Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 8 (1967), No. 4, 663--682

Persistent URL: http://dml.cz/dmlcz/105140

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8,4 (1967)

LIMITS OF FUNCTORS AND REALISATIONS OF CATEGORIES *

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The present paper is in a close connection with [2]. We show that the results of [2] remain, in essential, valid after enlarging the discussed system of functors in order to close it with respect to arbitrary limits and colimits over small categories, and after allowing infinite systems of functors in description of discussed categories.

Paragraphs 1 and 2 deal with limits and colimits of systems of functors (in particular, of set functors). Paragraph 3 contains, after some technical lemmas, the description of the enlarged system of functors and some theorems about it. The main theorems are formulated and proved in § 4.

The notation of [1] and [2] is preserved with the exception that we write $S((F_\iota, \Delta_\iota)_{\iota \in J})$ instead of $\gamma(\{(F_\iota, \Delta_\iota) \mid \iota \in J\})$ (in that point was the notation of [1] and [2] inconsistent).

I am indebted to P. Vopěnka and Z. Hedrlín for valuable discussions and advises. The essence of the proof of Theorem 4.3 is due to P. Vopěnka.

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* Supported by the Alexander von Humboldt-Stiftung

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§ 1. Limits of transitive systems of functors.

1.1. **Definition**: Let C be a small category. For every object a from C let there be given a functor F_a : $: \mathcal{K} \to \mathcal{L}$ (always covariant or always contravariant), for every morphism $\varphi : a \to b$ from C let there be given a transformation $\tau_\varphi : F_a \to F_b$ such that

a) if $\varphi = id_a$, then τ_φ is the identical transformation of F_a;

b) $\tau_{\varphi\psi} = \tau_\varphi \circ \tau_\psi$.

Such a system of functors and transformations will be termed a transitive system (over C) and denoted by $(F_a, \tau_\varphi)_C$. The limit of a transitive system $(F_a, \tau_\varphi)_C$ is a system $(\tau_a : F \to F_a)_{a \in obj\, C}$ of transformations such that

α) for every $\varphi : a \to b$, $\tau_b = \tau_\varphi \circ \tau_a$,

β) if $(\vartheta_a : G \to F_a)$ is a system of transformations such that $\vartheta_b = \tau_\varphi \cdot \vartheta_a$ for any φ, then there exists exactly one transformation $\vartheta : G \to F$ such that $\vartheta_a = \tau_a \circ \vartheta$ for every a.

Dually the colimit of a transitive system (F_a, τ_φ) is defined.

1.2. **Remarks**: 1) The limit (colimit, resp.) of (F_a, τ_φ) is, hence, formally the limit (colimit, resp.) of the "functor" Φ from C into the "category of all functors from $\mathcal{K}$ into $\mathcal{L}$ and their transformations" defined by $\Phi(a) = F_a$, $\Phi(\varphi) = \tau_\varphi$.

2) Evidently, both limit and colimit are determined up to a natural equivalence.

1.3. **Theorem**: Let (F_a, τ_φ) be a transitive system. Let there exist for every object X of $\mathcal{L}$, a limit (colimit, resp.) of the functor $\Phi^X : \mathcal{C} \to \mathcal{L}$ defined by $\Phi^X(a) = F_a(X)$, $\Phi^X(\varphi) = \tau_\varphi^X$. Choose, for every X , $\lim \Phi^X = (\varkappa_a^X : F(X) \to F_a(X))$ ($\mathrm{colim}\, \Phi^X = (\varkappa_a^X : F_a(X) \to F(X))$). Then for every morphism $f: X \to Y$ there is exactly one $F(f) : F(X) \to F(Y)$ such that $F_a(f)\varkappa_a^X = \varkappa_a^Y F(f)$ ($F(f)\varkappa_a^X = \varkappa_a^Y F_a(f)$ in the case of colimit for every a . The correspondence F just described is a functor and $(\varkappa_a : F \to F_a)$ is a limit ($(\varkappa_a : F_a \to F)$ is a colimit) of $(F_a, \tau_\varphi)_{\mathcal{C}}$.

<u>Remark</u>: The statement is formulated for the covariant case. The reformulating for the contravariant case is evident.

<u>Proof</u>: We have $F_a(f)\cdot\varkappa_a^X : F(X) \to F_a(Y)$ and, for every $\varphi : a \to b$,

$$\tau_\varphi^Y \cdot F_a(f) \cdot \varkappa_a^X = F_b(f)\cdot\tau_\varphi^X\cdot\varkappa_a^X = F_b(f)\cdot\varkappa_b^X \ .$$

Thus, there exists exactly one $F(f) : F(X) \to F(Y)$ with $\varkappa_a^Y F(f) = F_a(f)\varkappa_a^X$ for every a . For $f : X \to Y$, $g : Y \to Z$ we obtain $\varkappa_a^Z F(g) F(f) = F_a(g)\varkappa_a^Y F(f) = F_a(g) F_a(f)\varkappa_a^X = F_a(gf)\varkappa_a^X$ for any a , and hence necessarily $F(gf) = F(g) F(f)$. Obviously $F(id) = id$. Evidently $\tau_\varphi \varkappa_a = \varkappa_b$ for any $\varphi : a \to b$ morphism in C .

Now, let $(\vartheta_a : G \to F_a)_{obj\, C}$ be a system of transformations with $\tau_\varphi \cdot \vartheta_a = \vartheta_b$ for any $\varphi : a \to b$. Thus, we have for every object X in $\mathfrak{K}$ $\tau_\varphi^X \vartheta_a^X = \vartheta_b^X$ and hence (as $(\varkappa_a^X : F(X) \to F_a(X))$ is a limit) there is exactly one ϑ^X with $\vartheta_a^X = \tau_a^X \vartheta^X$ for any a .

Now, let $f: X \longrightarrow Y$ be a morphism. We have (for every object a in C)

$$\tau_a^Y F(f) \vartheta^X = F_a(f) \tau_a^X \vartheta^X = F_a(f) \vartheta_a^X = \vartheta_a^Y G(f) = \tau_a^Y \vartheta G(f)$$

and hence $F(f) \vartheta^X = \vartheta^Y G(f)$, so that ϑ is a transformation. Similarly with colimits.

1.4. <u>Notation</u>: If $\tau : F \to G$ is a transformation and H a functor, we denote by τH the transformation $FH \to GH$ defined by $(\tau H)^X = \tau^{H(X)}$, by $H\tau$ the transformation $HF \to HG$ defined by $(H\tau)^X = H(\tau^X)$.

1.5. <u>Theorem</u>: Let $(\varkappa_a : F \to F_a)_{obj\, C}$ be a limit $((\varkappa_a : F_a \to F)_{obj\, C}$ a colimit) of $(F_a, \tau_\varphi)_C$. Then $(\varkappa_a G : FG \to F_a G)$ is a limit $((\varkappa_a G : F_a G \to FG)$ is a colimit) of $(F_a G, \tau_\varphi G)_C$.

<u>Proof</u>: will be done for the limits and covariant functors, the other cases are analogous. First, we have $\varkappa_a G :$ $: FG \to F_a G$, and for every $\varphi : a \to b$,

$(\tau_\varphi G \circ \varkappa_a G)^X = ((\tau_\varphi \varkappa_a) G)^X = (\varkappa_b G)^X$ for every X, i.e. $\tau_\varphi G \circ \varkappa_a G = \varkappa_b G$.

On the other hand let $(\vartheta_a : H \to FG)$ be a system such that $(\tau_\varphi G) \vartheta_a = \vartheta_b$ for every $\varphi : a \to b$. Thus, for a firm X we have $\tau_\varphi^{G(X)} \vartheta_a^X = \vartheta_b^X$ (for any $\varphi :$ $: a \to b$) and hence there exists exactly one $\vartheta^X : H(X) \to$ $\to F(G(X))$ such that $\vartheta_a^X = \varkappa_a^{G(X)} \vartheta^X$ $(= (\varkappa_a G)^X \vartheta^X)$ for every a. Now, let $f: X \to Y$ be a morphism. We have

$$\varkappa_a^{G(Y)} (FG(f) \vartheta^X) = F_a G(f) \varkappa_a^{G(X)} \vartheta^X = F_a G(f) \vartheta_a^X = \vartheta_a^Y H(f) =$$
$$= \varkappa_a^{G(Y)} (\vartheta^Y H(f))$$

for every a and hence $FG(f)\vartheta^{x} = \vartheta^{y}H(f)$. Thus, ϑ is a transformation of H into FG . Evidently, ϑ is uniquely determined.

1.6. Lemma: Let C,D be small categories, $(F_a, \tau_\varphi)_C$, $(G_b, \vartheta_\psi)_D$ transitive systems, $\Phi : C \to D$ a covariant functor. For every object a in C let there be given a transformation $\varkappa_a : F_a \longrightarrow G_{\Phi(a)}$. Let, for every morphism $\varphi : a \to b$ in C , $\vartheta_{\Phi(\varphi)} \varkappa_a = \varkappa_b \tau_\varphi$.

Let $(\tau_a : F_a \to F)$ be a colimit of $(F_a, \tau_\varphi)_C$, $(\vartheta_b : G_b \to G)$ colimit of $(G_b, \vartheta_\psi)_D$. Then there is exactly one transformation $\varkappa : F \to G$ such that

$$\vartheta_{\Phi(a)} \varkappa_a = \varkappa \tau_a \qquad \text{for every } a .$$

If Φ is the identical functor of C and every $\varkappa_a$ a natural equivalence, then $\varkappa$ is a natural equivalence.

Proof: follows immediately by the fact that for every $\varphi : a \longrightarrow b$

$$(\vartheta_{\Phi(b)} \varkappa_b) \tau_\varphi = \vartheta_{\Phi(b)} \vartheta_{\Phi(\varphi)} \varkappa_a = \vartheta_{\Phi(a)} \varkappa_a .$$

§ 2. Limits of special systems of set functors. Transfinite powers of covariant functors.

2.1. Since the category of sets and all their mappings is both complete and cocomplete, it follows by Theorem 1.3 that any transitive system of set functors has both limit and colimit. In the present section, we shall deal with two particular cases of the category C .

First let all the morphisms of C be identities. For every object a of C let there be given set functors F_a . For any set X is the cartesian product $\underset{a \in C}{\times} F_a(X)$ to-

gether with the projection $p_b^X : \underset{C}{\times} F_a(X) \longrightarrow F_b(X)$, defined by $p_b^X((x_a)_{a \in C}) = x_b$ the limit of $(F_a(X))_{a \in C}$. If f: X ⟶ Y is a mapping, we have $p_b^Y(\times F_a(f)) = F_b(f)\, p_b^X$ for any b (where $\underset{C}{\times} F_a(f)$ is defined by $(\times F_a(f))((x_a)_{a \in C}) = (F_a(f)(x_a))_{a \in C}$. Thus, by 1.3, the just described functor $\underset{C}{\times} F_a$ is, together with the evident transformation p_a, a limit(product) of the system $(F_a)_C$.
Similarly we see easily that $\underset{C}{\vee} F_a$ defined by

$$(\underset{C}{\vee})(X) = \cup\{F_a(X) \times (a) \mid a \in C\},\ (\underset{C}{\vee} F_a)(f)(x, b) = (F_b(f)(x), b)$$

is a colimit (coproduct, join) of the system $(F_a)_C$.

<u>Remark</u>: If C consists of two objects, we see that the product of $(F_i)_{i \in 2}$ is $F_0 \times F_1$ and join is $F_0 \vee \vee F_1$ from [2].

2.2. Now, let C be a directed set, i.e. a small category such that, for any two objects, the set $M(a, b) \cup \cup M(b, a)$ consists of at most one element, and that for any two objects a,b there is an object c with $M(a, c) \neq \emptyset \neq M(b, c)$. Write a ≤ b provided $M(a, b) \neq \emptyset$. The relation ≤ determines the category C. Now, let $(F_a, \tau_\varphi)_C$ be a transitive system. Since the morphisms $\varphi : a \to b$ in C are determined by the objects a,b, we write $\tau_\varphi = \tau_{ab}$.
For a given X, $(\tau_a^X : F_a(X) \longrightarrow F(X))_{a \in obj\, C}$, defined by $F(X) = \underset{C}{\vee} F_a(X)/_E$, $(x, y) \in E \Longleftrightarrow (x = (x', a)\ \&\ y =$

$$= (y', b)\ \&\ (\exists c \geq a, b,\ \tau_{ac}^X x' = \tau_{bc}^X y')),\ \tau_a^X(x) = = [(x, a)]$$

($[\xi]$ is the equivalence class containing ξ), a colimit of $(F_a(X), \tau^X_{ab})_C$.

By 1.3, we may define, for $f: X \to Y$, a mapping $F(f):$ $: F(X) \to F(Y)$ by $F(f)[x] = [F_a(f)(x')]$ (where $x = (x', a)$), i.e. by

$$F(f)\tau^X_a = \tau^Y_a F_a(f)$$

and we obtain a functor F such that $(\tau_a : F_a \to F)_{a \in obj\, C}$ is a colimit of $(F_a, \tau_{ab})_C$.

2.3. Lemma: If all the τ_{ab} are monotransformations, then all the τ_a are monotransformations.

Proof: is easy.

2.4. Lemma: Let the assumptions of the first part of lemma 1.6 be satisfied. Let, moreover, C,D be directed sets and let for every $b \in D$ there be an $a \in C$ with $b \leq \Phi(a)$. Let F_a, G_a be set functors. If all the $\varkappa_a$ are monotransformations (epitransformations, natural equivalences), then $\varkappa$ is a monotransformation (epitransformation, natural equivalence).

Proof: We have (see 1.6) $\varkappa^X[x] = [\varkappa^X_a x']$ for $x = (x', a)$. Let all the $\varkappa_a$ be monotransformations. If $\varkappa^X[x] = \varkappa^X[y]$ ($y = (y', b)$), there is a $d \in D$ such that $\Phi(a) \leq d$, $\Phi(b) \leq d$ and

$$\vartheta^X_{\Phi(a),d}\, \varkappa^X_a(x') = \vartheta^X_{\Phi(b),d}\, \varkappa^X_b(y').$$

There is a $c \in C$ with $d \leq \Phi(c)$. We have

$$\vartheta^X_{\Phi(a)\Phi(c)}\, \varkappa^X_a(x') = \vartheta^X_{d\Phi(c)}\, \vartheta^X_{\Phi(a)d}\, \varkappa^X_a(x') = \vartheta^X_{d\Phi(c)}\, \vartheta^X_{\Phi(b)d}\, \varkappa^X_b(y') =$$
$$= \vartheta^X_{\Phi(b)\Phi(c)}\, \varkappa^X_b(y')$$

and consequently $\varkappa^X_c \tau^X_{ac}(x') = \varkappa^X_c \tau^X_{bc}(y')$, and hence,

$\tau^X_{ac}(x') = \tau^X_{bc}(y')$. Thus, $[x] = [y]$.

Let all the $\varkappa_a$ be epitransformations. Let $[y] \in G(X)$, $y = (y', b)$. Let $\Phi(a) \geq b$. $\varkappa^X_a : F_a(X) \rightarrow G_{\Phi(a)}(X)$ is a mapping onto and hence $\vartheta_{b\Phi(a)}(y') = \varkappa^X_a(x)$ for some $x \in F_a(X)$. We have $\varkappa^X[(x, a)] = [\vartheta_{b\Phi(a)}(y')] = [y]$.

2.5. __Definition__: Let $\tau : I \rightarrow F$ be a monotransformation. The functors $(F, \tau)^\alpha$ for ordinals α are defined inductively as follows:

1) $(F, \tau)^0 = I$, τ_{00} is the identical transformation of I.

2) If $(F, \tau)^\beta$, $\tau_{\beta\gamma}$ are defined for $\beta \leq \gamma < \alpha$, we define $(F, \tau)^\alpha = F \circ (F, \tau)^\beta$, $\tau_{\alpha\alpha}$ identical, $\tau_{\gamma\alpha} = \tau(F, \tau)^\beta \cdot \tau_{\gamma\beta}$ provided $\alpha = \beta + 1$, $(\tau_{\beta\alpha} : (F, \tau)^\beta \rightarrow (F, \tau)^\alpha)_{\beta < \alpha} = colim((F, \tau)^\beta, \tau_{\beta\gamma})_{\beta \leq \gamma < \alpha}$, $\tau_{\alpha\alpha}$ identical, provided α is a limit ordinal.

In the following we write often concisely F^α instead of $(F, \tau)^\alpha$.

2.6. __Theorem__: $(F, \tau)^\beta \cdot (F, \tau)^\alpha \cong (F, \tau)^{\alpha+\beta}$.

__Proof__: will be done by induction by β. For $\beta = 0$ we have $F^0 \cdot F^\alpha = I \cdot F^\alpha = F^{\alpha+0}$. Define $\varkappa_0$ as the identical transformation of F^α. Let there be found, for $\iota < \beta$, natural equivalences $\varkappa_\iota : F^\iota \circ F^\alpha \rightarrow F^{\alpha+\iota}$ such that, for $\iota < \iota' < \beta$,

$$\tau_{\alpha+\iota, \alpha+\iota'} \circ \varkappa_\iota = \varkappa_{\iota'} \circ (\tau_{\iota\iota'} F^\alpha).$$

First, let $\beta = \gamma + 1$. We have $F^\beta = F \circ F^\gamma$ and $F^{\alpha+\beta} = F \circ F^{\alpha+\gamma}$. Put $\varkappa_\beta = F(\varkappa_\gamma)$. We have

$\tau_{\alpha+\iota, \alpha+\beta} = \tau F^{\alpha+\gamma} \circ \tau_{\alpha+\iota, \alpha+\gamma}$, $\tau_{\iota\beta} = \tau F^\gamma \circ \tau_{\iota\gamma}$.

Since always $\varkappa_\beta = F(\varkappa_\gamma)$, we have $\varkappa_\beta^X \tau^{F^\gamma F^\alpha(X)} =$
$= \tau^{F^{\alpha+\gamma}(X)} \cdot \varkappa_\gamma^X$, and hence $\varkappa_\beta \circ (\tau F^\gamma F^\alpha) = (\tau F^{\alpha+\gamma}) \cdot \varkappa_\gamma$.
We have

$$\varkappa_\beta(\tau_{\iota\beta} F^\alpha) = \varkappa_\beta((\tau F^\gamma) F^\alpha) \cdot \tau_{\iota\gamma} F^\alpha = (\tau F^{\alpha+\gamma}) \varkappa_\gamma (\tau_{\iota\gamma} F^\alpha) =$$
$$= \tau F^{\alpha+\gamma} \cdot \tau_{\alpha+\iota,\alpha+\gamma} \cdot \varkappa_\iota = \tau_{\alpha+\iota,\alpha+\beta} \circ \varkappa_\iota .$$

If β is a limit ordinal, define $\varkappa_\beta$ as the transformation $\varkappa$ from 1.6. We have $(\tau_{\iota\beta} : F^\iota \to F^\beta)_{\iota<\beta} =$
$= colim\ (F^\iota, \tau_{\iota\iota'})_{\iota \le \iota' < \beta}$, and hence, by Theorem 1.5, $(\tau_{\iota\beta} F^\alpha : F^\iota F^\alpha \to F^\beta F^\alpha) = colim (F^\iota F^\alpha, \tau_{\iota\iota'} F^\alpha)$. We have further $(\tau_{\iota,\alpha+\beta} : F^\iota \to F^{\alpha+\beta}) = colim\ (F^\iota, \tau_{\iota\iota'})_{\iota \le \iota' < \alpha+\beta}$.
Thus, since we have for $\iota \le \iota' < \beta$

$$\tau_{\alpha+\iota,\alpha+\iota'} \circ \varkappa_\iota = \varkappa_{\iota'} \cdot \tau_{\iota\iota'} F^\alpha ,$$

we obtain, by 1.6, $\tau_{\alpha+\iota,\alpha+\beta} \circ \varkappa_\iota = \varkappa_\beta \cdot \tau_{\iota\beta} F_\alpha$.
By lemma 2.4, $\varkappa_\beta$ is a natural equivalence.

2.7. Remark: In [2], a superfluous notion of "nice functor" was defined. Every set functor is nice, since for every one-to-one mapping $f: X \to Y$ (for every mapping f of X onto Y, resp.) there exists a mapping $g: Y \to X$ with $gf = id_X$ ($fg = id_Y$, resp.). We shall use the preserving of monomorphisms and epimorphisms by set functors without further mentioning.

2.8. Lemma: Let $\tau : I \to F$, $\vartheta : I \to G$ be monotransformations, $\varkappa : F \to G$ a mono-(epi-)transformation with $\varkappa\tau = \vartheta$. Then there exist mono-(epi-) transformations $\varkappa_\alpha : (F, \tau)^\alpha \to (G, \tau)^\alpha$ such that $\varkappa_0$ is the identity transformation of I, $\varkappa_1 = \varkappa$, and we have

$$\varkappa_\alpha \circ \tau_{\beta\alpha} = \vartheta_{\beta\alpha} \circ \varkappa_\beta$$

whenever $\alpha \leq \beta$.

<u>Proof</u>: Let $\varkappa_\beta$ be found for any $\beta < \alpha$. Let $\alpha = \beta + 1$. Put $\varkappa_\alpha = \varkappa G^\beta \circ F\varkappa_\beta$. We have

$$\varkappa_\alpha \circ \tau_{\gamma\alpha} = \varkappa G^\beta \circ F\varkappa_\beta \circ \tau F^\beta \circ \tau_{\gamma\beta} = \varkappa G^\beta \circ \tau G^\beta \circ \varkappa_\beta \circ \tau_{\gamma\beta} =$$

$$= (\varkappa\tau) G^\beta \circ \vartheta_{\gamma\beta} \circ \varkappa_\gamma = \vartheta G^\beta \circ \vartheta_{\gamma\beta} \circ \varkappa_\gamma = \vartheta_{\gamma\alpha} \circ \varkappa_\gamma .$$

For limit ordinals α immediately by 2.4.

2.9. <u>Remarks</u>: Let F be a covariant set functor. Then the transformations $\tau : I \to F$ are in a one-to-one correspondence with the elements of F(1) $(1 = \{\emptyset\})$. In fact, define, for $a \in F(1)$, $T(a) : I \to F$ by $T(a)^X(x) = F(\xi_x^X)(a)$, where $\xi_x^X : 1 \to X$, $\xi_x^X(\emptyset) = x$.

On the other hand, define, for $\tau : I \to F$, $A(\tau) = \tau^1(\emptyset)$. We have $AT(a) = T(a)^1(\emptyset) = F(\xi_\emptyset^1)(a) = F(id)(a) = a$, $(TA(\tau))^X(x) = F(\xi_x^X)(A\tau) = F(\xi_x^X)\tau^1(\emptyset) = \tau^X \xi_x^X(\emptyset) = \tau^X(x)$.

We see easily, that the monotransformations are exactly that T(a) with $F(\xi_\emptyset^2)(a) \neq F(\xi_1^2)(a)$. Further, we see easily that the following three statements about set functors are equivalent: I. F is faithfull, II. $F(\xi_\emptyset^2) \neq F(\xi_1^2)$, III. There exists a monotransformation $\mu : I \to F$.

Since in the following the powers of $(P^-)^2$ play an important role, we shall show, that they are (up to natural equivalence) independent on the choice of monotransformation. We have $(P^-)^2(1) = P^-(\{\emptyset, 1\}) = \{\emptyset, 1, \{1\}, 2\}$. $T(\emptyset)$ and T(2) are evidently no monotransformation. Define

$\varkappa : (P^-)^2 \to (P^-)^2$ by $\varkappa^X(\mathfrak{M}) = \{N \mid X \setminus N \in \mathfrak{M}\}$;
since $\varkappa \circ \varkappa$ is the identity transformation, $\varkappa$ is a natural equivalence. We have $\varkappa \circ T(1) = T(\{1\})$ and consequently $((P^-)^2, T(1))^\infty$ and $((P^-)^2, T(\{1\}))$ are naturally equivalent by 2.8.

§ 3. T B -functors.

In the following, the term functor means always a set functor.

3.1. Lemma: If $F_a = F$ for every a in C, we have

$$\bigvee_C F_\iota = K_C \circ F, \quad \bigtimes_C F_\iota \cong Q_C \circ F .$$

Proof is trivial.

3.2. Lemma: Let $\mu : F' \to G$ be a monotransformation and $\eta : F' \to F$ and epitransformation. Then there exist a functor H, monotransformation $\nu : F \to H$ and an epitransformation $\varepsilon : G \to H$

Proof: Define an equivalence r(X) on G(X) as follows:

$$a \notin \mu^X(F'(X)) \Rightarrow ((a, b) \in r \iff a = b),$$

$$a = \mu^X(a') \Rightarrow ((a, b) \in r \iff (b = \mu^X(b') \,\&\, \eta^X(a') = \eta^X(b')))$$

Put $H(X) = G(X)/r(X)$. If f: X $\to$ Y is a mapping, a, $b \in G(X)$ and $(a, b) \in r(X)$, we see easily that $(G(f)(a), G(f)(b)) \in r(Y)$.
Thus, we may define $H(f) : H(X) \to H(Y)$ by $H(f)[a] = [G(f)(a)]$ (the square brackets designate the equivalence class containing a given element). Evidently, H is a

functor. Define $\varepsilon^X : G(X) \to H(X)$ by $\varepsilon^X(x) = [x]$. We have $H(f)\,\varepsilon^X(x) = H(f)[x] = [G(f)(x)] = \varepsilon^Y G(f)(x)$. Thus, ε is a transformation of G into H , evidently an epitransformation.

Now, define $\nu : F \to H$ as follows. For $x \in F(X)$ take an $a \in F'(X)$ with $\eta^X(a) = x$ and put $\nu^X(x) = [\mu^X(a)]$ (if $\eta^X(a) = x = \eta^X(b)$, we have $(\mu^X(a), \mu^X(b)) \in \varkappa$). Let f: X → Y be a mapping. We have $H(f)\,\nu^X(x) = H(f)[\mu^X(a)] = [G(f)\mu^X(a)] = [\mu^Y F'(f)(a)]$ for some a such that $\eta^X(a) = x$. Consequently,

$$[\mu^Y F'(f)(a)] = \nu^Y \eta^Y F'(f)(a) = \nu^Y F(f)\eta^X(a) = \nu^Y F(f)(x).$$

If $\nu^X(x) = \nu^X(y)$, we have $x = \eta^X(a)$, $y = \eta^X(b)$ and $[\mu^X(a)] = [\mu^X(b)]$. Since μ is one-to-one, we must have $\eta^X(a) = \eta^X(b)$, i.e. x = y .

The proof for contravariant functors is quite analogous.

3.3. <u>Theorem</u>: F < G (see [2]) if and only if there is a functor H , a monotransformation $\mu : F \to H$ and an epitransformation $\eta : G \to H$.

<u>Proof</u>: Let F < G . Then there exists a sequence of functors $F_0, F_1, \dots, F_{n+1}$, monotransformations $\mu_i : F_{2i} \to F_{2i+1}$ $(i \geq 0)$ and epitransformations $\eta_i : F_{2i} \to F_{2i-1}$ $(i \geq 0)$ (epitransformations $\eta_i : F_{2i+1} \to F_{2i}$ and monotransformations $\mu_i : F_{2i+1} \to F_{2i+2}$, respectively) such that $F_0 = F$, $F_{n+1} = G$. Let n be the least natural number with this property. n ≥ 2 leads easily to a contradiction with 3.2. If n = 1 , we may in the first case put $H = F_1$, in the second case we use, again, lemma 3.2. If n = 0 , we may repeat one of the functors and consider its identical transformation.

The reverse implication is trivial.

3.4. Lemma: Let $\alpha_a : F_a \to G_a \ (a \in C)$ be mono-(epi-)transformations. Then $\alpha : {}_C\bigvee F_a \to {}_C\bigvee G_a$ and $\beta :$ $: {}_C\bigtimes F_a \to {}_C\bigtimes G_a$ defined by

$$\alpha^X(x, a) = (\alpha_a^X(x), a), \quad \beta^X((x_a)_{a\in C}) = (\beta_a^X(x_a))_{a\in C}$$

are mono-(epi-)transformations.

Proof is trivial.

3.5. Theorem: Let $F_a < G_a$ for all $a \in C$. Then

$$_C\bigvee F_a < {}_C\bigvee G_a, \quad {}_C\bigtimes F_a < {}_C\bigtimes G_a .$$

Proof follows immediately by Theorem 3.3 and Lemma 3.4.

3.6. Theorem: Let C be a small category, $(F_a, \tau_\varphi)_C$ a transitive system. Let $(\tau_a : F \to F_a)_{a \in obj\, C}$ be a limit, $(\vartheta_a : F_a \to F')_{a\in obj\, C}$ a colimit of $(F_a, \tau_\varphi)_C$. Then there exists a monotransformation $\mu : F \to {}_{obj\, C}\bigtimes F_a$ and an epitransformation $\eta : {}_{obj\, C}\bigvee F_a \to F'$.

Proof follows by 1.3, proof of Freyd's theorem ([3], Chapter II, Th.2.4, p.45) and its dualisation.

3.7. Lemma: Let $\tau : I \to F$ be a monotransformation, α, β ordinals, $\alpha \le \beta$. Then $(F, \tau)^\alpha \le (F, \tau)^\beta$.

Proof: By lemma 2.3, all the $\tau_{\alpha\beta}$ (see 2.5) are monotransformations.

3.8. Lemma: Let a monotransformation $\tau : I \to F$ have the property that whenever for $f: Z \to X$ and $B \subset X$ holds $f(Z) \cap B = \emptyset$, then $F(f)(F(Z)) \cap \tau^X(B) = \emptyset$. Then

$$V_A \circ (F, \tau)^\alpha < (F, \tau)^\alpha \circ V_A$$

for any set A and any ordinal α.

Proof: First, we prove that the described property of

τ remains valid for every $\tau_{o\alpha} : I \to (F, \tau)^{\alpha}$. For $\alpha = 1$ we have $\tau_{o1} = \tau$. Let all the $\tau_{o\beta}$ with $\beta < \alpha$ have the property. If $\alpha = \beta + 1$, we have

$$F^{\alpha}(f)(F^{\alpha}(Z)) \cap \tau^{X}_{o\alpha}(B) = F(F^{\beta}(f))(F(F^{\beta}(Z)) \cap \tau^{F^{\beta}(X)}(\tau_{o\beta}(B)) = \emptyset .$$

In fact, put $f' = F^{\beta}(f)$, $Z' = F^{\beta}(Z)$, $X' = F^{\beta}(X)$, $B' = \tau_{o\beta}(B)$; we have $f'(Z') \cap B' = \emptyset$ by the inductive hypothesis. Now, let α be a limit ordinal, $x \in F^{\alpha}(f)(F^{\alpha}(Z)) \cap \tau^{X}_{o\alpha}(B)$. Thus, $x = F^{\alpha}(f)(z)$ for some $z \in F^{\alpha}(Z)$ and $z = \tau^{Z}_{\beta\alpha} y$ for some $\beta < \alpha$. We obtain $x = F^{\alpha}(f)\, \tau^{Z}_{\beta\alpha}(y) = \tau^{X}_{\beta\alpha} F_{\beta}(f)(y)$.

On the other hand, $x = \tau^{X}_{o\alpha}(b) = \tau^{X}_{\beta\alpha}\, \tau^{X}_{o\beta}(b)$. Since $\tau^{X}_{\beta\alpha}$ is one-to-one, we obtain $F_{\beta}(f)(y) = \tau^{X}_{o\beta}(b)$, which is a contradiction.

Now, it suffices to prove that, if there is a monotransformation $\tau : I \to F$ with the described property, then there exists a monotransformation $\mu : V_A \circ F \to F \circ V_A$. Define $\mu^{X} : F(X) \vee A \to F(X \vee A)$ by

$$\mu^{X}(x, 0) = F(j_X)(x), \quad \mu^{X}(a, 1) = \tau^{X \vee A}(a, 1),$$

where j_X is the natural embedding of X into X $\vee$ A . Both $\mu^{X} \mid F(X) \times (0)$ and $\mu^{X} \mid A \times (1)$ are one-to-one. Thus, since $j_X(X) \cap (A \times (1) = \emptyset$ and consequently $F(j_X)(F(X)) \cap \tau^{X \vee A}(A) = \emptyset$, μ is one-to-one. We see easily that $(F \circ V_A)(f) \circ \mu^{X} = \mu^{Y} \circ (V_A \circ F)(f)$ for any f: X $\to$ Y .

3.9. <u>Theorem</u>: For any set A and any ordinal α holds

$$V_A \circ ((P^-)^2)^{\alpha} < ((P^-)^2)^{\alpha} \circ V_A .$$

<u>Proof</u>: Define $\tau : I \to (P^-)^2$ by $\tau^{X}(x) = \{M \mid x \in M \subset X\}$. Let f: Z $\to$ X , $a \in (P^-)^2(f)((P^-)^2(Z)) \cap \tau^{X}(B)$,

i.e. $\mathfrak{a} = (P^-)^2(f)(\mathfrak{m})$, $\mathfrak{m} \in (P^-)^2(Z)$, $\mathfrak{a} = \{N \mid b \in N \subset X\}$, $b \in B$. Hence, $\mathfrak{a} = \{N \mid N \subset X,\ f^{-1}(N) \in \mathfrak{m}\}$.
Thus, $b \in N$ if and only if $f^{-1}(N) \in \mathfrak{m}$. If $B \cap f(Z) = \emptyset$, we have $f^{-1}(N) = f^{-1}(N - B)$ for any $N \subset Z$ and hence $b \in (b) - B$, which is a contradiction.

3.10. __Theorem__: For every ordinal α there is an ordi- α' and a set A such that

$$((P^-)^2)^{\alpha} \circ P^- < P^- \circ ((P^-)^2)^{\alpha'} \circ V_A .$$

__Proof__ will be done by induction. For $\alpha = 1$ it suffices to put $\alpha' = 1$, $A = \emptyset$. Let the statement hold for all $\beta < \alpha$. If $\alpha = \beta + 1$, we have

$$((P^-)^2)^{\alpha} \circ P^- = (P^-)^2 \circ ((P^-)^2)^{\beta} \circ P^- <$$
$$< (P^-)^2 \circ P^- \circ ((P^-)^2)^{\beta'} \circ V_A = P^- \circ ((P^-)^2)^{\beta'+1} \circ V_A .$$

Let α be a limit ordinal. Put $\alpha'' = \sup_{\beta<\alpha} \beta'$, $A' = \bigcup_{\beta<\alpha} A(\beta)$ (where $A(\beta)$ is such that $((P^-)^2)^{\beta} \circ P^- < ((P^-)^2)^{\beta'} \circ V_{A(\beta)}$ holds). By lemma 3.7 and by [2] (3.7, 4.3) we have $((P^-)^2)^{\beta} \circ P^- < P^- \circ ((P^-)^2)^{\alpha''} \circ V_{A'}$ __for every__ $\beta < \alpha$. Thus, by 1.5, 3.1, 3.5, 3.6, 3.7, 3.9 and by [2] (4.5 – 4.8) $((P^-)^2)^{\alpha} \circ P^- < \bigvee_{\alpha} ((P^-)^2)^{\beta} \circ P^- < K_{\alpha} \circ P^- \circ ((P^-)^2)^{\alpha} \circ V_{A'} < (P^-)^2 \circ V_{\alpha} \circ P^- \circ ((P^-)^2)^{\alpha''} \circ V_{A'} < (P^-)^2 \circ P^- \circ ((P^-)^2)^{\alpha''} \circ V_{\alpha \vee A'} = P^- \circ ((P^-)^2)^{\alpha'} \circ V_A$,

where $\alpha' = \alpha'' + 1$, $A = \alpha \vee A'$.

3.11. __Definition__: Transfinitely constructive functors (concisely TC-functors) are defined recursively as follows:

(a) I, V_A, K_A, Q_A, P_A, P^+ are TC-functors (for any set A)

(b) If F, G are TC-functors, $F \circ G$ is a TC-functor.

(c) If $(F_\alpha, \tau_\varphi)_C$ is a transitive system over a small category C, F_a TC-functors, $(\tau_a : F \to F_a)$ a limit (resp. $(\tau_a : F_a \to F)$ a colimit) of $(F_a, \tau_\varphi)_C$ then F is a TC-functor.

(d) If F,G are TC-functors, one of them covariant and the other contravariant, then $\langle F, G \rangle$ is a TC-functor.

A functor F is said to be transitively bounded (TB-functor) if there exists a TC-functor G with $F < G$.

3.12. <u>Theorem</u>: For every TB-functor F there is an ordinal α and a set A such that

$$F < (P^-)^i \circ ((P^-)^2)^\alpha \circ V_A$$

(i = 0 for covariant, i = 1 for contravariant F).

<u>Proof</u>: It suffices to prove the statement for TC-functors. It holds for $I, V_A, \ldots, P^+$ by [2] (§ 5). Let $F < (P^-)^i \circ ((P^-)^2)^\alpha \circ V_A$, $G < (P^-)^j \circ ((P^-)^2)^\beta \circ V_A$. Thus, $F \circ G <$

$$< (P^-)^i \circ ((P^-)^2)^\alpha \circ V_A \circ (P^-)^j \circ ((P^-)^2)^\beta \circ V_B < (P^-)^i \circ ((P^-)^2)^\alpha \circ (P^-)^j \circ ((P^-)^2)^\beta \circ V_{A \vee B}$$

by [2] (§ 4; the results of this paragraph shall be used in the following without further mentioning) and 3.9. If j = = 0, we obtain, by 2.6, $F \circ G < (P^-)^i \circ ((P^-)^2)^{\alpha+\beta} \circ V_{A \cup B}$, if $j \neq 0$, we obtain by 2.6, 3.9 and 3.10,

$$F \circ G < (P^-)^{i+j} \circ ((P^-)^2)^{\alpha'} \circ V_C \circ ((P^-)^2)^\beta \circ V_{A \vee B} < (P^-)^{i+j} \circ ((P^-)^2)^{\beta+\alpha'} \circ V_{A \vee B \vee C} .$$

Let $(\tau_a : F \to F_a)$ be a limit $(\tau_a : F_a \to F)$ a colimit of a transitive system $(F_a, \tau_\varphi)_C$. Then we have $F_a < (P^-)^i \circ ((P^-)^2)^{\alpha_a} \circ V_{A_a} < (P^-)^i \circ ((P^-)^2)^\alpha \circ V_a$,

where $\alpha = \sup \alpha_a$, $A = \bigcup A_a$. By 2.6, 3.4, 3.6, 3.9 we obtain, for a limit, $F < \underset{obj\,C}{\times} F_a < Q_{obj\,C} \cdot (P^-)^i \circ ((P^-)^2)^\alpha \circ V_A <$

$< (P^-)^2 \circ (P^-)^i \circ ((P^-)^2)^{\alpha} \circ V_{obj\, C \vee A} \cong (P^-)^i \circ ((P^-)^2)^{\alpha+2} \circ V_{obj\, C \vee A}$.

Similarly for a colimit.

Now, let $F < (P^-)^i \circ ((P^-)^2)^{\alpha} \circ V_A$, $G < (P^-)^j \circ ((P^-)^2)^{\beta} \circ V_B$, $i \neq j$. Put $\gamma = \max(\alpha, \beta)$, $C = A \cup B$. We obtain $F < (P^-)^i \circ ((P^-)^2)^{\gamma} \circ V_C$, $G < P^- \circ (P^-)^i \circ ((P^-)^2)^{\gamma} \circ V_C$ and hence $\langle F, G \rangle < P^- \circ Q_2 \circ (P^-)^i \circ ((P^-)^2)^{\gamma} \circ V_C < (P^-)^k \circ ((P^-)^2)^{\sigma} \circ V_D$

by [2] (Theorem 5.6) and 3.9.

3.13. __Metatheorem__: The system of all TB-functors is closed upon compositions, forming of limits and colimits (over small categories), the operation $\langle -, - \rangle$, subfunctors and factor-functors.

__Proof__: This follows by definition of majorisation abd by the proof of 3.12.

§ 4. __Realisation of categories determined by TB-functors and their boundability__.

4.1. __Lemma__: Let J be a set. Let F_ι $(\iota \in J)$ be covariant set functors, Δ_ι $(\iota \in J)$ types. Then there exist sets A_ι , B_ι $(\iota \in J)$ such that

$$S((F_\iota, \Delta_\iota)_{\iota \in J}) \rightrightarrows S((K_{A_\iota} \circ Q_{B_\iota} \circ F_\iota)_{\iota \in J}) .$$

__Lemma__: If F_ι $(\iota \in J)$ are covariant set functors, we have

$$S((F_\iota)_{\iota \in J}) \rightrightarrows S(\bigvee_J F_\iota) .$$

__Proofs__ of this lemmas are done in [2] (6.2 and 6.3). The finiteness of J from the formulation in [2] plays no role.

4.2. <u>Theorem</u>: Let F_ι $(\iota \in J)$ be TB-functors, Δ_ι $(\iota \in J)$ types. Then there exists an ordinal α and a set A such that

$$S((F_\iota, \Delta_\iota)_{\iota \in J}) \rightrightarrows S(((P^-)^2)^\alpha \circ V_A).$$

<u>Proof</u>: By [2] (Theorem 1.5)

$$S((F_\iota, \Delta_\iota)_{\iota \in J}) \rightrightarrows S((G_\iota, \Delta_\iota)_{\iota \in J})$$

where G_ι are covariant TB-functors. Now, we obtain the statement by 4.1 and 3.12 and [2] (6.1).

4.3. <u>Theorem</u>: Let C be an ordered set, $(F_a, \tau_{ab})_C$ a transitive system of covariant functors. Let all the F_a be selective (see [1]) and faithful, let τ_{ab} be mono-transformations. Let $(\tau_a : F_a \to F)_{obj\, C}$ be a colimit of $(F_a, \tau_{ab})_C$. Then F is selective.

<u>Proof</u>: Let Δ be a type. Since F_a are selective, there exist full embeddings $\Phi_a : S(I, \Delta) \longrightarrow S(I, \Delta^a)$

$(\Delta^a = \{\alpha^a_\beta \mid \beta < \gamma^a\})$ such that $\square \cdot \Phi_a = F_a \circ \square$ ($\square$ designates the natural forgetful functors). We shall construct a $\Phi : S(I, \Delta) \longrightarrow S(I, \bar{\Delta})$ such that $\square \cdot \Phi = F \circ \square$. Put $\bar{\Delta} = \{\alpha^a_\beta, \alpha_a \mid a \in obj\, C, \beta < \gamma^a\}$, where α^a_β was defined above, $\alpha_a = 1$ for every a. We shall not index Δ directly by ordinals to simplify the notation.

It is evident how the notation should be changed to obtain the type in an ordinary form.

Let (X,r) be an object of $S(I, \Delta)$. We have $\Phi_a(X, \varkappa) = (F_a(X), \varkappa^a)$, where $\varkappa^a = (\varkappa^a_\beta)$ is a relational system of the type Δ^a. We define a relational system

$\bar{\kappa} = (\bar{\kappa}^a_\beta, \bar{\kappa}_a)_{a \in obj\, C,\, \beta < \gamma^a}$ on $F(X)$ as follows:

$$\varphi \in \bar{\kappa}^a_\beta \Longleftrightarrow \exists \psi \in \kappa^a_\beta, \quad \varphi = \tau^X_a \circ \psi \quad ,$$

$$\bar{\kappa}_a = \tau^X_a (F_a(X)) .$$

Let $f: X \longrightarrow Y$ be an r s-compatible mapping. If $\varphi \in \bar{\kappa}^a_\beta$, we have $\varphi = \tau^X_a \circ \psi$ for some $\psi \in \kappa^a_\beta$. By the definition of $F(f)$ we have $F(f)\tau^X_a \psi = \tau^Y_a F_a(f)\psi \in \bar{s}$, since $F_a(f)$ is $r^a s^a$-compatible. If $\xi \in \bar{\kappa}_a$, we have $\xi = \tau^X_a(\eta)$ for some $\eta \in F_a(X)$. Thus, $F(f)(\xi) = F(f)\tau^X_a(\eta) = \tau^Y_a(F_a(f)(\eta)) \in \bar{s}_a$

Thus, $F(f): F(X) \longrightarrow F(Y)$ is $\bar{r}\,\bar{s}$-compatible. We may define $\Phi : S(I, \Delta) \longrightarrow S(I, \bar{\Delta})$ by $\Phi(X, \kappa) = (F(X), \bar{\kappa})$, $\Box \cdot \Phi(f) = F \circ \Box(f)$. Φ is evidently a one-to-one functor and it remains to be proved that it maps $S(I, \Delta)$ onto a full subcategory of $S(I, \bar{\Delta})$.

Let $g: F(X) \longrightarrow F(Y)$ be an $\bar{r}\,\bar{s}$-compatible mapping. Hence, first, it is $\bar{r}_a \bar{s}_a$-compatible, i.e. $g(\tau^X_a F_a(X)) \subset \subset \tau^Y_a F_a(Y)$ Consequently, by 2.3, for every $x \in F_a(X)$ there is exactly one $g_a(x) \in F_a(Y)$ with $g\tau^X_a(x) = = \tau^Y_a g_a(x)$. Thus, we obtained for every $a \in obj\, C$ a mapping $g_a : F_a(X) \longrightarrow F_a(Y)$. Let $\psi \in \kappa^a_\beta$. We have $\tau^X_a \psi \in \bar{\kappa}^a_\beta$ and hence $\tau^Y_a g_a \psi = g \tau^X_a \psi \in \bar{s}^a_\beta$ and hence there is a $\varphi \in s^a_\beta$ with $\tau^Y_a g_a \psi = \tau^Y_a \varphi$ Since τ^Y_a are (see 2.3) one-to-one, we have $g_a \psi = = \varphi \in s^a_\beta$. Thus, g_a is $r^a s^a$-compatible and hence there exists an r s-compatible $f_a: X \longrightarrow Y$ with $g_a = F_a(f_a)$.

Let a,b be objects from C . There is a c such that $a \leq \leq c$, $b \leq c$. We have $\tau^Y_a F_a(f_a) = \tau^Y_a \tau^Y_{ac} F_a(f_c) = \tau^Y_c F_c(f_c)\tau^X_{ac} =$

$= \tau_a^Y g_c \tau_{ac}^X = g \tau_c^X \tau_{ac}^X = g \tau_a^X = \tau_a^Y g_a = \tau_a^Y F_a(f_a)$.

Since τ_a^Y is one-to-one, we obtain $F_a(f_c) = F_a(f_a)$. Since F_a is faithful, $f_c = f_a$. Similarly, $f_b = f_c$. Thus, all the mappings f_a are equal to a unique mapping f and we obtain $F(f)\tau_a^X = \tau_a^Y F_a(f) = \tau_a^Y g_a = g \tau_a^X$ for every a and hence $F(f) = g$.

4.4. <u>Theorem</u>: Under the assumption (M) (see [1]) about the set theory, in particular, if there are no measurable cardinals, the following statement holds: Let $\mathfrak{K}$ be a category which may be fully embedded into some $S((F_\iota, \Delta_\iota)_{\iota \in J})$ with TB-functors F_ι . Then $\mathfrak{K}$ is boundable.

<u>Proof</u>: By 4.2, $\mathfrak{K} \overset{=}{\longrightarrow} S((P^-)^2 \cdot V_A)$ for some ordinal α and some set A . By [1] (theorems 2 and 8) and by 4.3, $((P^-)^2)^\alpha \cdot V_A$ is selective (under the assumption (M)). Thus, the statement follows by [2] (4.2) and [1] (Theorems 2 and 6).

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(Received July 21, 1967)