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REMARKS ON THE DIFFERENTIABILITY OF MAPPINGS IN LINEAR NORMED SPACES

Josef KOLOMÝ, Václav ZIZLER, Praha

1. Introduction. In last time a great attention is paid in developing of differential calculus for metric and non-metric structures. Various definitions of differentials and derivatives have been proposed for instance by J. Sebastião e Silva [1], G. Marinescu [2], A.R. Fischer [3], J. Gil de Lamadrid [4], E. Dubinsky [5], M. Sova [6], G.A. Reid [7], V.I. Averbuch, O.G. Smoljanov [8], H.H. Keller [9] and others.

The purpose of this paper is to develop (cf. a footnote on p.53 [10, § 3]) the theory of weak differential and derivatives in linear normed spaces, examine these definitions to derive some basic relations among them and to establish some properties of mappings owing these differentials. The connection between weak differentials and derivatives is also considered and some sufficient conditions for linearity and continuity of such differentials are derived. Some assertions concerning these questions are pronounced also for Gateaux and Fréchet differentials.

The notions of weak differentials and derivatives seem to be sometimes convenient if we deal with differential operators in inner-product spaces, for instance in W_2^1. This note is a continuation of our considerations [11], [12].

2. **Notations and definitions.** Let X, Y be linear normed spaces, $(X \to Y)$ the space of all linear continuous mappings of X into Y, X^* (or Y^*) a dual space of X (or Y), (x, e^*) the value of $e^* \in X^*$ at the point $x \in X$. Let $D_R = \{x \in X : \|x\| \leq R\}$ denote the closed ball in X of radius $R > 0$ about the origin; S_R the boundary of D_R. We shall use the symbols "$\longrightarrow$" and "$\xrightarrow{w}$" to denote the strong and weak convergence in X, Y (or in X^*, Y^*) respectively. Throughout this paper by a word "space" there is meant a real space. A mapping $F : X \to Y$ of X into Y is said to be weakly (demi-) continuous [14] at $x_0 \in X$ if $x_n \xrightarrow{w} x_0$ ($x_n \longrightarrow x_0$ implies $F(x_n) \xrightarrow{w} F(x_0)$. A mapping $F : X \to Y$ is called compact (weakly compact) on a set $M \subset X$ if for every bounded subset $E \subset M$ the set $F(E)$ is compact (weakly compact). By $VF(x_0, h)$ (by $DF(x_0, h)$) we denote the Gâteaux (a liner Gâteaux) differential of a mapping $F : X \to Y$ at $x_0 \in X$ respectively. By $dF(x_0, h)$ we shall understand the Fréchet differential of F at $x_0 \in X$ ($h \in X$) cf. [10,chapt.I]. A mapping $F : X \to Y$ is said to possess the Gâteaux (Fréchet) derivative at $x_0 \in X$ if $DF(x_0, h)$ ($dF(x_0, h)$) is bounded in h on some S_R.

3. Now we introduce (cf. also a footnote on p.53 [10, § 3]) the notions of weak Gâteaux, Fréchet differentials and derivatives, examine these definitions to derive some basic

relations among them and some properties of mappings having these differentials.

Definition 1. We shall say that a mapping $F: X \to Y$ possesses at $x_0 \in X$ a weak Gâteaux differential $\hat{V} F(x_0, h)$ if

$$\lim_{t \to 0} \left(\frac{F(x_0 + th) - F(x_0)}{t}, e^* \right) = (\hat{V} F(x_0, h), e^*)$$

exists for every $h \in X$ and $e^* \in Y^*$.

If $\hat{V} F(x_0, h)$ is a linear (i.e. additive and homogeneous) mapping in h, we denote it by $\hat{D}F(x_0, h)$. We shall say that a mapping $F: X \to Y$ has a weak Gâteaux derivative $\hat{F}'(x_0)$ at $x_0 \in X$ if $\hat{D}F(x, h)$ is bounded on some S_R.

Proposition 1. Suppose that a mapping $F: X \to Y$ possesses on a convex subset $E \subset X$ a weak Gâteaux differential $\hat{V} F(x, h)$. If $x, x+h \in E$ are two arbitrary points of E, then

(1) $(F(x+h) - F(x), e^*) = (\hat{V} F(x + \tau h, h), e^*)$,

where e^* is an arbitrary point of Y^* and $0 < \tau = \tau(e^*) < 1$.

Proof. Set $\varphi(x, e^*) = (F(x), e^*)$. Then

$$\frac{1}{t}[\varphi(x+th, e^*) - \varphi(x, e^*)] = \left(\frac{1}{t}[F(x+th) - F(x)], e^*\right).$$

From this equality we conclude that there exist $V\varphi(x, h, e^*) = (\hat{V} F(x, h), e^*)$. Since $\varphi(x+h, e^*) - \varphi(x, e^*) = V\varphi(x+\tau h, h, e^*)$, $0<\tau<1$, we obtain at once (1).

Proposition 2. Suppose that a mapping $F: X \to Y$ possesses a weak Gâteaux differential $\hat{V} F(x, h)$ in some neighbourhood $U(x_0)$ of $x_0 \in X$. Assume that

$\hat{V} F(x, h)$ is demicontinuous at the point $x_0 \in X$ for every (but fixed) $h \in X$.

Then $\hat{V}F(x_0, h) = \hat{D}F(x_0, h)$, $h \in X$.

Proof: depends on Proposition 1 and arguments similar to that [10, § 3].

Remark 1. If $F : X \to Y$ is a linear mapping of X into Y demicontinuous at $\theta \in X$, then F is continuous in X. Indeed, it is easy to show that F maps each bounded subset of X into a bounded subset of Y (as the weak boundedness is equivalent to the strong one). Linearity of F gives the continuity.

Analysing the proof of Theorem 3.1 [10] and using Remark 1 it is easy to show that this theorem holds in the more general setting:

Proposition 3. Let the following conditions be fulfilled: 1) A mapping $F : X \to Y$ has a Gâteaux differential $VF(x, h)$ in some neighbourhood $U(x_0)$ of $x_0 \in X$ and $VF(x, h)$ is demicontinuous at the point $x_0 \in X$ for an arbitrary (but fixed) $h \in X$. 2) $VF(x_0, h)$ is demicontinuous at $h = \theta$ ($\|\theta\| = 0$).

Then $VF(x_0, h) = F'(x_0)h$, where $F'(x_0)$ denotes the Gâteaux derivative of F at x_0.

Definition 2. A mapping $F : X \to Y$ is said to have a weak Fréchet differential $\hat{d}F(x_0, h)$ at $x_0 \in X$ if

$$(F(x_0+h) - F(x_0), e^*) = (\hat{d}F(x_0, h), e^*) + (\hat{\omega}(x_0, h), e^*)$$

holds for any $e^* \in Y^*$, where $\hat{d}F(x_0, h)$ is linear in h and

$$\lim_{\|h\|\to 0} \frac{|(\hat{\omega}(x_0, h), e^*)|}{\|h\|} = 0 .$$

If $\hat{d} F(x_0, h)$ is bounded on some S_R, we shall say that F possesses a weak Fréchet derivative $\hat{F}(x_0)$ at x_0.

Proposition 4. The following assertions are valid: a) If F has a weak Gâteaux differential at $x_0 \in X$, then F is demicontinuous at x_0 under an arbitrary direction h, i.e. $\lim_{t\to 0} |(F(x_0+th)-F(x_0), e^*)| = 0$

for any $e^* \in Y^*$. b) If F possesses a weak Fréchet derivative $\hat{F}'(x_0)$ at x_0, then F is demicontinuous at $x_0 \in X$.

Proof. For instance b). Suppose that $h_n \in X$, $h_n \to 0$. For any $e^* \in Y^*$ $|(\hat{\omega}^*(x_0, h_n), e^*)| \leq K \|h_n\|$. So that

$$|(F(x_0+h_n) - F(x_0), e^*)| \leq$$

$$\leq \|e^*\| \, \|\hat{F}'(x_0)\| \, \|h_n\| + K \|h_n\| \longrightarrow 0 .$$

Hence $F(x_0 + h_n) \xrightarrow{w} F(x_0)$.

Proposition 5. Suppose that $F : X \to Y$ has a weak Gâteaux differential $\hat{V} F(x_0, h)$ at $x_0 \in X$. Then $\hat{V} F(x_0, h)$ is demicontinuous at $h = 0$ under an arbitrary direction u ($\|u\| = 1$) if and only if F is demicontinuous at x_0 under the direction u.

Proof. Suppose that F is demicontinuous at x_0

under the direction u. Then for any $e^* \in Y^*$ and $\frac{\varepsilon}{2} > 0$ there exists a positive number δ_1 such that for $|t| < \delta_1$ there is $|(F(x_0 + tu) - F(x_0), e^*)| < \frac{\varepsilon}{2}$. Moreover there exists $\delta_2(u) > 0$ such that for $|t| < \delta_2$ there is $|(\hat{\omega}(x_0, tu), e^*)| < \frac{\varepsilon}{2}|t|$. Set $\delta = Min(\delta_1, \delta_2)$, then for $|t| < \delta$ we have

$$|(\hat{V}F(x_0, tu), e^*)| \leq |(F(x_0 + tu) - F(x_0), e^*)| + $$

$$+ |(\hat{\omega}(x_0, tu), e^*)| < \frac{\varepsilon}{2}(1 + \delta_2).$$

If $\lim\limits_{t \to 0} |(\hat{V}F(x_0, tu), e^*)| = 0$ for any $e^* \in Y^*$, then the inequality

$$|(F(x_0 + tu) - F(x_0), e^*)| \leq |(\hat{V}F(x_0, tu), e^*)| + $$

$$+ |(\hat{\omega}(x_0, tu), e^*)|$$

implies immediately the first assertion of our proposition.

Theorem 1. Let X be a Banach space, Y a linear normed space, $F : X \longrightarrow Y$ a demicontinuous mapping of X in Y. Suppose that there exists $\hat{V}F(x, h)$ in some neighbourhood $U(x_0)$ of $x_0 \in X$. Assume that $\hat{V}F(x, h)$ is demicontinuous at the point x_0 for an arbitrary (but fixed) $h \in X$.

Then $\hat{V}F(x_0, h) = \hat{F}'(x_0)h$, where $\hat{F}'(x_0)$ denotes a weak Gâteaux derivative of F at x_0.

Proof. According to Proposition 2 $\hat{V} F(x_0, h) = \hat{D} F(x_0, h)$. Let $e^* \in Y^*$ be any element of Y^*. Then

$(\frac{1}{t}[F(x_0 + th) - F(x_0)], e^*)$ is a continuous linear functional in h on X; because $h_n \to h$ implies

$(\frac{1}{t}(F(x_0+th_n) - F(x_0)), e^*) \to (\frac{1}{t}(F(x_0+th) - F(x_0)), e^*)$.

If $t_n \to 0$ as $n \to \infty$, then

$(\hat{D}F(x_0,h), e^*) = \lim_{n\to\infty} (\frac{1}{t_n}[F(x_0+t_n h) - F(x_0)], e^*)$

and hence $(\hat{D}F(x_0,h), e^*)$ is a point limit of linear continuous functionals defined on X. According to Baire's theorems $(\hat{D}F(x_0, h), e^*)$ is continuous in $h \in X$. Hence $\hat{D}F(x_0, h)$ is demicontinuous in $h \in X$. According to Remark 1, $\hat{D}F(x_0, h)$ is continuous in $h \in X$. Thus $\hat{D}F(x_0, h) = \hat{F}'(x_0)h$. This concludes the proof.

The following theorem gives sufficient conditions for the weak and strong continuities of the smooth mapping in linear normed spaces. In such way this theorem completes the results of [13, Theorems 1,2,5].

Theorem 2. Let X, Y be linear normed spaces, $F: W \to Y$ a mapping of convex bounded subset $W \subset X$ in Y. Suppose that F has a weak Fréchet derivative $\hat{F}'(x)$ on W having the property that $\hat{F}'(x)$ is compact on W. Then F is weakly continuous. Moreover, if F is compact on W, then F is strongly continuous on W.

Proof. Use the arguments similar to that of [10, § 4.3].

Theorem 3. Let X, Y be linear normed spaces, $F: X \to Y$ a mapping having a weak Fréchet differential $\hat{d}F(x_0, h)$ (or the Fréchet differential $dF(x_0, h)$) at $x_0 \in X$. If F is weakly compact in some neighbourhood $U(x_0)$ of x_0, then $\hat{d}F(x_0, h)$ (or $dF(x_0, h)$) is weakly compact on X.

Proof. We prove our assertion for a weak differential $\hat{d}F(x_0, h)$. Suppose contrary, there exist a positive number ε_0, a linear functional $e_0^* \in Y^*$, $\|e_0^*\| = 1$ and a bounded sequence $\{h_n\} \in X$, $(\|h_n\| \leqq R)$ such that

$$|(\hat{d}F(x_0, h_n) - \hat{d}F(x_0, h_m), e_0^*)| \geqq \varepsilon_0 .$$

Choose a positive number t such that $x_0 + t h_n \in U(x_0)$ for every n $(n = 1, 2, \dots)$. We have

$$|(F(x_0 + t h_n) - F(x_0 + t h_m), e_0^*)| \geqq$$

$$\geqq t\, |(\hat{d}F(x_0, h_n) - \hat{d}F(x_0, h_m), e_0^*)| -$$

$$- |(\hat{\omega}(x_0, t h_n), e_0^*)| - |(\hat{\omega}(x_0, t h_m), e_0^*)| .$$

But this inequality leads to a contradiction with our assumption.

Definition 3. We shall say that a mapping $F: X \to Y$ has a weak bounded differential $\hat{d}F(x_0, h)$ if: a) for any $e^* \in Y^*$ there exists

$$\lim_{t\to 0}\left(\frac{1}{t}[F(x_0+th)-F(x_0)], e^*\right)=(\hat{d}F(x_0, h), e^*)$$

uniformly with respect to $\|h\| = 1$. b) $\hat{d}F(x_0, h)$ is a bounded mapping on S_1.

<u>Theorem 4</u>. Let X, Y be linear normed spaces, $F: X \to Y$ a mapping of X into Y having in some neighbourhood $U(x_0)$ of $x_0 \in X$ a weak Gâteaux differential $\hat{V}F(x_0, h)$. Suppose that for any $e^* \in Y^*$

$$\lim_{t\to 0}(\hat{V}F(x_0+th, h) - \hat{V}F(x_0, h), e^*) = 0$$

holds uniformly with respect to $h \in S_R$ and that $\hat{V}F(x_0, h)$ is bounded in h on S_1.

Then F possesses a weak bounded differential $\hat{d}F(x_0, h)$ at $x_0 \in X$.

<u>Proof</u>. The proof depends on Proposition 1 and the arguments similar to that of the proof of Theorem 1 [12].

We shall say that $VF(x, h)$ is continuous at $x_0 \in X$ and strongly continuous at $h \in D_1$ ($\|x\| \leq 1$) $\subset X$ jointly if $x_n \to x_0$, $h_n \xrightarrow{w} h$, $h_n \in D_1$, $h \in D_1$ imply $VF(x_n, h_n) \to VF(x_0, h_0)$. The following theorem shows that the assumptions of Theorem 2 [11] can be weakened.

<u>Theorem 5</u>. Let X be a reflexive Banach space, Y a linear normed space, $F: X \to Y$ a mapping of X into Y. Suppose that F possesses the Gâteaux differential $VF(x, h)$ in a convex neighbourhood $U(x_0)$ of $x_0 \in X$. If $VF(x, h)$ is continuous at $x_0 \in X$ and strongly continuous in $h \in D_1$ jointly,

then F possesses the Fréchet derivative $F'(x_0)$ at x_0, $VF(x_0, h) = F'(x_0)h$ and $F'(x_0)$ is weakly compact on D_1.

Proof. From our assumptions it follows that $VF(x_0, h) = DF(x_0, h) = F'(x_0)h$, where $F'(x_0)$ denotes the Gâteaux derivative of F at x_0. Since X is reflexive, $F'(x_0)$ is weakly compact on X. Let ε be an arbitrary positive number, h a fixed (but arbitrary) element of X. Then there exists a constant $\delta_1(\varepsilon) > 0$ such that if $|t| < \delta_1(\varepsilon)$, then

$$(2) \qquad \left\| \frac{1}{t}\omega(x_0, th) \right\| < \varepsilon ,$$

where $\omega(x_0, th) = F(x_0 + th) - F(x_0) - VF(x_0, th)$.

Suppose on the contrary that

$$(3) \qquad \lim_{t \to 0} \left\| \frac{1}{t}\omega(x_0, th) \right\| = 0$$

is not uniform on $\|h\| = 1$, $h \in X$. Then there exists $\varepsilon_0 > 0$ with the following property: for every n $(n = 1, 2, \dots)$ there exist $h_n \in X$, $\|h_n\| = 1$ and t_n such that $0 < |t_n| < \frac{1}{n}$ and

$$(4) \qquad \left\| \frac{1}{t_n}\omega(x_0, t_n h_n) \right\| \geq \varepsilon_0 .$$

Since X is reflexive and $\|h_n\| = 1$, passing to a subsequence $\{h_{n_k}\}$ we have that $h_{n_k} \xrightarrow{w} h_0 \in X$.

We have

$$F(x_0+t_{n_k}h_{n_k})-F(x_0)=F'(x_0)t_{n_k}h_{n_k}+\omega(x_0,t_{n_k}h_{n_k});$$

(5)

$$F(x_0+t_{n_k}h_0)-F(x_0)=F'(x_0)t_{n_k}h_0+\omega(x_0,t_{n_k}h_0).$$

Let $e_n^* \in Y^*$ be any arbitrary elements of Y^*. From (5), using the mean-value theorem, Hahn-Banach theorem and adding and substracting $F'(x_0)t_{n_k}h_0$, we obtain

$$\left\|\frac{1}{t_{n_k}}\omega(x_0,t_{n_k}h_{n_k})\right\| \leqq \left\|\frac{1}{t_{n_k}}\omega(x_0,t_{n_k}h_0)\right\| +$$

$$+\|F'(x_0+\alpha_{n_k}t_{n_k}h_{n_k})h_{n_k}-F'(x_0)h_0\| +$$

$$+\|F'(x_0+\beta_{n_k}t_{n_k}h_0)h_0-F'(x_0)h_0\| +$$

$$+\|F'(x_0)h_0-F'(x_0)h_{n_k}\|,$$

where $0<\alpha_{n_k}<1$, $0<\beta_{n_k}<1$, $(k=1,2,\dots)$.

Since $t_{n_k}\to 0$, $h_{n_k}\xrightarrow{w} h_0$, $\|h_{n_k}\|=1$ and $0<\alpha_{n_k}t_{n_k}\|h_{n_k}\|=\alpha_{n_k}t_{n_k}\to 0$ as $k\to\infty$, $x_0+\alpha_{n_k}t_{n_k}\to x_0$ and hence $F'(x_0+\alpha_{n_k}t_{n_k}h_{n_k})h_{n_k}\to F'(x_0)h_0$.

Thus

$$\left\|\frac{1}{t_{n_k}}\omega(x_0,t_{n_k}h_{n_k})\right\|\to 0$$

whenever $k\to\infty$. But this contradicts with (4). This completes the proof.

We shall say that a weak Gâteaux differential $\hat{V}F(x,h)$ is demicontinuous at $x_0\in X$ and weakly continuous on D_1 $(\|x\|\leqq 1)\subset X$ jointly if $x_n\to x_0$, $h_n\xrightarrow{w} h$, h_n, $h\in D_1$ imply $\hat{V}F(x_n,h_n)\xrightarrow{w}\hat{V}F(x_0,h_0)$.

Theorem 6. Let X be a reflexive Banach space, Y a linear normed space, $F: X \to Y$ a mapping of X into Y. Suppose that F possesses a weak Gâteaux differential $\hat{V}F(x,h)$ in a convex neighbourhood $U(x_0)$ of $x_0 \in X$. If $\hat{V}F(x,h)$ is demicontinuous at $x_0 \in X$ and weakly continuous on D_1 jointly, then F possesses a weak Fréchet derivative $\hat{F}'(x_0)$ at x_0 and $\hat{F}'(x_0)$ is weakly compact on D_1.

Proof. To prove this theorem use Proposition 2 and similar arguments as in proof of Theorem 5.

Definition 4. We shall say that a mapping $F: X \to Y$ has a local weak uniform Gâteaux differential $\hat{V}F(x,h)$ in B_R $(\|x\| < R)$ if for any $\varepsilon > 0$, $x_0 \in B_R$, $e^* \in Y^*$ and $h \in X$ there exist two positive constants $\delta(\varepsilon, x_0, e^*, h)$, $\eta(\varepsilon, x_0, e^*, h)$ such that if $0 < |t| < \delta$, then

$$|(\frac{1}{t}\hat{\omega}(x,th), e^*)| < \varepsilon$$

for every $x \in B(x_0, \eta) \cap B_R$, where

$$\hat{\omega}(x,th) = F(x+th) - F(x) - \hat{V}F(x,th),\ B(x_0,\eta) =$$
$$= \{x \in X : \|x - x_0\| < \eta\}.$$

We shall say that a mapping $F: X \to Y$ is said to be locally uniformly demicontinuous in $B_R \subset X$ if for every $x \in B_R$ there exists a neighbourhood $U(x_0)$ of x on which F is uniformly demicontinuous (cf.[13]).

Theorem 7. Suppose that a mapping $F: X \to Y$ has in B_R a weak Gâteaux differential $\hat{V}F(x,h)$ which is demicontinuous in $x \in B_R$. Then F possesses a local weak uniform Gâteaux differential in B_R. Conversely, if F

is locally uniformly demicontinuous in B_R and possesses a local weak uniform Gâteaux differential $\hat{V} F(x, h)$ in B_R, then $\hat{D} F(x, h)$ is demicontinuous in $x \in B_R$.

<u>Proof</u>. Use the arguments similar to that of [11].

References

[1] J.SEBASTIÃO e SILVA: Le calcul différentiel et intégral dans les espaces localement convexes réels on complexes. Atti dei Lincei 20(1956), 743-750.

[2] G. MARINESCU: Differentielles de Gâteaux et Fréchet dans les espaces localement convexes. Bull.Math.Soc. Sci.Math.et phys.RPR,1(1957),No 1,77-86.

[3] H.R. FISCHER: Differentialkalkül für nicht-metrische Strukturen. Ann.Acad.Sci.Fenn. H 247(1957), 1-14.

[4] J.GIL de LAMADRID: Topology of mappings and differentiation-processes. Illinois J.Math.3(1959),408-420.

[5] E. DUBINSKY: Differential equations and differential calculus in Montel spaces. Trans.Am.Math.Soc. 110(1964),1-21.

[6] M. SOVA: Some conditions on the differentiability in linear topological spaces. Czech.Math.J.3(16), 1966,339-362.

[7] G.A. REID: Concepts on differentiability and analyticity on certain classes of topological groups. Proc.Cambr.Phil.Soc.61(1965),347-379.

[8] V.I. AUERBUCH, O.G. SMOLJANOV: Differencirovanie v linejnych topologičeskich prostranstvach. Dokl.Ak.Nauk SSSR 173(1967),No 4,735-738.

[9] H.H. KELLER: Über die Probleme, die bei einer Differentialrechnung in topol.Vektorräume auftreten H.P.Künzi,A.Pfluger, Festband zum 70. Geburtstag von Rolf Nevanlinna. Springer 1966.

[10] M.M. VAJNBERG: Variacionnyje metody issledovanija nelinejnych operatorov. Moskva 1956.

[11] J. KOLOMÝ: On the differentiability of mappings in functional spaces. Comment.Math.Univ.Carolinae 8(1967),No 2,315-329.

[12] V. ZIZLER: On the differentiability of mappings in Banach spaces. Comment.Math.Univ.Carolinae 8(1967), No 3, 415-430.

[13] J. KOLOMÝ: A note on the continuity properties of nonlinear operators. Comment.Math.Univ.Carolinae 8(1967),No 3,503-514.

[14] E.F. BROWDER: Variational boundary value problems for quasi-linear elliptic equations of arbitrary order. Proc.Nat.Acad.Sci.U.S.A. 50(1963),No 1, 31-37.

[15] M.A. KRASNOSELSKIJ: Položitelnyje rešenija operatornych uravněnij. Moskva 1962.

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