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ON ISOMORPHISM OF QUASI-ISOMORPHIC TORSION FREE ABELIAN GROUPS

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In this paper we shall give a full description of all completely decomposable torsion free Abelian groups with the property that any two such groups are quasi-isomorphic if and only if they are isomorphic. By the word "group" we shall always mean an additively written torsion free Abelian group.

Definition 1. Two groups G and H are said to be quasi-isomorphic if there exist two positive integers m, n and subgroups S, T of G and H respectively such that

(1) $mG \subseteq S \subseteq G$; $nH \subseteq T \subseteq H$ and $S \cong T$.

We write $G \doteq H$.

Definition 2. We say that a group G is an IQ-group if it is isomorphic to every group H such that $G \doteq H$.

Definition 3. We say that the group G has the IQp-property if $G \cong H$ for all subgroups H of G with $pG \subseteq H \subseteq G$.

Lemma 1. A group G is an IQ-group if and only if G has the IQp-property for every prime p.

Proof. Only the sufficiency must be proved. It is easy to see that $G \doteq H$ if and only if there exist a positive integer k and a subgroup U of G such that $kG \subseteq$

$\leq U \leq G$ and $H \cong U$. If $k = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_m^{k_m}$, then a simple induction by m shows that G is an IQ-group if and only if, for every prime p and every subgroup H of G with $p^n G \leq H$ for suitable positive integer n , there is $G \cong H$. To prove the sufficiency of the condition of Lemma 1 we apply the induction by n . (The details of the proofs are left to the reader.)

<u>Definition 4</u>. A group G is called completely decomposable if it is a direct sum of rank one groups: $G = \sum_{\iota \in I}^{d} J_\iota$.

<u>Notation</u>. The p-height of an element g of the group G is denoted by $h_p^G(g)$. If τ is a height, then $\hat{\tau}$ will be the type to which the height τ belongs. By $T(G)$ we denote the set of the types of all direct summands J_ι of a completely decomposable group $G = \sum_{\iota \in I}^{d} J_\iota$. $\hat{\tau}(g)$ will denote the type of the element g in the group G (more precisely $\hat{\tau}^G(g)$). $\hat{\tau}(G)$ denotes the type set of the group G ,i.e., the set of the $\hat{\tau}(g)$ for all $g \in G$, and finally $G(\hat{\tau}) = \{g \in G;\ \hat{\tau}(g) \geq \hat{\tau}\}$.

It is well known that, for every type $\hat{\tau}$, $G(\hat{\tau})$ is a pure subgroup of G .

In the following we shall use:

K o v á c s theorem (see [2], theorem B): If G is a completely decomposable group such that $T(G)$ is inversely well ordered (in the natural order of the types), then G is an IQ-group.

<u>Remark</u>. Let M be an arbitrary set of the types. By $G^*(M)$ we denote the subgroup of the group G generated by all the elements of G , the types of which are greater or

equal than all the elements from M . It is easy to see that, in a completely decomposable group $G=\sum_{\iota\in I}^{d} J_\iota$, $G^*(M)$ is just the direct sum of those J_ι , the types of which are greater or equal than all the elements of the set M .

Lemma 2. Let p be a prime number and G a completely decomposable group with the IQp-property. Then, for an arbitrary type set M , the factor-group $\bar{G} = G/G^*(M)$ has also the IQp-property.

Proof. Let $\bar{H} \subseteq \bar{G}$ and $p\bar{G} \subseteq \bar{H}$. By the isomorphism theorem, there exists a subgroup H of G such that $G^*(M) \subseteq H \subseteq G$ and $H/G^*(M) \cong \bar{H}$. Further, $G/H \cong$

$\cong G/G^*(M) \big/ H/G^*(M) \cong \bar{G}/\bar{H}$,so that $pG \subseteq H$. By hypothesis there exists an isomorphism φ of G onto H . It is easy to see that $G^*(M) = H^*(M)$ (because $G^*(M) \subseteq H$ implies $G^*(M) \subseteq H^*(M)$). Moreover, the type of an element is an isomorphism invariant, hence $(G^*(M))\varphi = G^*(M)$, and this fact completes the proof of the Lemma.

Lemma 3. Let G be an arbitrary group, H its subgroup such that $pG \subseteq H$. Then H is a q-pure in G for all primes $q \neq p$.

Proof. Let the equation $q^k x = h,\ h \in H$ be solvable in G . From the relation $(p, q^k) = 1$ it follows that there exist integers r, s such that $pr + q^k s = 1$, and then $x = r(px) + s(q^k x) \in H$.

Theorem 1. Let G be a completely decomposable IQ-group. Then every infinite increasing sequence $\{\hat{\tau}_n\}$ of the elements from $T(G)$ has the following property: For every prime p , the inequality $\tau_n(p) \neq \infty$ holds for a finite number of n's only.

<u>Proof</u>. Suppose conversely that there exists an infinite increasing sequence $\{\hat{\tau}_n\}$ of the elements from $T(G)$ and a prime p such that $\tau_n(p) \neq \infty$ for all integers n .

By Lemma 1, the group G is isomorphic to each its subgroup H with $pG \leq H$. The same property has the group $\overline{G} = G/_{G*(M)}$ where $M = \{\hat{\tau}_n\}$ (see Lemma 2). It is easy to see that the group $\overline{G}$ is isomorphic to a completely decomposable direct summand G_1 of G such that:

(2) $$\hat{\tau}_n \in T(G_1) \qquad n = 1, 2, \dots \ ;$$

(3) No element $\hat{\tau}$ with $\hat{\tau} > \hat{\tau}_n$, for all integers n , is in $T(G_1)$.

Let $G = \sum_{\iota \in I}{}_d\, J_\iota$ be a completely decomposable group, the type set $T(G)$ of which has the properties (2) and (3). Theorem 1 will be proved by constructing a subgroup H of G with $pG \leq H \leq G$, H being not isomorphic to G . For this, we denote by J_i that rank one direct summand of the given direct decomposition of G for which $\hat{\tau}(J_i) = \hat{\tau}_i$, and put

(4) $$U = \sum_{i=1}^{\infty}{}_d\, J_i \ ; \qquad V = \sum_{\substack{\iota \in I \\ \iota \neq i}}{}_d\, J_\iota$$

so that $G = U \dotplus V$.

In each J_i , we choose an element u_i with $h_p^G(u_i) = 0$. Now, define the subgroup H of G by the formula

(5) $H = \{ V ; pU ; u_i - u_{i+1} , \quad i = 1, 2, \dots \}$.

Clearly, $pG \subseteq H$. First of all, we shall prove

(6) $$u_1 \notin H .$$

Suppose conversely that $u_1 \in H$. Then

(7) $$u_1 = v + pu + \sum_{i=1}^{n-1} b_i \quad (u_i - u_{i+1}) .$$

Because $u \in U$, there exist integers m , a_i ; $i = 1, 2, \dots, n$; such that $mu = \sum_{i=1}^{n} a_i u_i$. In view of $h_p^G (u_i) = 0$ we may suppose $(m, p) = 1$. From (7), it immediately follows

(8) $$m u_1 = mv + p \sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n-1} m b_i \quad (u_i - u_{i+1})$$

with $(m, p) = 1$.

In view of the independence of the elements v and u_i ; $i = 1, 2, \dots, n$; the equality (8) holds if and only if

(9) $$\begin{aligned} mv &= 0 \\ pa_1 + mb_1 &= m \\ pa_2 - mb_1 + mb_2 &= 0 \\ &\vdots \\ pa_{n-1} - mb_{n-2} + mb_{n-1} &= 0 \\ pa_n - mb_{n-1} &= 0 \end{aligned}$$

From the last equation it follows that $p \mid b_{n-1}$, then $p \mid b_{n-2}$ from the preceding one, etc. Thus we obtain that $p \mid b_i$; $i = 1, 2, \dots, n-1$; and the second equation now yields $p \mid m$, which contradicts our hypothesis (8). This contradiction proves (6).

Clearly $h_p^H(x) \leq h_p^G(x)$ for all $x \in H$. Further, if $h_p^G(x) = \infty$ then necessarily $x \in V$, hence $h_p^H(x) = \infty$. From this and from Lemma 3 we conclude that the type of each element from H is the same in H as in G.

Suppose that the group H is completely decomposable:

$$(10) \qquad H = \sum_{\alpha \in A}{}^{d} I_\alpha .$$

Because $pu_1 \in H$, the element pu_1 has a non-zero component in finitely many of A_α's. H_1 be the direct sum of those direct summands I_α of the group H, in which pu_1 has a non-zero component, and H_2 be the direct sum of all the other direct summands I_α of the group H, so that $H = H_1 \dotplus H_2$ is true.

From (3) and from the finiteness of $\hat{\tau}(H_1)$ it follows that there exists $\hat{\tau}_j$ so that

$$(11) \qquad \hat{\tau}_j \nleq \hat{\tau} \qquad \text{for all } \hat{\tau} \in \hat{\tau}(H_1) .$$

From this fact it follows that pu_j has a zero component in every direct summand of H_1, hence $pu_j \in H_2$.

Further, $u_1 - u_j = (u_1 - u_2) + (u_2 - u_3) + \dots + (u_{j-1} - u_j) \in H$, hence we may write $u_1 - u_j = h_1 + h_2$, $h_i \in H_i$, $i = 1, 2$.

Then $pu_1 - pu_j = ph_1 + ph_2$ and finally $pu_1 = ph_1$ (by the definition of H_i, $i = 1, 2$). But G is torsion free, hence $u_1 = h_1 \in H$ which contradicts (6). The proof of the theorem is now complete.

Let G' (H') be a maximal p-divisible subgroup of G (H). If $G \cong H$ and φ is an isomorphism of G onto H, then it may be easily shown that $G'\varphi = H'$. We shall use this simple fact in the proof of the following

Lemma 4. Let p be a prime and G a group which is the direct sum of a p-divisible group G_1 and a p-reduced group G_2, $G = G_1 \dotplus G_2$. Then G has the IQp-property if and only if G_2 has the IQp-property.

Proof. First of all, let G have the IQp-property, and let H_2 be a subgroup of G_2 with $pG_2 \subseteq H_2 \subseteq G_2$. If we put $H = G_1 \dotplus H_2$, then $pG \subseteq H$, so that by hypothesis there exists an isomorphism φ of G onto H. Because G_1 is the maximal p-divisible subgroup of both G and H, there is $G_1\varphi = G_1$ and $G_2 \cong G/G_1 \cong G\varphi/G_1\varphi = H/G_1 \cong H_2$.

Conversely, let G_2 have the IQp-property and let H be a subgroup of G with $pG \subseteq H \subseteq G$. From the p-divisibility of G_1 it follows $G_1 = pG_1 \subseteq pG \subseteq H$, hence $H = G_1 \dotplus (G_2 \cap H) = G_1 \dotplus H_2$. Further, $pG_2 \subseteq pG \subseteq H$ and $pG_2 \subseteq G_2$, so that $pG_2 \subseteq G_2 \cap H = H_2$. By hypothesis we have $G_2 \cong H_2$ and now it may be easily proved that $G \cong H$, too.

Lemma 5. Let G be a p-reduced, completely decomposable group such that $T(G)$ satisfies the maximum condition, and let $T(G)$ contain two incomparable types which are maximal in $T(G)$. Then G contains a subgroup H with $pG \subseteq H$ and $G \ncong H$.

Proof. Let $\hat{\tau}_1, \hat{\tau}_2$ be two incomparable types from $T(G)$ which are maximal in $T(G)$. Denote by U_1 that rank one direct summand of G (in a given direct decomposition) the type of which is $\hat{\tau}_1$, by U_2 that rank one direct summand of G the type of which is $\hat{\tau}_2$, and by G' the direct sum of all the other direct summands of G. Hence $G = U_1 \dotplus U_2 \dotplus G'$. Because U_1 and U_2 are not p-divisible, there exist two elements $u_1 \in U_1$ and $u_2 \in$ $\in U_2$ such that $h_p^{\sigma}(u_i) = 0, \quad i = 1, 2$.

Define the subgroup H of G :

$$(12) \qquad H = \{ G'; \, p\, U_1 ; \, p\, U_2 ; \, u_1 - u_2 \} .$$

Clearly $pG \subseteq H$. Firstly, let us show

$$(13) \qquad u_1 \notin H .$$

Let $u_1 \in H$. By (12) we may write

(14) $u_1 = g' + p u_1' + p u_2' + k (u_1 - u_2)$, where $g' \in$ $\in G'$, $u_i' \in U_i$; $i = 1, 2$.

Now there exist integers n, m, a', b' such that $n u_1' =$ $= a' u_1$, $m u_2' = b' u_2$, and we may suppose that $(n, p) = 1$ and $(m, p) = 1$. Then, for $l = [m, n]$ it holds $(l, p) = 1$, too, and there exist integers a, b such that $l u_1' = a u_1$, $l u_2' = b u_2$. Multiplying (14) by l, we get

$$(15) \qquad l u_1 = l g' + p a u_1 + p b u_2 + k l (u_1 - u_2) .$$

In view of the independence of the elements g', u_1, u_2, the equality (15) holds if and only if

$$(16) \qquad \begin{aligned} l g' &= 0 \\ pa + kl &= l \\ pb - kl &= 0 \end{aligned}$$

From the last equality it follows that $p \mid kl$, hence the second equation gives $p \mid l$, which is a contradiction. Hence (13) is true.

Now suppose that H is completely decomposable: $H =$ $= \sum_{\lambda \in \Lambda}^{d} I_\lambda$. Denote by H_1 the direct sum of all H_λ

the type of which is $\hat{\tau}_1$, and by H_2 the direct sum of all the other direct summands of H . Clearly $H = H_1 \dot{+} H_2$. From the incomparability and maximality of types it follows, by (12),

$$(17) \qquad p u_1 \in H_1 \ ; \quad p u_2 \in H_2 \ .$$

Further, $u_1 - u_2 \in H$ implies $u_1 - u_2 = h_1 + h_2$ where $h_i \in H_i$. Multiplying by p , we get $pu_1 - pu_2 = ph_1 + ph_2$. But then $pu_1 = ph_1$, and, by the torsion free character of G , $u_1 = h_1 \in H_1 \subseteq H$ which contradicts (13). This contradiction proves Lemma 5.

<u>Theorem 2</u>. Let G be a completely decomposable IQ-group. Then, for any two incomparable types $\hat{\tau}_1$, $\hat{\tau}_2$ from $T(G)$, we have $\sup \{\tau_1, \tau_2\} = (\infty, \infty, \dots, \infty, \dots)$.

<u>Proof</u>. For an arbitrary prime p we denote by $G_1^{(p)}$ the direct sum of all p-divisible rank one direct summands of G (in a given complete decomposition), and by $G_2^{(p)}$ the direct sum of all the other rank one direct summands of G . Clearly, $G = G_1^{(p)} \dot{+} G_2^{(p)}$ where $G_1^{(p)}$ is p-divisible and $G_2^{(p)}$ p-reduced.

It suffices to prove that $T(G_2^{(p)})$ is ordered for every prime p . Suppose conversely that there exists a prime number p such that $T(G_2^{(p)})$ is not ordered. For this prime, denote $G_i^{(p)}$ simply by G_i ; $i = 1, 2$. By Theorem 1, the set $T(G_2)$ satisfies the maximum condition, so that there exist two incomparable types $\hat{\tau}_1$, $\hat{\tau}_2$ such that, for every $\hat{\tau} \in T(G_2)$ for which $\hat{\tau} > \hat{\tau}_1$ implies $\hat{\tau} > \hat{\tau}_2$ and the set of all types $\hat{\tau} \in T(G_2)$, $\hat{\tau} \geq \sup \{\hat{\tau}_1, \hat{\tau}_2\}$ is ordered. It is easy to see that, for the group $\bar{G} = G_2 / G_2^*(M)$ where

$M = \{\hat{\tau}_1, \hat{\tau}_2\}$, all the conditions of Lemma 5 are fulfilled, so that the group $\overline{G}$ contains a subgroup $\overline{H}$ such that $p\overline{G} \subseteq \overline{H}$ and $\overline{G} \ncong \overline{H}$. On the other hand, applying Lemmas 1, 4 and 2, we get $\overline{G} \cong \overline{H}$. This contradiction proves our theorem.

<u>Theorem 3</u>. A completely decomposable group G is an IQ-group if and only if the following two conditions are fulfilled:

(α) If $\{\hat{\tau}_n\}$ is an infinite increasing sequence of elements from $T(G)$ then for every prime p the inequality $\tau_n(p) \neq \infty$ holds for a finite number of n's only.

(β) For any two incomparable types $\hat{\tau}_1$, $\hat{\tau}_2$ from $T(G)$, there is $\sup\{\tau_1, \tau_2\} = (\infty, \infty, \dots, \infty, \dots)$.

<u>Proof</u>. The conditions (α) and (β) are necessary by Theorems 1 and 2. Now we shall prove the sufficiency of the conditions (α) and (β).

Let p be an arbitrary prime. Let G_1 be the direct sum of all p-divisible direct summands (in a given direct decomposition) of G, and G_2 be the direct sum of all the other direct summands of G. Hence, $G = G_1 \dotplus G_2$, G_1 is p-divisible and G_2 p-reduced. By condition (α), $T(G_2)$ fulfils the maximum condition and by (β) $T(G_2)$ is ordered. Then by Kovács's theorem G_2 is an IQ-group. By Lemma 1 G_2 has the IQp-property. Then by Lemma 4 G has the IQp-property, too. Because p was an arbitrary prime, G is the IQ-group by Lemma 1.

A simple consequence of Theorem 3 is:

<u>Theorem 4</u>. A completely decomposable group G with ordered type set $T(G)$ is an IQ-group if and only if the

the condition (α) from Theorem 3 holds.

R e f e r e n c e s

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