \dots)$. Then $x_n \xrightarrow{w} x_0$ and in view of weak lower-semicontinuity of f at x_0, $f(x_0) \leqq \underline{\lim}_{n\to\infty} f(x_n)$. But from (1) it follows that

$$f(x_0) \leqq \underline{\lim}_{n\to\infty} f(x_n) \leqq A_0 < f(x_0)$$

which is a contradiction. This concludes the proof.

Proposition 4. Let X be a linear normed space, $f: X \to E_1$ a weakly lower-semicontinuous functional on X. If f is bounded below on X, then there exists a sequence $\{f_n\}$ of functionals $f_n : X \to E_1$ weakly continuous on X and such that $f_n \nearrow f$.

Proof. Let x be an arbitrary point of X and $A_x < f(x)$. According to Proposition 3 there exist $\sigma_x > 0$ and $e^*_{A_x,x} \in X^*$ such that if $|(z-x, e^*_{A_x,x})| < \sigma_x$, then $f(z) > A_x$. Set $f_n(x) = \inf_{y \in X}\{f(y) + n|(y-x, e^*_{A_x,x})|\}$ and use the arguments similar to that [10, chapt.6].

Remark. We can replace X (as a domain of f or f) by an arbitrary convex closed subset of X. The generalized Dini's theorem [1, § 22] is valid under the following weaker assumptions: X is a reflexive linear normed space, $f_n(f)$ weakly lower-semicontinuous (w.upper s.) on D_R ($\|x\| \leq R$), $f_n \nearrow f$ on D_R.

3. Convex functionals. We prove the following

Theorem 3. Let X be a separable linear normed space, $f: X \to E_1$ a convex finite functional on X. Suppose that there exists an open subset $U \neq \emptyset$ of X such that f is upper bounded on U/in particular, assume that f is upper-semicontinuous on X/. Then the set Z of all $x \in X$ where the Gâteaux derivative $f'(x)$ of f exists is a G_δ-set.

Proof. By [13, chapt.II] f is continuous on X. The one-sided Gâteaux differential $V_+f(x,h)$ exists for each $x \in X$ and $h \in X$ [11]. By lemma 2 [9] $V_+f(x,.)$ is continuous at $h=0$ for each $x \in X$. Since $V_+f(x,h)$ is subadditive [11] in $h \in X$ and $V_+f(x,0)=0$, $V_+f(x,h)$ is continuous in $h \in X$ for each $x \in X$. Repeating the considerations of the first part of the proof of Th.6 [9] we see that the set Z of all $x \in X$ where the Gâteaux differential $Vf(x,h)$ exists is a G_δ-set. But convexity and continuity of f imply that $Vf(x,h) = Df(x,h) = f'(x)h$ for each $x \in Z$.

Proposition 6. Let X be a linear normed space, $f: X \to E_1$ a convex functional continuous at some point $x_0 \in X$. If there exists the Gâteaux differential $Vf(x_0, h)$ then

f possesses the Gâteaux derivative $f'(x_0)$ at x_0 .

Remark. Under the assumptions of Theorem 8 [9] suppose that f is also finite on X . Then the assertion b) of Th.8 holds as follows: the one-sided Gâteaux differential $V_+ f(x_0, h)$ is continuous and weakly lower-semicontinuous in h on X . In fact, $V_+ f(x_0, h)$ is subadditive [11] in $h \in X$ and by lemma 2 [9] continuous at $h = 0$. Since $V_+ f(x_0, 0) = 0$, $V_+ f(x_0, h)$ is continuous on X . Being continuous and convex [11] in $h \in X$, $V_+ f(x_0, h)$ is weakly lower-semicontinuous on X .

For some recent results concerning the weaker notion than the Gâteaux derivative of convex functions see [12] and the papers cited here.

Theorem 4. Let X be a linear normed space, $f: X \longrightarrow E_1$ a convex functional on X . If f is lower-semicontinuous at 0 , then f is bounded below on any closed ball $D_R (\|x\| \leq R)$. Moreover, if $f(0)$ is finite and $f(x) \leq c$ for each $x \in X$ with $\|x\| = R$, then f is bounded in D_R and continuous in $B_R (\|x\| < R)$.

Proof. Since f is lower-semicontinuous at 0 , for $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that if $\|x\| \leq \delta_0$ then $f(x) \geq f(0) - \varepsilon_0$. If $R \leq \delta_0$, then the first assertion follows at once from the last inequality. Suppose that $R > \delta_0$ and let x be an arbitrary point of D_R with $\|x\| > \delta_0$. Then $\frac{x \delta_0}{\|x\|} \in D_{\delta_0} (\|x\| \leq \delta_0)$

and

$$f(0) - \varepsilon_0 \leq f\left(\frac{x \delta_0}{\|x\|}\right) . \tag{1}$$

Since $\frac{\delta_0}{\|x\|} \in (0,1)$ and f is convex,

$$f\left(\frac{x\delta_0}{\|x\|}\right) = f\left(\left(1-\frac{\delta_0}{\|x\|}\right)0 + \frac{\delta_0}{\|x\|}x\right) \leq$$

$$\leq \left(1-\frac{\delta_0}{\|x\|}\right)f(0) + \frac{\delta_0}{\|x\|} f(x) .$$

From (1) it follows that

$$(2) \qquad f(x) \geq -\frac{\varepsilon_0}{\delta_0} R + f(0)$$

for each $x \in D_R$ with $\|x\| > \delta_0$. But for each $x \in X$ with $\|x\| \leq \delta_0$ we have that $f(x) \geq f(0) - \varepsilon_0$. Since $f(0) - \varepsilon_0 > f(0) - \frac{R}{\delta_0}\varepsilon_0$, (2) holds for every $x \in D_R$. To prove the second assertion it is sufficient to show that f is upper bounded in D_R.

Assume $f(0) \geq 0$ and $x \neq 0$ is an arbitrary point of D_R with $\|x\| < R$. Then $0 < \frac{\|x\|}{R} < 1$, $\frac{Rx}{\|x\|} \in S_R$ $(\|x\| = R)$ and hence $f\left(\frac{Rx}{\|x\|}\right) \leq c$.

We have

$$f(x) = f\left(R\frac{x}{\|x\|}\frac{\|x\|}{R}\right) \leq \left(1-\frac{\|x\|}{R}\right) f(0) +$$

$$+ \frac{\|x\|}{R} f\left(\frac{Rx}{\|x\|}\right) \leq f(0) + C .$$

Thus in this case f is bounded on D_R. If $f(0) < 0$, we set $g(x) = f(x) - f(0)$. Then g is convex, lower-semicontinuous at 0 and $g(0) = 0$. Moreover, $g(x) \leq c - f(0)$ for each $x \in X$ with $\|x\| = R$. Using the above result to g, we see that g and hence f is bounded on D_R. Being bounded on D_R according to Theorem 2 [13, II, § 5], f is continuous in B_R.

Remark. For some results concerning the boundedness and continuity properties of nonlinear functionals see [1,chapt. I]. We recall the well-known result of I.M.Gelfand [14]. If f is a lower-semicontinuous seminorm (i.e. subadditive and $f(\alpha x) = |\alpha| f(x)$ for every α) on Banach space X, then f is bounded and hence continuous on X. Suppose that $f: X \rightarrow E_1$ is subadditive positive homogeneous and upper-semicontinuous at some $x_0 \in X$, where X is a linear normed space. Then f is bounded and continuous. In fact, subadditivity and positive homogeneity of f imply convexity. According to Corollary 1 [13,chapt.II] f is continuous. But this implies the boundedness of f.

Remark. When this note was already prepared to press, I acquainted with the paper [15], where F.E.Browder firstly has established the second assertion of Proposition 2 (see [15], Theorem 3) by an another way.

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