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Commentationes Mathematicae Universitatis Carolinae, Vol. 8 (1967), No. 4, 753--755

Persistent URL: http://dml.cz/dmlcz/105146

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8, 4 (1967)

CATEGORY OF COMMUTATIVE GROUPOIDS IS BINDING

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Let $\mathfrak{A}(1,1)$ denote the category of all (universal) algebras with two unary operations and their homomorphisms. Similarly, $\mathcal{C}(2)$ means the category of all commutative groupoids together with their homomorphisms.

A category $\mathcal{K}$ is called representable in a category $\mathcal{L}$ if there exists an isofunctor $\Phi : \mathcal{K} \rightarrow \mathcal{L}$ onto a full subcategory of $\mathcal{L}$.

A category in which $\mathfrak{A}(1,1)$ is representable is called binding. (Cf.[1]).

By methods very similar to those used in [2] it is possible to prove

<u>Theorem 1</u>. $\mathfrak{A}(1,1)$ is representable in $\mathcal{C}(2)$.

One of the corollaries is the following

<u>Theorem 2</u>. On every set X there is a binary operation such that (i) $\omega(x,y) = \omega(y,x)$ for any x and y in X, (ii) the algebra $(X;\omega)$ has only the identity mapping of X as an endomorphism.

<u>Proof of Theorem 1</u>. Let $A = (X; \varphi, \psi)$ be an object of $\mathfrak{A}(1,1)$. Put $Z = X \cup \{z_1(X), z_2(X), z_3(X)\}$, where $X \cap \{z_1(X), z_2(X), z_3(X)\} = \emptyset$. Put $\Phi(A) = (Z;\omega)$, ω being the binary operation on Z defined as follows:

(We write z_i instead of $z_i(X)$.)

$\omega(z, z_3) = \omega(z_3, z) = z_3$ for every z in Z, $z \neq z_3$

$\omega(x, z_1) = \omega(z_1, x) = \varphi(x)$ for every $x \in X$

$\omega(x, z_2) = \omega(z_2, x) = \psi(x)$ for every $x \in X$

$\omega(x, y) = z_3$ for every $x \in X$, $y \in X$

$\omega(z_i, z_j) = z_3$ for $i \neq j$ or $i = j = 1$

$\omega(z_2, z_2) = z_1$

$\omega(z_3, z_3) = z_2$

Let $A' = (X'; \varphi', \psi')$ be another object of $\mathcal{O}\!\mathcal{l}(1,1)$, $\Phi(A') = (X' \cup \{z_1(X'), z_2(X'), z_3(X')\}; \omega')$, let $f: A \to A'$ be a morphism of $\mathcal{O}\!\mathcal{l}(1,1)$. Define $\Phi(f): \Phi(A) \to \Phi(A')$ by

$$\Phi(f)(x) = f(x) \quad \text{for every } x \text{ in } X$$

$$\Phi(f)(z_i(X)) = z_i(X') \quad \text{for } i = 1, 2, 3 .$$

Clearly, $\Phi(f)$ is morphism in $\mathcal{C}(2)$ and, in addition, Φ is a one-to-one functor.

It remains to prove that its image is a full subcategory of $\mathcal{C}(2)$.

In the sequel, ω and ω' will be designated by juxtaposition.

Take $g: \Phi(A) \to \Phi(A')$ - a morphism of $\mathcal{C}(2)$. Provided $g(z_3) \in X'$, we have $g(z_2) = g(z_3 z_3) = g(z_3) g(z_3) = z_3'$, further $g(z_3) = g(z_2 z_3) = z_3' g(z_3) = z_3'$ - a contradiction.

Similar computation can be used for the proof of $g(z_3) \neq z_i'$, $i = 1, 2$. Thus only $g(z_3) = z_3'$ is possible. The last fact yields immediately $g(z_2) = z_2'$ and $g(z_1) = z_1'$.

If $g(x) = z_2'$ for some x in X , then $g(z_3) =$

$= g(xx) = z_2' z_2' = z_1'$ - a contradiction, similarly $g(x) \neq z_3'$ is obtained. Finally, suppose $g(x) = z_1'$. It is $g(\varphi(x)) = g(x z_1) = z_1' z_1' = z_3'$. As $\varphi(x) \in X$, this is a contradictory to the preceding statement.

Hence $g(X) \subseteq X'$, $g(z_i) = z_i'$ for i = 1,2,3. Define a mapping $f : X \to X'$ by $f(x) = g(x)$. It is $f(\varphi(x)) = g(\varphi(x)) = g(x z_1) = g(x) z_1' = \varphi'(g(x)) = \varphi'(f(x))$, and we obtain the same result for ψ and ψ'. Thus f is a morphism in $\mathfrak{A}(1,1)$, $\Phi(f) = g$.

Proof of Theorem 2. a) If X is infinite the statement is an easy consequence of [3],[2] and the construction given in the proof of Theorem 1.

b) Let X be finite, $X = \{x_1, \dots, x_n\}$. The binary symmetric operation will be defined by $\omega(x_i, x_j) = x_1$ for $i \neq j$ or $i = j = n$

$\omega(x_i, x_i) = x_{i+1}$ for $i = 1, \dots, n-1$.

A little computation concludes the proof.

I am indebted to A.Pultr for the suggestion of the problem.

References

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(Received November 13.1967)