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RIGID UNDIRECTED GRAPHS WITH GIVEN NUMBER OF VERTICES

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Throughout the present paper we use the term "graph" for a finite non-oriented graph $G = (X, R)$ with $|X| > 1$. A mapping $f : X \to X$ is termed an endomorphism of a graph (X,R) if $(f(x), f(y)) \in R$ whenever $(x, y) \in R$. An endomorphism is said to be an automorphism, if $f(X) = X$. A graph is said to be rigid, if it has no non-identical endomorphisms. The notions of a multigraph, homeomorphism of multigraphs, etc. are used in the sense of [2]. We use the following notation (for a graph G = (X,R)):

$R(x) = \{y \in X : (x, y) \in R\}$, $i(x) = |R(x)|$, $i(X) = \max\limits_{x \in X} i(x)$,
$M_G = \{x \in X : i(x) > 2\}$, $P_G = \{x \in M_G : |R(x) \cap M_G| > 2\}$, $G(x) =$
$= (X - \{x\}, R - \{(x, y) : y \in R(x)\})$.

Denote by $\gamma(G)$ the chromatic number of G. Denote by $\{x_1 \to y_1, x_2 \to y_2, \dots, x_n \to y_n\}$ the mapping $f : X \to X$ defined by $f(x_i) = y_i$ $(i = 1, 2, \dots n)$, $f(x) = x$ otherwise. Write $\{x \leftrightarrow y\}$ instead of $\{x \to y, y \to x\}$. We sometimes say that x is joined with y if $(x, y) \in R$.

It is proved in [1] that there is no rigid graph (X,R) with $|X| \leq 7$, while there are rigid graphs with any greater number of vertices. The present paper deals with number of edges. Namely it is shown (Theorem 1) that there is no rigid (X,R) with $|R| \leq 13$ and that (Theorem 2) for every n > 13 there is a rigid (X,R) with $|R| = n$.

Lemma 1. a) No full k-graph is rigid.

b) If G is rigid and $\gamma(G) \leq k$, then G contains no full k-graph.

c) If $G = (X, R)$, $|X| = 5$ and G contains no full 5-graph, then $\gamma(G) \leq 4$.

d) If $G = (X, R)$, $|X| = 5$ and G contains no full 4-graph, then $\gamma(G) \leq 3$.

e) If $G = (X, R)$, $|X| = 6$ and G contains no full 6-graph, then $\gamma(G) \leq 5$.

f) If $G = (X, R)$, $|X| = 6$ and G contains no full 5-graph, then $\gamma(G) \leq 4$.

g) If $G = (X, R)$, $|X| = 6$ and G contains no full 4-graph, then $\gamma(G) \leq 3$ or G is isomorphic to G^*.

h) Let $\gamma(P_G, R \cap P_G \times P_G) = 3$. Then $\gamma(G) = 3$.

i) Let $G = (X, R)$ be rigid, $|P_G| \leq 5$. Then G contains no full 3-graph.

j) Let $G = (X, R)$ be rigid, $|P_G| = 6$. Then either G contains no full 3-graph, or $(P_G, R \cap P_G \times P_G)$ is isomorphic to G^*.

Proof. Statements a), b) and c) are evident. Let us prove d): If $i(X) \leq 2$, we have evidently $\gamma(G) \leq 3$. If there is a vertex a with $i(a) = 3$, there are $b, c \in R(a)$ with $(b, c) \notin R$ and hence $\gamma(G) \leq 3$. If we have $i(a) = 4$ for some a, there are vertices p, r, s in R(a) such that $(p, r) \notin R$, $(r, s) \notin R$, $(p, s) \notin R$, or vertices t,u,v,w in R(a) such that $(t, u) \notin R$, $(v, w) \notin R$. Thus, we have always $\gamma(G) \leq 3$. Similarly we may prove e) and f). To prove g), let us assume that G contains no full 4-graph and that $\gamma(G) = 4$ (by f), $\gamma(G) \leq 4$). Thus, $i(X) > 2$. If $i(X) = 3$, $i(a) = 3$, we either may colour all vertices of R(a) equally (then $\gamma(G) \leq 3$), or there exist b,c,d $\in$ R(a) with $(b, c) \in R$, $(b, d) \notin R$ and then there is a vertex $e \in X - [R(a) \cup \{a\}]$ with $(e, c) \notin R$ and, again, $\gamma(G) \leq$ ≤ 3. If there is an $a \in X$ with $i(a) = 4$, the subgraph $(A, R \cap A \times A)$ (where $A = \{a\} \cup R(a)$) is 3-coloured by d) and we may colour $b \in X - A$ and a equally, which is a contradiction. Thus, there exists an $a \in X$ with $i(a) =$ $= 5$. The question is, now, equivalent with looking for a five-point graph with the chromatic number 3 without full 3-graphs. We see easily that this is exclusively the 5-cycle. Thus, G is isomorphic with G^* . The proof of h) is easy: first, colour P , then M - P and finally X - M . i) and j) follow by a)- h).

<u>Lemma 2</u>. Let $G = (X, R)$ be a rigid graph, $|X| = n$. Then

a) $i(x) \geq 2$ for every $x \in X$, $i(X) > 2$.

b) $i(x) \neq n-2$ for every $x \in X$.

c) If $i(X) = n-1$, then $|P_G| \geq 6$. $|P_G| = 6$ only if $(P_G, R \cap$ $\cap P_G \times P_G)$ is isomorphic to G^*.

d) G contains no even cycle $x_1, x_2, \dots, x_{2k}, x_1$ such that $i(x_j) = 2$ for $j = 1, 2, \dots, k-1$.

e) G contains no cycle $x_1, \dots, x_k, x_1$ such that $i(x_j) =$ $= 2$ for $j = 2, \dots k$.

f) If $n \leq 15$, then G is connected.

g) We cannot denote some k points of X by $x_1, \dots, x_k$ in such a way that $\{x_i \to x_{k+1-i}, i = 1, 2, \dots k\}$ is an automorphism; in particular, there are no vertices a,b,c,d with $i(c) =$ $= i(d) = 3$ and $(a, c) \in R, (a, d) \in R, (b, c) \in R, (b, d) \in R, (c, d) \in R$.

<u>Proof</u>. a),b) are proved in [1], c) follows by Lemma 1 i), j). The other statements are evident. E.g., f) is a consequence of [1], since if $|X| < 16$ there is either a one-point

component or a component with more than one and less than 8 vertices.

Theorem 1. There is no rigid $G=(X,R)$ with $|R|\leq 13$. This follows by the following lemmas:

Lemma 3. Let $G=(X,R)$ be rigid. Then $|R|>|X|+1$.

Proof By L.2 a), $|R|>|X|$. If $|R|=|X|+1$, M_G consists either of one vertex a with $i(a)=4$, or of vertices b,c with $i(b)=i(c)=3$. In the first case, some of the components is homeomorphic to the multigraph A_1, and we obtain a contradiction by L.2 e); in the second case, some component is homeomorphic either with A_2 (and hence G is not rigid by L.2 e)) or with A_3, and then there are two ways of odd or two ways of even length between b and c, which is in contradiction with L.2 d).

Lemma 4. There is no rigid $G=(X,R)$ with $|X|=n\leq 11$ and $|R|=n+2$.

Proof. Let there be such a G. Then G is connected by 2f) and, by 2a), there are the following possibilites for M_G:

α) $M_G=\{a\}$, $i(a)=6$

β) $M_G=\{a,b\}$, $i(a)=5$, $i(b)=3$

γ) $M_G=\{a,b\}$, $i(a)=i(b)=4$

δ) $M_G=\{a,b,c\}$ $i(a)=4$, $i(b)=i(c)=3$

ε) $M_G=\{a,b,c,d\}$ $i(a)=i(b)=i(c)=i(d)=3$

The graphs satisfying α), β) or γ) lead (similarly as did A_1, A_2, A_3) to a contradiction with L.2e or 2d. In the case of δ) we obtain, with the exception of the non-rigid graphs following evidently from 2e and 2d, a graph homeomorphic to the multigraph B_1. Similarly, in the case ε), G should be homeomorphic to either B_2 or B_3. If G homeomorphic to B_1 is rigid, then necessarily $p_1(ac)+p_2(ac)$

and $p_1(ab)+p_2(ab)$ are odd by 2d and greater than 1 by 1i (where $p_i(xy)$ signifies the number of vertices on the i-th edge joining x and y). Then, however, G is not rigid which may be seen from the Table 1 (in the last row there is written the lemma by which there is a non-trivial endomorphism). Analogically, we may treat the case with G homeomorphic to B_2 - see Table 2 (in the fifth column the corresponding endomorphism is marked concisely). It remains to prove that all graphs homeomorphic to B_3 are not rigid for $n \leq 11$. Put

$$M = max\,[p(ab)+p(bc)+p(ca);\ p(ab)+p(bd)+p(ad);\ p(bc)+p(cd)+p(bd);\ p(ac)+p(ad)+p(cd)]$$

$N = min\,[\ldots]$.

If we have $M = N = 0$, the graph is not rigid by 1a. If $N = 1$, the graph is not rigid by 2d. Hence, we have to examine the graphs with $M \geqq N \geqq 2$ and $n \leq 11$; this is described in the Table 3.

Lemma 5. There is no rigid $G = (X, R)$ with $|X| = n \leq 10$ and $|R| = n + 3$.

Proof. Such a graph would be connected and M_G would satisfy some of the following conditions:

α) $M_G = \{a\}$, $i(a) = 8$

β) $M_G = \{a, b\}$, $i(a) = 7$ $i(b) = 3$

γ) $M_G = \{a, b\}$, $i(a) = 6$ $i(b) = 4$

δ) $M_G = \{a, b, c\}$, $i(a) = 6$ $i(b) = i(c) = 3$

ε) $M_G = \{a, b\}$, $i(a) = i(b) = 5$

ζ) $M_G = \{a, b, c\}$, $i(a) = 5$ $i(b) = 4$ $i(c) = 3$

η) $M_G = \{a, b, c, d\}$, $i(a) = 5$ $i(b) = i(c) = i(d) = 3$

$\vartheta)$ $M_G = \{a, b, c\}$, $i(a) = i(b) = i(c) = 4$

$\iota)$ $M_G = \{a, b, c, d\}$, $i(a) = (ib) = 4$ $i(c) = i(d) = 3$

$\varkappa)$ $M_G = \{a, b, c, d, e\}$, $i(a) = 4$ $i(b) = i(c) = i(d) = i(e) = 3$

$\lambda)$ $M_G = \{a, b, c, d, e, f\}$, $i(x) = 3$ for every $x \in M_6$

Cases $\alpha) - \zeta)$ cannot occur by L.2e,2d. In $\eta)$ and $\vartheta)$ excluding the evidently non-rigid graphs, we obtain types of graphs which are not rigid for $n \leq 10$ (e.g., for graphs homeomorphic to C_1 : If such a graph is rigid, we have, by L.2d and 1i, $p_1(a,b) + p_2(ab) \geq 3$ and $p_1(ad) + p_2(ad) + p(ac) + p(cd) \geq 4$, which is, for $n \leq 10$, impossible). Analogically in the case $\iota)$, after excluding evidently non-rigid graphs, there remain only graphs homeomorphic to C_2 ; similarly, in the case $\varkappa)$, homeomorphic to C_3 or C_4 , and in the case $\lambda)$ homeomorphic to C_5 or C_6 . We shall prove that the graphs homeomorphic to $C_2 - C_6$ are not rigid whenever $n \leq 10$. We obtain, by L.2d and 1i, for rigid graphs homeomorphic to C_2 the following inequalities:

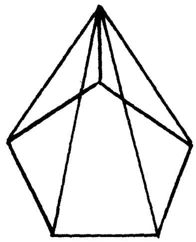
$$p_1(ab) + p_2(ab) \geq 3, \quad p(ac) + p(ad) + p(cd) \geq 2,$$

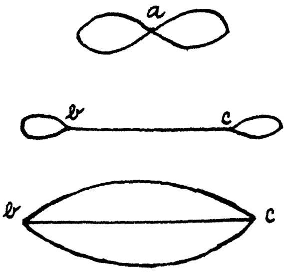
$$p(bc) + p(bd) + p(cd) \geq 2, \quad p_1(ab) + p_2(ab) + p(ac) + p(bc) \geq 4,$$

$$p_1(ab) + p_2(ab) + p(ad) + p(bd) \geq 4$$

Consequently (up to an isomorphism), either $p_1(ab) + p_2(ab) = 3$, $p(ad) = p(cd) = p(bd) = 1$, or $p_1(ab) + p_2(ab) = 3$, $p(bd) = p(ac) = p(cd) = 1$. Such graphs are non-rigid by L.2g. We may deal similarly the graphs homeomorphic to C_3 . If a rigid graph is homeomorphic to C_4 , we have $p(ab) + p(be) + p(ae) \geq 2$, $p(ac) + p(cd) + p(ad) \geq 2$. Since $n \leq 10$, we obtain (up to an isomorphism) $p(ab) + p(be) + p(ae) = 2$.

G^*

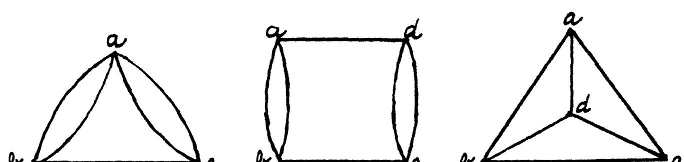




A_1

A_2

A_3



B_1 B_2 B_3

	$n=9$			$n=10$			$n=11$								
$p_1(ac)$	0	0	1	0	0	1	0	0	1	1	2	2	0	0	1
$p_2(ac)$	3	3	2	3	3	2	5	5	4	4	3	3	3	3	2
$p_1(ab)$	0	1	1	0	1	1	0	1	0	1	0	1	0	1	1
$p_2(ab)$	3	2	2	3	2	2	3	2	3	2	3	2	3	2	2
$p(bc)$	0	0	0	1	1	1	0	0	0	0	0	0	2	2	2
	2g	2d	2g	2g	2d	2g	2d	1i	2d	2d	2d	2d	2g	2d	2d

Table 1.

	$n=10$			$n=11$		
$p_1(ab)$	0	0	1	0	0	1
$p_2(ab)$	3	3	2	3	3	2
$p_1(cd)$	0	1	1	0	1	1
$p_2(cd)$	3	2	2	3	2	2
$p(ad)$	0	0	0	1	1	1
$p(bc)$	0	0	0	0	0	0
	2g	2d	2g	2g	[illegible]	2g

Table 2.

	M=N=2		M = 3				M = 4															
$p(ab)$	2	1	3	2	1	1	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$p(bc)$	0	1	0	1	1	1	0	1	1	1	1	2	2	2	2	1	1	1	1	1	1	1
$p(ca)$	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
$p(cd)$	2	1	~	~	1	1	~	~	2	1	2	2	1	2	2	~	1	0	2	1	1	0
$p(ad)$	0	1	~	~	1	1	0	0	0	1	1	0	1	0	1	0	1	1	1	2	1	2
$p(bd)$	0	0	~	~	0	1	0	0	1	0	0	0	~	1	0	0	0	1	0	0	1	1
	2g	2g	2d	2d	2g	2g	2g	2d	2g	2d	2d	2g	2d	2d	2d	2g	2d	2g	2d	2g	2g	2d

Table 3.

	M = 5									M = 6				M = 7
$p(ab)$	5	4	3	3	2	2	2	2	2	4	3	3	2	3
$p(bc)$	0	1	2	1	2	2	2	2	2	1	2	2	2	2
$p(ca)$	0	0	0	1	1	1	1	1	1	1	1	1	2	2
$p(cd)$	~	~	~	~	0	1	1	2	1	1	1	0	1	0
$p(ad)$	~	~	~	~	0	0	1	0	0	0	0	1	0	0
$p(bd)$	~	~	~	~	~	0	0	0	1	0	0	0	0	0
	2d	2d	2d	2d	2g	2d	2g	2d	2d	2g	2d	2d	2g	2g

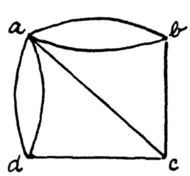
Table 3. (Cont.)

$p(ab)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	2	2	2	2	2
$p(ae)$	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
$p(be)$	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	2	0	0	0	0	0
$p(ac)$	1	0	1	1	2	2	~	0	0	~	1	1	1	1	1	~	2	2	~	1	2	0	0	0
$p(cd)$	1	1	0	1	~	0	2	0	0	~	0	0	1	1	1	2	~	0	~	~	~	1	1	2
$p(ad)$	0	1	1	1	~	0	~	2	2	2	1	1	0	0	1	~	~	0	~	~	~	1	2	1
$p(bc)$	0	~	0	0	0	1	~	0	1	0	0	1	0	0	0	~	0	1	~	~	~	0	0	0
$p(ed)$	~	~	1	0	~	0	~	0	0	~	0	0	1	0	0	~	~	0	~	~	~	1	0	0
	$2\overset{i}{d}$	2d	b-d	2d	2d	2i	$2\overset{i}{d}$	2d	e-x	2d	2g	2g	e-x	2d	2g	2d	2d	2d	$2\overset{i}{d}$	2d	2d	e-x	2d	2d

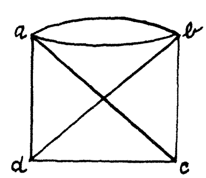
Table 4.

$p(af)$	2	2	2	2	0	0	0	1	1	1	1
$p(ae)$	0	0	0	0	2	2	2	1	1	1	0
$p(ef)$	0	0	0	0	0	0	0	0	0	0	1
$p(bc)$	2	0	1	1	~	0	0	1	0	1	1
$p(cd)$	0	~	1	0	~	0	1	0	1	1	1
$p(bd)$	0	~	0	1	0	2	1	1	1	0	0
	2g	2d	$a \to d$ $e \to b$	$f \to d$ $e \to c$	2d	2g	$c \to a$ $b \to f$	2g	2g	2g	2g

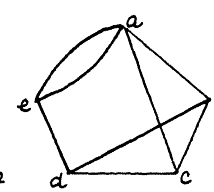
Table 5.



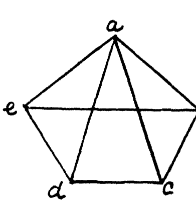
C_1



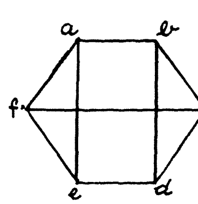
C_2



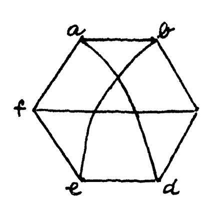
C_3



C_4



C_5



C_6

The possible cases are described in the Table 4 (the last row, again, contains indications of non-rigidity). The graphs homeomorphic to C_5 are not isomorphic to G^*. Thus, if some of them is rigid, we have by L.1j and 2d $p(af)+p(ae)+p(ef) = 2$, $p(bc)+p(cd)+p(bd) = 2$, hence, $p(ab) = 0$, $p(cf) = 0$ and $p(ed) = 0$. The possibilities are examined in the Table 5. Finally, we see easily by L.2d or lemma 2g that the graphs homeomorphic to C_6 are not rigid for $m = 8, 9, 10$.

Lemma 6. There is no rigid $G = (X, R)$ with $|X| = m \leq 9$ and $|R| = m + 4$.

Proof. We shall discuss only that cases of M_G which do not lead evidently to a non-rigid graph (as did the cases $\alpha) - \vartheta)$ in previous lemma).

$\alpha)$ $M_G = \{a, b, c, d, e, f\}$, $i(a) = 5$, $i(b) = i(c) = i(d) = i(e) = i(f) = 3$

$\beta)$ $M_G = \{a, b, c, d, e\}$, $i(a) = i(b) = i(c) = 4$, $i(d) = i(e) = 3$

$\gamma)$ $M_G = \{a, b, c, d, e, f\}$, $i(a) = i(b) = 4$, $i(c) = i(d) = i(e) = i(f) = 3$

$\delta)$ $M_G = \{a, b, c, d, e, f, g\}$, $i(a) = 4$, $i(b) = \dots = i(g) = 3$

$\varepsilon)$ $M_G = \{a, b, c, d, e, f, g, h\}$, $i(x) = 3$

for every $x \in M_G$

In case $\alpha)$, [1] yields $m \geq 8$, so that, whilst $|M_G| = 6$, we have $|P_G| < 6$ or $|P_G| = 6$ and G is not isomorphic to G^*; now, we may use Lemma 1i and j. In case $\beta)$ we obtain, excluding evidently non-rigid graphs, G homeomorphic to D_1, which is a contradiction, since, by L.2i and 2d $p(ab)+p(ae)+p(be) \geq 2$, $p(ac)+p(ae)+p(ec) \geq 2$ and

$p(bc)+p(cd)+p(bd) \geq 2$, so that $p(ae) = 2$. Since also $p(bc)+p(ab)+p(ac) \geq 2$, we have $p(bc) = 2$, and, due to the triangle abd, G contradicts L.1i. If G satisfies $\gamma)$, it is homeomorphic to D_2 or D_3, which is not possible, since, in D_2, we have $p(ab) = 2$ (similarly as in D_1), and, for $n \leq 9$, some of the mappings $\{c \to d\}$, $\{d \to e\}$, $\{e \to c\}$ is an endomorphism, and the graphs homeomorphic with D_3 are non-rigid by L.2g, 1i, and 2d (we have $p_1(ab) + p_2(ab) = 3$). Case $\delta)$ does not occur, since - after excluding the obviously non-rigid graphs - G should be homeomorphic to D_4 and $\{e \to f\}$ would be an endomorphism (there is $|P_G| \leq 6$ and G does not contain G^*; we may use L.1i and j and, by L.2d, $p(ab)+p(bc)+p(ac) = 2$). In case $\varepsilon)$ we have, for $n = 9$, $|P_G| \leq 6$ and G contains no subgraph isomorphic to G^*. Thus, by L.2i,j, and d, G is homeomorphic to either D_5 and D_6. We see easily that G is then not rigid. If $n = 8$, then either G contains no triangle and hence it is isomorphic to D_5 and D_6, or G contains a triangle (e.g., $(a,b) \in R$, $(b,c) \in R$, $(a,c) \in R$). By L.2g, we have $R(a) \cap R(b) - \{c\} = \emptyset$ (otherwise $\{a \leftrightarrow b\}$ is an endomorphism), similarly for $R(b)$, $R(c)$ and $R(a)$, $R(c)$. Let, e.g., $R(a) = \{b, c, d\}$, $R(b) = \{a, c, e\}$, $R(c) = \{a, b, f\}$. Then there is either $R(g) = \{d, e, f\}$, or $R(g) = \{d, e, h\}$, i.e., we obtain non-rigid graphs D_7 and D_8.

<u>Lemma 7</u>. There is no rigid $G = (X, R)$ with $n = |X| = 8$ and $|R| = 13$.

<u>Proof</u>. By Lemmas 2f, 2b and 2c it suffices to investigate the connected graphs with $i(X) < 6$ ($i(X) = 6$ is

impossible by 2b and $i(X) = 7$ implies $|P_G| < 6$ in contradiction with 2c). Excluding the cases of graphs non-rigid by L.1i,j and 2d (with $n = 8$), we obtain the following possibilities for M :

α) $M_G = \{a,b,c,d,e,f\}$ $i(a) = i(b) = 5$, $i(c) = \ldots = i(f) = 3$

β) $M_G = \{a,b,c,d,e,f\}$ $i(a) = 5$, $i(b) = i(c) = 4$, $i(d) = i(e) = i(f) = 3$

γ) $M_G = \{a,b,c,d,e,f,g\}$ $i(a) = 5$, $i(b) = 4$, $i(c) = \ldots = i(g) = 3$

δ) $M_G = X$ $i(a) = 5$ for some $a \in X$, $i(x) = 3$ otherwise

ε) $M_G = \{a,b,c,d,e,f,g\}$ $i(a) = i(b) = i(c) = 4$,

$i(d) = \ldots = i(g) = 3$

φ) $M_G = X = \{a,b,c,d,e,f,g,h\}$ $i(a) = i(b) = 4$, $i(x) = 3$ otherwise .

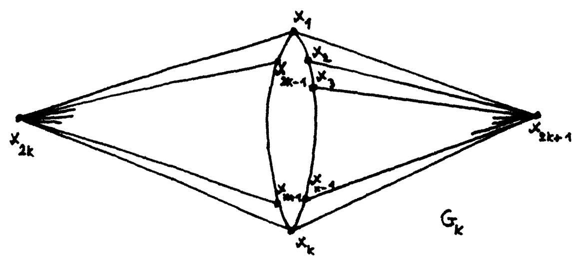
In case α), G is obviously non-rigid by L.1i,j and 2d, whenever $(P_G, P_G \times P_G \cap R)$ is not isomorphic to G^* ; otherwise it is not rigid by L.2e.

Similarly, in case β), the only posibility to be investigated is that of $(P_G, R \cap P_G \times P_G)$ isomorphic to G^*. Then, G is isomorphic to E_1 or E_2 and hence non-rigid.

In case γ), G is not rigid whenever $|P_G| < 6$, or $|P_G| = 6$ and $(P_G, P_G \times P_G \cap R)$ is not isomorphic to G^*. If this should not occur, we must have $R(h) = \{a,b\}$ (where $h \in X$, $i(h) = 2$). By Lemma 2d, $R(a) \cap R(b) = \{h\}$, and consequently $(a,b) \in R$. Thus, G contains the triangle abh , and $\gamma(G) = 3$.(By Lemma 2g there are two distinct vertices $c, d \in R(a)$ such that $(c,d) \notin R$. Thus, G may be coloured as follows: $\gamma(a) = \gamma(f) = 1$, $\gamma(b) = \gamma(c) =$

$=\gamma(d)=2, \gamma(e)=\gamma(h)=3, \gamma(g)$ either 1 or 3 , where $R(a)=\{b, c, d, e, h\}$, $R(b)=\{a, f, g, h\}$.)In case $\delta)$ put $R(a)=\{b, c, d, e, f\}$ and $X-R(a)=\{g, h\}$.

We have necessarily $(g, h)\in R$, otherwise $\{g\to a\}$ is an endomorphism. Investigating all possibilities ($|R(g)\cap R(h)|=2$, $|R(g)\cap R(h)|=1$ and $R(g)\cap R(h)=\emptyset$) we see that G is never rigid by L.2g. In case $\varepsilon)$, for $|P_G|<7$, by L.1i,j, G is either non-rigid or G contains no triangle. We are going to prove that this is the case for $|P_G|=7$, too. Let $h\in X$, $i(h)=2$. Consequently, $x\in R(h)$ implies $i(x)=4$. Let, e.g., $R(h)=\{a, b\}$. By L.2d, G is either non-rigid or $(a, b)\in R$ and $R(a)\cap R(b)=\{h\}$. Let $R(a)=\{b, c, d, h\}$, $R(b)=\{a, f, g, h\}$. If $i(e)=4$, by L.2g, G is not rigid. We may assume that $R(e)=\{c, d, f\}$. G is not rigid, if $(c, d)\in R$ (by 2g) or if $R(c)\cap R(d)-\{a, e\}\neq\emptyset$ ($\{c\to d\}$ or $\{d\to c\}$ is an endomorphism). There remains (up to an isomorphism) the graph G with $R(c)=\{a, e, g\}$, $R(d)=\{a, e, f\}$ which is 3-coloured. By L.1b, hence, a rigid G satisfying $\varepsilon)$ contains no triangle. Thus, it is homeomorphic with E_3 . This is a contradiction, since some of the mappings $\{a\to b\}$, $\{b\to c\}$, $\{c\to a\}$ is then an endomorphism. In case $\varphi)$ suppose first $(a, b)\notin R$. By L.2g, $|R(a)\cap R(b)|\neq 4$. If $R(a)\cap R(b)=\{c, d, e\}$ (e.g., $R(a)=\{c, d, e, g\}$, $R(b)=\{c, d, e, h\}$), we have $R(f)\cap R(a)\subsetneqq R(f)$ (otherwise $\{f\to a\}$ is an endomorphism), similarly $R(f)\cap R(b)\subsetneqq R(f)$, so that, up to an isomorphism, $R(f)=\{g, h, c\}$. In both cases $(d, e)\in R$ or $(d, e)\notin R$ (i.e. $(d, g)\in R$, $(e, h)\in R$), the graphs are, by 2g, not rigid. If $R(a)\cap R(b)=\{c, d\}$, G is non rigid by 2 g



(if $(c,d) \in R$ or $R(c) \cap R(d) - \{a,b\} \neq \emptyset$) or by 1b (we investigate easily both possibilities $R(c) = \{a,b,e\}$, $R(d) = \{a,b,f\}$ and $R(c) = \{a,b,e\}$, $R(d) = \{a,b,g\}$, where $e, f \in R(a)$ and $g \in R(b)$). Provided $(a,b) \in R$, the graphs with $|R(a) \cap R(b)| = 3$ are obviously not rigid. If $|R(a) \cap R(b)| = 2$ and $R(a) = \{b,c,d,e\}$, $R(b) = \{a,c,d,f\}$, the graphs are, by 1b, not rigid in all the subcases: $R(c) = \{a,b,e\}$, $R(d) = \{a,b,g\}$; $R(c) = \{a,b,h\}$, $R(d) = \{a,b,g\}$. Similarly for $|R(a) \cap R(b)| = 1$, and for $R(a) \cap R(b) = \emptyset$ all graphs are not rigid by Lemmas 1b and 2g.

Now, Theorem 1 follows from [1].

Theorem 2. If n is an integer greater than 13, there is a rigid (X,R) with $|R| = n$.

First, define, for every $k \geq 4$ a graph $G_k(X_k, R_k)$ as follows:

$$X_k = \{x_i;\ i = 1, \dots, 2k+1\},\quad R_k = \{(x_1, x_2), (x_2, x_3), \dots$$

$$\dots (x_{2k-2}, x_{2k-1})\,(x_{2k-1}, x_1)\} \cup$$

$$\cup \{(x_{2k}, x_i);\ i = k, k+1, \dots 2k-1, 1\} \cup$$

$$\cup \{(x_{2k+1}, x_i);\ i = 1, 2, \dots k\}$$

Lemma 8. a) Let f be an automorphism of (X,R). Then $i(f(x)) = i(x)$ for every $x \in X$.

b) Let G' be a subgraph of G. Then $\gamma(G') \leq \gamma(G)$.

c) Let $Y \subset X$, and let there exist an endomorphism of G into $G' = (Y, R \cap Y \times Y)$.

Then $\gamma(G') = \gamma(G)$.

d) For every natural $k \geq 4$, $\gamma(G_k) = 4$, and the system of all the endomorphisms of G_k consists of the identity and the automorphism $g_k = \{x_j \to x_{k+1-j}\ (j = 1, 2, \dots k),\ x_j \to x_{3k-j}\ (j = k+1, \dots 2k-1)\}$.

Proof. a),b),c) are trivial. d): We have evidently $\gamma(G_k) = 4$, while $\gamma(G') < 3$ for every $G' \subsetneq G$. Thus, by c),a), every endomorphism of G_k is an automorphism and $f(x_{2k}) = x_{2k}$, $f(x_{2k+1}) = x_{2k+1}$. Hence, $R(x_{2k}) \cap R(x_{2k+1})$ must be mapped onto itself, which leads either to the identity or to g_k.

Proof of Theorem 2: First, let $n \geq 18$. Then $n = 4k + z$ where $k \geq 4$ and z is equal to $2,3,4$ or 5. Let us construct graphs $G_k^{(z)} = (X_k^{(z)}, R_k^{(z)})$ as follows:

$X_k^{(z)} = X_k \cup \{x_{2k+2}\}$ for $z = 2, 3, 4, 5$, $R_k^{(2)} = R_k \cup \{(x_{2k+2}, x_2), (x_{2k+2}, x_{k+1})\}$, $R_k^{(3)} = R_k^{(2)} \cup \{(x_{2k+2}, x_1)\}$,

$R_k^{(4)} = R_k^{(3)} \cup \{(x_{2k+2}, x_{2k-1})\}$, $R_k^{(5)} = R_k^{(4)} \cup \{(x_{2k+2}, x_{2k-2})\}$.

We have evidently $|R_k^{(z)}| = 4k + z = n$. We see easily that always, for $k > 4$,

(1) $\gamma(G_k^{(z)}) = 4$,

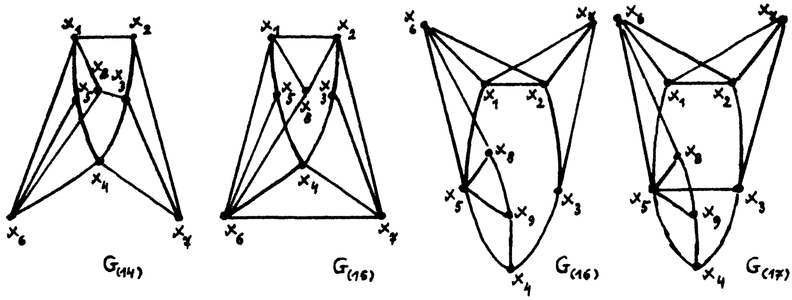
(2) $\gamma(G_k^{(z)}(x_i)) = 3$ for $i = 1, 2, \dots 2k+1$, $\gamma(G_k^{(z)}(x_{2k+2})) = 4$,

(3) $\gamma(G) \leq 3$ for every proper subgraph of $G_k^{(z)}(x_{2k+2})$.

Let $k > 4$. Let f be an endomorphism of $G_k^{(z)}$. By L.8b and 8c, f is an automorphism or a mapping onto

$G_k^{(z)}(x_{2k+2}) = G_k$. Anyway, f / X_k is an endomorphism of G_k and hence it is either the identity or g_k . If f is a mapping onto $G_k^{(z)}(x_{2k+2})$, we have $f(x_{2k+2}) \in R(f(x_2)) \cap$ $\cap R(f(x_{k+1})) = \emptyset$ in both cases of f / X_k , which is a contradiction. If f is an automorphism, we have $R(g_k(x_2)) \cap R(g_k(x_{k+1})) = R(x_{k-1}) \cap R(x_{2k-1}) = \emptyset$ while $R(x_2) \cap R(x_{k+1}) = x_{2k+1}$. Thus, f is the identity.
The proof for $k = 4$ and $z = 2,3,4$ is quite analogous. For $k = 4$ and $z = 5$, $G_4^{(5)}$ has, besides $G_4^{(5)}(x_{10})$, another 4-coloured graph, $G_4^{(5)}(x_8)$. All other proper subgraphs are again 3-coloured. If f is an endomorphism of $G_4^{(5)}$, it is, by L.8b and c, either a mapping onto $G_4^{(5)}(x_8)$ or onto $G_4^{(5)}(x_{10})$ or an automorphism. In the first case, $f / X_4^{(5)} - \{x_8\}$ is an automorphism, so that, by Lemma 8a, $f(x_{10}) = x_{10}$, further $f(x_9) = x_9$ (x_9 is the only vertex with $i(x) = 4$ in $X_4^{(5)} - R(x_{10})$),etc. We see easily that $f / X_4^{(5)} -$ $- \{x_8\}$ is an identity, which is a contradiction, since $f(x_8) \in R(f(x_7)) \cap R(f(x_4)) = R(x_7) \cap R(x_4) = \emptyset$.

In the remaining two cases we may proceed analogously as we did in the case of $k > 4$.
It remains to find rigit graphs with $14 \leq n \leq 17$. Such are, e.g., the following graphs $G_{(n)}$:



Evidently $G_{(16)}$ (and hence also $G_{(17)}$) is a 4-coloured graph and we see easily that all proper subgraphs of $G_{(17)}$ (and hence also of $G_{(16)}$) are 3-coloured. Thus every endomorphism f of $G_{(16)}$ is an automorphism and we have at $G_{(16)}$ by L.8a $f(x_5) = x_5$, further $f(x_2) = x_2$ (since $\{x_2\} = \{x_i \mid i(x_i) = 4\} - R(x_5)$) and $f(x_9) = x_9$ (unique vertex with $i(x) = 3$ in $R(x_5)$) and $f(x_8) = x_8$ (unique vertex with $i(x) = 3$ joined with y such that $i(y) = 4$). Now, we see easily that f is the identity. Similarly with $G_{(17)}$. Graphs $G_{(14)}$ and $G_{(15)}$ are again 4-coloured. Their unique 4-coloured proper subgraphs are $G_{(14)}(x_8)$ and $G_{(15)}(x_8)$. By 8c, an endomorphism, f of $G_{(14)}$ is either a mapping onto $G_{(14)}(x_8)$ or an automorphism. In the first case, $f / \{x_1, \dots x_7\}$ is an automorphism and hence some of the mappings

$\{x_5 \leftrightarrow x_6\}$, $\{x_3 \leftrightarrow x_7\}$, $\{x_6 \leftrightarrow x_5$, $x_3 \leftrightarrow x_7\}$, $\{x_5 \leftrightarrow x_3$, $x_6 \leftrightarrow x_7$, $x_1 \leftrightarrow x_2\}$, $\{x_5 \leftrightarrow x_7$, $x_6 \leftrightarrow x_3$, $x_1 \leftrightarrow x_2\}$.

Anyway, $R(f(x_1)) \cap R(f(x_6)) \cap R(f(x_3)) = \emptyset$, which is a contradiction. In the second case, we may prove easily that f is the identity. Similarly with $G_{(15)}$.

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