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FIXED POINT THEOREMS BASED ON LERAY-SCHAUDER DEGREE Svatopluk FUČÍK, Praha

1. Introduction. This note deals with the existence of the solution of the equation Fx = x . The technique used here is the so-called Leray-Schauder topological degree. This method has been largely used by Altman [1], J. Cronin [2], Leray and Schauder [6] and others.

In this paper there is given a generalization of Altman's result [1] (see Remark 2). Theorem 8 is the extending of Theorem 7 for a special case.

Other theorems are well known, but the proofs are based on topological degree theory (see Theorem 1 and Theorem 10). These theorems were proved in [7], [4] respectively, by an another way.

Using the theory of topological degree and the properties of Minkovski's function we prove Schauder's fixed-point theorem for a convex set (compare [2], p.139).

2. Terminology and notations used in this paper. Banach space is denoted by X, its norm by I I and θ is the zero element in X. The symbol $f\colon M \longrightarrow X$ denotes mapping defined on $M \subset X$ with range in X. Let be I identity mapping in X and P be completely continuous operator defined on $M \subset X$. The ball K_R in Banach space is set of all x,

 $\| \mathbf{x} \| < \mathbf{R}$ and $\mathbf{S}_{\mathbf{R}}$ is its boundary. Let \mathbf{M} be subset of \mathbf{X} and $\mathbf{f} \colon \mathbf{M} \longrightarrow \mathbf{X}$. The symbol $\overline{\mathbf{M}}$ denotes closure of \mathbf{M} (in norm of \mathbf{X}), $\partial \mathbf{M}$ its boundary and $\mathbf{f}(\mathbf{M})$ the image under the transformation \mathbf{f} .

3. Brouwer's and Leray-Schauder's degree. Let G be a bounded and open subset of X, $f \colon \overline{G} \to X$ and $z_o \in X - f(\partial G)$ (arbitrary but fixed). If X is finite dimensional Banach space let f be continuous mapping on \overline{G} and in another case let be f = I - F, where F is a completely continuous operator.

Degree d[f, G, z] has following properties:

I. d[I, G, z, I] = 1 for $z, \in G$.

II. $d[I, G, z_o] = 0$ for $z_o \notin \overline{G}$.

III. Let G_1 , G_2 be bounded and open subsets of X, f: $: \overline{G_1} \cup G_2 \longrightarrow X \quad \text{and} \quad z_o \in X - (f(\partial G_1) \cup f(\partial G_2)), G_1 \cap G_2 = \emptyset$ Then is $d[f, G_1 \cup G_2, z_o] = d[f, G_1, z_o] + d[f, G_2, z_o]$.

IV. Let $d \, \Box \, f, \, G, \, z_o \,] \neq 0$. Then exists $x_o \in G$ such that $f(x_o) = z_o$.

V. Finite dimensional case. Let $f(x,t): \overline{G} \times \langle 0,1 \rangle \to X$ be continuous mapping such that for all $x \in \partial G$ and for all $t \in \langle 0,1 \rangle$ is $f(x,t) \neq x$, $\in X$. Then d[f(x,t),G,x] is constant on $\langle 0,1 \rangle$.

Infinite dimensional case. Let $F: \overline{G} \to X$ be completely continuous operator such that for all $x \in \partial G$ and for all $t \in (0, 1)$ is $(I - tF)x + z, \in X$. Then $d \in I, G, z, J = d \cap I - F, G, z, J$.

4. Fixed point theorems in finite dimensional space.

Theorem 1 (17): Let f be a continuous mapping of \overline{K}_{g} into finite dimensional space X and let m be any constant. Suppose that f satisfies one of the following two conditions:

- (1) For $x \in S_0$ such that f(x) = ax is $a \le m$,
- (2) For $x \in S_R$ such that f(x) = ax is a > m.

Then there exists at least one element $x_o \in \overline{K}_R$ such that $f(x_o) = mx_o$.

<u>Proof</u>: Let the condition (1) be held. We assume that for all $x \in S_R$ is $f(x) \neq mx$ and we define H(x,t) = tx - (1-t)(f(x) - mx)

for all $x\in \overline{K}_R$ and for $t\in (0,1)$. The function H(x,t) satisfies assumptions in V and we have (by V and I):

 $d[-f(x)+mx, K_R, \theta] = d[I, K_R, \theta] = 1$.

The theorem is proved by the property IV. Analogously for condition 2.

Theorem 2 ([7]): Let f be continuous mapping of \overline{K}_{g} into X satisfying one of the following two conditions:

- (3) For each $x \in S_R$ there is $\|x f(x)\|^2 > \|f(x)\|^2 + \|x\|^2$.
- (4) For each $x \in S_R$ there is $\|x f(x)\|^2 \le$ $\leq \|f(x)\|^2 + \|x\|^2.$

Then there exists a point $x_o \in \overline{K}_g$ such that $f(x_o) = 0$.

<u>Proof</u>: Let the condition (3) be held. We suppose there exists $x \in S_R$ such that f(x) = ax. The condition (3) implies that $a \le 0$ and the condition (1) is satisfied. Analogously for the condition (4).

Theorem 3 ([7]): Let f be a continuous mapping of $\overline{\mathbb{K}}_R$ into X satisfying one of the following conditions:

- (5) For each $x \in S_R$ there is $\|f(x) x\|^2 \ge \|f(x)\|^2 \|x\|^2$.
- (6) For each $x \in S_R$ there is $\|f(x) x\|^2 \le \|f(x)\|^2 \|x\|^2$.

Then there exists a point $x_0 \in \overline{K}_R$ such that $f(x_0) = x_0$.

<u>Proof</u>: Define for all $x \in \overline{K}_R$ g(x) = x - f(x). The function g satisfies the conditions of Theorem 2.

Theorem 4 (Rothe): If f is a continuous mapping of \overline{K}_R into X such that $f(S_R) \subset \overline{K}_R$, then there exists $x_0 \in \overline{K}_R$ such that $f(x_0) = x_0$.

<u>Proof</u>: The assumptions in this theorem imply the condition (5) of Theorem 3.

Theorem 5 (Brouwer): Let $M \subset X$ be homeomorphic with \overline{K}_R and f be a continuous mapping of M into X such that $f(M) \subset M$. Then there exists $x_o \in M$ such that $f(x_o) = x_o$.

<u>Proof:</u> Let be $\mathbf{M} = \overline{\mathbf{K}}_R$. The assertion is valid by Theorem 4. Let be $\mathbf{M} = \mathbf{h}(\overline{\mathbf{K}}_R)$ where h is homeomorphism. Then $\mathbf{h} + \mathbf{h}^{-1} (\overline{\mathbf{K}}_R) \subset \overline{\mathbf{K}}_R$ and there exists $\mathbf{w}_o \in \overline{\mathbf{K}}_R$ such that $\mathbf{h} + \mathbf{h}^{-1} (\mathbf{w}_o) = \mathbf{w}_o$. The point $\mathbf{x}_o = \mathbf{h}^{-1} (\mathbf{w}_o)$ satisfies the equation $\mathbf{f}(\mathbf{x}_o) = \mathbf{x}_o$.

Theorem 6: If f is a continuous mapping of \overline{K}_R into X, there exist a point $\times_o \in \overline{K}_R$ and a real number $\lambda_o > 0$ such that $f(\times_o) = \lambda_o \times_o$.

<u>Proof</u>: If there exists point $x_0 \in \overline{K}_R$ such that

 $f(\mathbf{x}_o) = \theta \quad \text{we have} \quad \lambda_o = 0 \quad . \quad \text{Let for each} \quad \mathbf{x} \in \overline{\mathbb{K}}_R$ be $f(\mathbf{x}) + \theta \quad . \quad \text{Define function } g \quad \text{for every } \mathbf{x} \in \overline{\mathbb{K}}_R \quad \text{as}$ $g(\mathbf{x}) = R \cdot \frac{f(\mathbf{x})}{\|f(\mathbf{x})\|} \quad . \quad \text{We obtain} \quad g_*(\overline{\mathbb{K}}_R) \subset \overline{\mathbb{K}}_R \quad \text{and}$ $\mathbf{x}_o \in \overline{\mathbb{K}}_R \quad \text{such that} \quad g_*(\mathbf{x}_o) = \mathbf{x}_o \quad \text{i.e.} \quad f(\mathbf{x}_o) = \lambda_o \times_o \quad ,$ where $\lambda_o = \frac{\|f(\mathbf{x}_o)\|}{R} \quad .$

5. Fixed point theorems in infinite dimensional space.

Lemma 1. Let p be a real number, p > 0, p ≠ 1

and a > b > 0. Let the inequality

 $[sqn(n-1)](a-b)^n \ge [sqn(n-1)](a^n-b^n)$ be satisfied. Then a=b.

Lemma 2. Let p be a real number, $p \neq 0$, 1 and a > b > 0. Then

[sqn(n-1)](a-b)ⁿ < [sqn(n-1)](aⁿ-bⁿ).

Theorem 7: Let F be a completely continuous operator of \overline{K}_R into X and p be a function of S_R into real numbers such that $\mu(x) > 0$ and $\mu(x) + 1$ for every $x \in S_R$. Let the following condition be fulfilled for each $x \in S_R$:

[sqn(p(x)-1)] $\|x - Fx\|^{p(x)} > [sqn(p(x)-1)](\|Fx|^{p(x)} - \|x\|^{p(x)})$

Then there exists $\mathbf{x}_{o} \in \overline{\mathbb{K}}_{R}$ such that $\mathbf{F}\mathbf{x}_{o} = \mathbf{x}_{o}$.

Proof: We suppose that for each $\mathbf{x} \in S_{R}$ is $\mathbf{F}\mathbf{x} + \mathbf{x}_{o}$.

Define the operator H(x,t) = x - tFx for every $\mathbf{x} \in \overline{\mathbb{K}}_{R}$ and each $t \in \langle 0, 1 \rangle$.

Lemma 1 proves that the assumptions of property V are valid and by property I and IV we obtain the assertion of Theorem 7.

Theorem 8: Let F be a completely continuous operator

of \overline{K}_R into X such that for every $x \in S_R$ there is $Fx \neq \theta$ and Fx + x.

Let p be a function of S_R into real numbers such that for each $x \in S_R$ there is $\rho(x) \neq 0$, $\rho(x) \neq 1$. Assume that for all $x \in S_R$

 $[sgn(n(x)-1)] \|x-Fx\|^{n(x)} \ge [sgn(n(x)-1)] (\|Fx\|^{n(x)}-\|x\|^{n(x)}).$

Then there exists the point $x_{,} \in K_{g}$ such that $Fx_{,} = x_{,}$.

Proof: As Theorem 7 but by using Lemma 2.

Remark 1: In Theorem 7 and 8 we can assume that F is defined on set \overline{G} , $\theta \in G$ and the conditions are satisfied on ∂G .

Remark 2: If p(x) = 2 in Theorem 7, we obtain Altman's fixed point theorem [1]. This Altman's condition is equivalent to $(FX, X) \le (X, X)$ in Hilbert space, where (\cdot, \cdot) is inner product.

Theorem 9 (Rothe): Let F be a completely continuous operator of \overline{K}_R into K such that $F(S_R) \subset \overline{K}_R$. Then there exists $x_o \in \overline{K}_R$ such that $Fx_o = x_o$.

Proof: As in finite dimensional case.

Theorem 10 ([4]): Let X be Hilbert space with inner product (\cdot, \cdot) , $v_i \in X$, f a mapping of \overline{K}_R into X satisfying one of the following two conditions:

- (7) For each $x \in S_R$ there is $(x, f(x)) \le (x, y)$ and I = f is a completely continuous operator.
- (8) For each $x \in S_R$ there is $(x, f(x)) \ge (x, y)$ and 1 + f is a completely continuous operator.

Then there exists $x_o \in \overline{K}_g$ such that $f(x_o) = y$.

Proof: Let the condition (7) be satisfied and let be

F = I + f - y. Then F is a completely continuous operator and for each $x \in S_R$ is $(Fx, x) = (x, x) + (f(x), x) - (x, y) \le (x, x)$ and the remark 2 proves this theorem. Analogously for the condition (8).

Remark 3: Let M be an open bounded subset of X, $\theta \in \mathbb{N}$, $\overline{\mathbb{M}}$ a convex set. Set $J(x) = \{a, a > 0, \frac{x}{a} \in \overline{\mathbb{M}}\}$

for every $x \in X$ and $t(x) = \inf_{\theta \in J(x)} a$. The function t has the following properties [3]:

- (11) x \(\vec{M}\) if and only if \(\tau(\times) > \sqrt{1}\).

 Theorem 11 (Schauder): Let M be an open bounded set of X, \(\vec{M}\) a convex set and F a completely continuous opera-

or X, M a convex set and F a completely continuous operator of \overline{M} into X such that $F(\overline{M}) \subset \overline{M}$. Then there exists $X \in \overline{M}$ such that $FX_0 = X_0$.

Proof: We assume that for all $X \in \partial M$ is $FX \neq X$

Proof: We assume that for all $x \in \partial$ M is $Fx \neq x$ and $\theta \in M$. Let exist $c_{\eta} \in (0,1)$ and $x_{\eta} \in \partial$ M such that $x_{\eta} = c_{\eta} F x_{\eta}$. Then $t(Fx_{\eta}) = t(\frac{x_{\eta}}{c_{\eta}}) = \frac{1}{c_{\eta}} > 1$, and

(11) implies $Fx_1 \notin \overline{M}$, and this contradicts our assumptions. The assertion is valid by property V, I and IV. If $G \notin M$ we must use a translation for the set M.

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