Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 9 (1968), No. 1, 121--131

Persistent URL: http://dml.cz/dmlcz/105162

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9,1 (1968)

TOLERANCE GRAPHS

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E.C. Zeeman [3] has introduced a concept of a tolerance on a set. M.A. Arbib [1,2] has used it for the investigation of finite automata. Obviously the tolerance can be used also in other branches of mathematics. In this paper it will be introduced into the graph theory.

The tolerance ξ on a set X is a relation on X that is reflexive and symmetric. A tolerance space (X, ξ) is a set X together with a tolerance ξ on it. In [3] the following assertion is proved.

A tolerance on X induces a tolerance on the lattice L_X of subsets of X as follows: Given $A, A' \subset X$, write $(A, A') \in \xi$ if $A \subset \xi A'$ and $A' \subset \xi A$. Then the relation ξ is a tolerance on L_X.

The symbol ξA denotes the set consisting of all elements $x \in X$ such that there exists an element $a \in A$ for which $(x, a) \in \xi$ holds. Analogously ξx is defined for $x \in X$.

Two tolerance spaces (X, ξ), (X', ξ') are called isomorphic if there is a one-to-one mapping φ of X onto X' such that $(x, y) \in \xi$ implies $(\varphi(x), \varphi(y)) \in \xi'$ for all x and y from X.

The graphs considered here will be non-directed graphs without loops, multiple edges and isolated vertices (except of the graphs of tolerance defined below in which the loops exist).

The ξ -tolerance graph is a graph G on whose vertex set U a tolerance ξ is given such that $u \in U$, $v \in U$, $(u, v) \in \xi$ implies $(\Gamma u, \Gamma v) \in \xi$.

By the symbol Γx we denote the set of all vertices of G joined with x by an edge. By e we shall denote the tolerance on U such that $(x, y) \in e$ for all x, y from U ; so e is the so-called universal relation on U . By σ we shall denote the tolerance on U such that $(x, y) \in \sigma$ if and only if x = y ; thus σ is the relation of identity on U .

<u>Theorem 1</u>. Let U be a set. The tolerances e and σ are the unique tolerances ξ on U such that every graph with the vertex set U is a ξ -tolerance graph.

<u>Proof</u>. If $\xi = e$, then evidently $(A, A') \in \xi$ for arbitrary two non-empty subsets of U , so (as we do not consider graphs with isolated vertices) $(\Gamma u, \Gamma v) \in \xi$ for each two vertices u and v from U , and, according to the definition, $(u, v) \in \xi$. If $\xi = \sigma$, then $(x, y) \in \xi$ implies x = y ,and we have $\Gamma x = \Gamma y$ which implies $(\Gamma x, \Gamma y) \in$ $\in \xi$. Now let $\xi \neq e$, $\xi \neq \sigma$. This means that there exist two vertices u, v from U such that $u \neq v$ and $(u, v) \in \xi$ and two vertices x,y from U such that $(x, y) \notin \xi$. If the vertices u, v, x, y are pairwise different, then we construct a graph G in which

$\Gamma u = \{x\}$, $\Gamma v = \{y\}$, $\Gamma x = U \dot{-} \{v, x\}$, $\Gamma y = U \dot{-} \{u, y\}$, $\Gamma z = U \dot{-} \{u, v, z\}$ for all $z \in U$ different from u,v,x,y . Such a graph is not a ξ -tolerance graph, because $(u, v) \in \xi$ and $(\Gamma u, \Gamma v) \notin \xi$. If x = u, then we put $\Gamma u = \{v, y\}$, $\Gamma v = \{u\}$, $\Gamma z = U \dot{-} \{u, v, y, z\}$, $\Gamma y = \{u\}$, for all $z \in U$ different from u,v,y . Analogously for u = y, v = x, v = y .

Denote $M(\xi) = \max |\xi m|$, $m(\xi) = \min |\xi u|$. By the symbol $\rho(u)$ we denote the degree of u in the graph G .

Theorem 2. Let u be a vertex of the ξ -tolerance graph G with the vertex set U . Let v be a vertex of G such that $(u, v) \in \xi$. Then

$$\rho(u)/M(\xi) \leqq \rho(v) \leqq \rho(u) M(\xi) .$$

Proof. The set Γv must be contained in $\xi \Gamma u$ which is the set of all z such that a $y \in \Gamma u$ exists for which $(z, y) \in \xi$. For each $y \in \Gamma u$, the number of such elements z is at most $M(\xi)$. As the number of elements of Γu is $\rho(u)$, the set $\xi \Gamma u$ contains at most $\rho(u) M(\xi)$ elements, and $\rho(v) \leqq \rho(u) M(\xi)$. The tolerance being symmetric, there is also $(v, u) \in \xi$, from which $\rho(u) \leqq \rho(v) M(\xi)$. Dividing this inequality by $M(\xi)$, we obtain $\rho(u)/M(\xi) \leqq \rho(v)$.

In [3] the following operations on graphs are defined: sum, product, Cartesian sum and Cartesian product.

Theorem 3. Let G_1, G_2 be ξ -tolerance graphs with the same vertex set U . Then the graph $G_3 = G_1 + G_2$ is a ξ -tolerance graph, too.

Proof. Let $\Gamma_1 x$, $\Gamma_2 x$, $\Gamma_3 x$, $\Gamma_4 x$ denote the sets of vertices joined by an edge with x in G_1 ,G_2 ,G_3 ,G_4 .

Let $u \in U$, $v \in U$, $(u, v) \in \xi$. As G_1, G_2 are ξ-tolerance graphs, there is $(\Gamma_1 u, \Gamma_1 v) \in \xi$, $(\Gamma_2 u, \Gamma_2 v) \in \xi$. In the graph G_3, we have $\Gamma_3 u = \Gamma_1 u \cup \Gamma_2 u$, $\Gamma_3 v = \Gamma_1 v \cup \Gamma_2 v$. So $\xi \Gamma_3 u = \xi \Gamma_1 u \cup \xi \Gamma_2 u$, $\xi \Gamma_3 v = \xi \Gamma_1 v \cup \xi \Gamma_2 v$, and the inclusions $\Gamma_1 u \subset \xi \Gamma_1 v$, $\Gamma_1 v \subset \xi \Gamma_1 u$, $\Gamma_2 u \subset \xi \Gamma_2 v$, $\Gamma_2 v \subset \xi \Gamma_2 u$ imply $\Gamma_3 u \subset \xi \Gamma_3 v$, $\Gamma_3 v \subset \xi \Gamma_3 u$, and $(\Gamma_3 u, \Gamma_3 v) \in \xi$.

Let (X_1, ξ_1), (X_2, ξ_2) be tolerance spaces. A tolerance ξ_3 is defined on the Cartesian product $X_1 \times X_2$ as follows: $([x_1, x_2], [y_1, y_2]) \in \xi_3$ if and only if $(x_1, y_1) \in \xi_1$, $(x_2, y_2) \in \xi_2$; it will be denoted by $[\xi_1, \xi_2]$.

<u>Theorem 4</u>. Let G_1 be a ξ_1-tolerance graph with the vertex set U_1 and G_2 a ξ_2-tolerance graph with the vertex set U_2. Then the graph $G_3 = G_1 + G_2$ is a ξ_3-tolerance graph, where $\xi_3 = [\xi_1, \xi_2]$.

<u>Proof</u>. Let $[x_1, x_2] \in U_1 \times U_2$, let $[y_1, y_2] \in \in \Gamma_3 [x_1, x_2]$. This means that $y_1 \in \Gamma_1 x_1$, $y_2 \in \Gamma_2 x_2$. Let $([x_1, x_2], [x_1', x_2']) \in [\xi_1, \xi_2]$; this means that $(x_1, x_1') \in \xi_1$, $(x_2, x_2') \in \xi_2$. As G_1 is a ξ_1-tolerance graph and G_2 a ξ_2-tolerance graph, we must have $(\Gamma_1 x_1, \Gamma_1 x_1') \in \xi_1$, $(\Gamma_2 x_2, \Gamma_2 x_2') \in \xi_2$, i.e., $\Gamma_1 x_1 \subset \xi_1 \Gamma_1 x_1'$, $\Gamma_1 x_1' \subset \xi_1 \Gamma_1 x_1$, $\Gamma_2 x_2 \subset \xi_2 \Gamma_2 x_2'$, $\Gamma_2 x_2' \subset \xi_2 \Gamma_2 x_2$, and thus $\Gamma_1 x_1 \times \Gamma_2 x_2 \subset \xi_1 \Gamma_1 x_1' \times \xi_2 \Gamma_2 x_2' = [\xi_1, \xi_2](\Gamma_1 x_1' \times \Gamma_2 x_2') = \xi_3 \Gamma_3 [x_1', x_2']$, $\Gamma_1 x_1' \times \Gamma_2 x_2' \subset \xi_1 \Gamma_1 x_1 \times \xi_2 \Gamma_2 x_2 = [\xi_1, \xi_2](\Gamma_1 x_1 \times \Gamma_2 x_2) = \xi_3 \Gamma_3 [x_1, x_2]$.

It is well-known that every binary symmetric relation on the set U can be represented by a non-directed graph with the vertex set U. Thus if we have a tolerance ξ on the set U, there exists a non-directed graph Ξ whose vertex set is U and two vertices u,v of Ξ are joined by an edge if and only if $(u, v) \in \xi$. The graph Ξ will be called the graph of the tolerance ξ. This graph has obviously a loop at each vertex and some vertices may be incident only with this loop.

Now, following [3], we define the product $G_1 \cdot G_2$ of two graphs (non-directed) G_1, G_2 with the same vertex set U as a directed graph G_3 in which a directed edge goes from u into v if and only if a vertex $w \in U$ exists such that there is an edge joining u and w in G_1 and an edge joining w and v in G_2. In these product graphs we consider also (directed) loops.

__Theorem 5__. Let (U, ξ) be a tolerance space, let Ξ be the graph of the tolerance ξ. Let G be a graph (without isolated vertices) with the vertex set U. Then G is a ξ-tolerance graph if and only if $G \cdot \Xi = \Xi \cdot G$.

__Proof__. Let G be a Ξ-tolerance graph and let the vertices u,v be joined by an edge in $\Xi \cdot G$. This means that there exists a vertex $w \in U$ such that u and w are joined by an edge in Ξ, i.e., $(u, w) \in \xi$, and w and v are joined by an edge in G. As G is a ξ-tolerance graph and $(u, w) \in \xi$, $v \in \Gamma w$, there must exist a vertex $z \in \Gamma u$ such that $(v, z) \in \xi$. But then u and z are joined by an edge in G and v and z in Ξ, so there is an edge from u into v in $G \cdot \Xi$. $\Xi \cdot G$ is a subgraph of $G \cdot \Xi$. Analogously we can prove that $G \cdot \Xi$ is a subgraph of $\Xi \cdot G$, and therefore $G \cdot \Xi = \Xi \cdot G$.

Now assume that $G \cdot \Xi = \Xi \cdot G$. Let $x \in U$, $x' \in U$, $(x, x') \in \xi$. Let $y \in \Gamma x$. Then there is an edge from x' into y in $\Xi \cdot G$. As $\Xi \cdot G = G \cdot \Xi$, there must be an edge from x' into y in $G \cdot \Xi$. Thus a vertex y' exists such that y' is joined by an edge with x' in G and $(y, y') \in \xi$.

This implies $\Gamma x \subset \xi \Gamma x'$ and, analogously, $\Gamma x' \subset \xi \Gamma x$.

Corollary. The graph Ξ of the tolerance ξ is a ξ-tolerance graph at an arbitrary tolerance space $(\mathcal{U}, \xi)$ (if we consider also graphs with loops as tolerance graphs).

Evidently Ξ commutes with itself considering the graph multiplication.

Now the tolerance ξ whose graph Ξ is regular of order r will be called the regular tolerance of order $r - 1$ (as in this case there is no need to count a loop twice). The regular tolerance of order 1 is the tolerance σ defined above. So we shall not investigate it. Take a regular tolerance of order 2. As mentioned above, the graph Ξ has a loop at each vertex, so if ξ is regular of order 2, the graph Ξ is a regular graph of order 1 with loops added at all vertices. Thus each of its components consists of two vertices joined by an edge and with loops at each of them (Fig.1).

Theorem 6. Let G be a ξ-tolerance graph with the vertex set U. Let ξ be regular of order 2, G be regular of order 1. Then the sum $G + \Xi$ is a graph, each of whose components is a quadrangle with loops added at all vertices (Fig.2) or an edge with its end vertices, a loop being added at each of them.

Proof. Let $(x, x') \in \xi$, $x \neq x'$ and let x be joined with a vertex y by an edge in G. Then x' must be joined in G with a vertex y' such that $(y, y') \in \xi$. If $y' = y$, the vertex y is joined in G with two different vertices x, x' which is a contradiction with the assumption that G is regular of order 1. So $y \neq y'$ and we have a quadrangle in $G + \Xi$ consisting of the edges joining the pairs x, x'

and y, y' in Ξ and the pairs x, y and x', y' in G. At each of the vertices x, x', y, y' there is a loop in Ξ and therefore also in $G + \Xi$. This quadrangle with loops added is a component of $G + \Xi$, because each vertex in it is incident exactly with two edges of Ξ and one edge of G, so it cannot be joined with any other vertex of $G + \Xi$. Now if $y = x'$, the vertices x and x' are joined by an edge of Ξ and by an edge of G, so they can be joined with any other vertex of U neither by an edge of Ξ nor by an edge of G.

Theorem 7. Let (U, ξ) be a tolerance space, let ξ be regular of degree 2. The necessary and sufficient condition for a regular graph G of degree 2 to be isomorphic with a ξ-tolerance graph with the vertex set U is that G have the number of vertices equal to the number of elements of U, and to each component of it which is a circuit with an odd number of vertices there exists an even number of components isomorphic with it (including this component itself).

Proof. Necessity of the condition concerning the number of vertices is clear. Now assume that there exists a ξ-tolerance graph G regular of degree 2, and let C be its component being a circuit with an odd number of vertices. Let $u_1, \dots, u_k$ be its vertices, k odd, and let the pairs u_i, u_{i+1} for $i = 1, \dots, k-1$ and the pair u_1, u_k be joined by an edge in C (i.e., in G). Assume that $(u_\ell, u_m) \in \xi$, where ℓ, m are two of the numbers $1, \dots, k$ and $\ell \neq m$. First of all, let $m = \ell + 1$. We have $\Gamma u_\ell = \{u_{\ell-1}, u_{\ell+1}\}$, $\Gamma u_{\ell+1} = \{u_\ell, u_{\ell+2}\}$ and $\Gamma u_\ell \subset \xi \Gamma u_{\ell+1}$, $\Gamma u_{\ell+1} \subset \xi \Gamma u_\ell$. So either $(u_{l-1}, u_l) \in \xi$, or $(u_{l-1}, u_{l+2}) \in \xi$. The first case is impossible, because ξ is regular of degree 2 and there are yet two vertices in the relation ξ with u_{l-1}, these are u_{l+1} and u_l itself (k being odd, it must be $u_{l-1} \neq u_{l+1}$). So $(u_{l-1}, u_{l+2}) \in \xi$. Analogously we shall prove that $(u_{l-2}, u_{l+3}) \in \xi$, $(u_{l-3}, u_{l+4}) \in \xi$ etc., generally $(u_{l-i}, u_{l+i+1}) \in \xi$ for all i's. But as k is odd, for $i = \frac{1}{2}(k-1)$, we have $u_{l-i} = u_{l+i+1}$. Denote by v the vertex such that $(u_{l-\frac{1}{2}(k-1)}, v) \in \xi$. All vertices of C except $u_{l-\frac{1}{2}(k-1)}$ are yet divided into pairs such that the vertices of one pair are in the relation ξ; and ξ is regular of degree 2. Thus v cannot belong to C. Let v belong to a component C' of G. As $u_{l-\frac{1}{2}(k-1)}$ is joined with $u_{l-\frac{1}{2}(k-1)+1}$, there must be a vertex w joined with v such that $(u_{l-\frac{1}{2}(k-1)+1}, w) \in \xi$. This means that either $w = u_{l+\frac{1}{2}(k-1)}$ or $w = u_{l-\frac{1}{2}(k-1)+1}$, therefore w belongs to C. But w is joined with v, so v belongs also to C, which is a contradiction.

Now let $u_m \neq u_{k+1}$, $u_l \neq u_{m+1}$. We have $\Gamma u_l = \{u_{l-1}, u_{l+1}\}$, $\Gamma u_m = \{u_{m-1}, u_{m+1}\}$. Analogously to the above considerations we prove that either $(u_{l-1}, u_{m-1}) \in \xi$ and $(u_{l+1}, u_{m+1}) \in \xi$, or $(u_{l-1}, u_{m+1}) \in \xi$, $(u_{l+1}, u_{m-1}) \in \xi$. Consider the second case. We can prove that then $(u_{l+i}, u_{m-i}) \in \xi$ for all i. Assume without the loss of generality that $m > l$. If $m - l$ is odd, then for $i = \frac{1}{2}(m - l + 1)$ we have $u_{l+i} = u_{m-i}$ and we can come to the contradiction as in the preceding case. If $m - l$ is even, then $u_{l-j} = u_{m+j}$, where $j = \frac{1}{2}(k - m + l)$, and we come again to a contradiction.

Now assume that $(u_{l+1}, u_{m-1}) \in \xi$, $(u_{l+1}, u_{m+1}) \in \xi$. Then we can prove that $(u_{l+i}, u_{m+i}) \in \xi$ for all i. Take $i = m - l$ (we assume $m > l$). So $l + i = m$, and we have $(u_m, u_{2m-l}) \in \xi$. The vertex u_m is in the relation ξ with u_l and with itself. Thus either $u_{2m-l} = u_l$, or $u_{2m-l} = u_m$. The first case implies $2m - l \equiv l \pmod{k}$, so $2m \equiv 2l \pmod{k}$, and, as k is odd, $m \equiv l \pmod{k}$, which means $m = l$, because $0 < l < m \leq k$. But we have excluded this case. The equality $u_{2m-l} = u_m$ implies $2m - l \equiv m \pmod{k}$, therefore also $m \equiv l \pmod{k}$. We have obtained a result that no two different vertices of C can be joined by an edge.

So $(u_1, v_1) \in \xi$, where v_1 is a vertex of a component $C' \neq C$ of the graph G. As u_1 is joined by an edge with u_2, the vertex v_1 must be joined with a vertex v_2 such that $(u_2, v_2) \in \xi$. The vertex v_2 belongs evidently to C'. We proceed further and assign to each u_i a vertex v_i of C' such that $(u_i, v_i) \in \xi$ and v_{i-1}, v_i are joined by an edge in C'. For $i \neq j$, $1 \leq i \leq k$, $1 \leq j \leq k$, we have $v_i \neq v_j$ because in the opposite case we should have $(v_i, u_i) \in \xi, (v_i, u_j) \in \xi, (v_i, v_i) \in \xi$, so the vertex v_i would be in the relation ξ with three different vertices, this being a contradiction with the assumption that G is regular of degree 2. We can also easily prove that v_1, v_k are joined in G. So the vertices $v_1, v_2, \dots, v_k$ form a circuit with k vertices which is evidently a component of G, i.e., it is equal to C'. The components with k vertices can be divided into such pairs $\{C, C'\}$, these pairs are pairwise disjoint, and their number is even. And as these components are circuits with the same number of vertices, they must be isomorphic with each other.

Now we shall prove the sufficiency of the condition. Let us have a regular graph G' of degree 2 with the vertex set U' satisfying the conditions. Now define a relation ξ' as follows. If C is a component of G' being a circuit with an even number p of vertices, and its vertices are $u_1, \ldots, u_p$, where u_i is joined by an edge with u_{i+1} for $i = 1, \ldots, p-1$ and u_p is joined with u_1, we put $(u_i, u_{i+p/2}) \in \xi'$. Now let q be an odd number. Divide all components of G' with q vertices (if any) into disjoint pairs (it is possible, because their number is even). If we have a pair $\{C', C''\}$, the vertices of C' are $v_1', \ldots, v_q'$, the vertices of C'' are $v_1'', \ldots, v_q''$ (going along the circuit), we put $(v_i', v_i'') \in \xi$ for $i = 1, \ldots, q$. And obviously we put also $(x, x) \in \xi$ for each vertex x of G'. The sets U and U' have the same number of vertices, the tolerances ξ and ξ' are both regular of degree 2, so the tolerance spaces $(U, \xi), (U', \xi')$ are isomorphic. If we map (U', ξ') isomorphically onto (U, ξ), the graph G' is evidently isomorphically mapped onto a ξ-tolerance graph (because G' is a ξ'-tolerance graph, which can be easily proved).

Note. In this proof the subscripts at u are taken modulo k.

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Tolerance graphs.

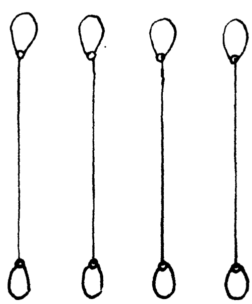

Fig. 1.

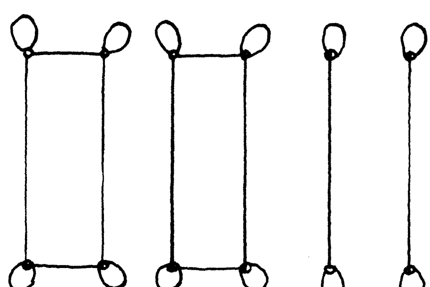

Fig. 2.

(Received January 10, 1968)