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Commentationes Mathematicae Universitatis Carolinae, Vol. 8 (1967), No. 4, 581--633

Persistent URL: http://dml.cz/dmlcz/105137

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8,4 (1967)

COMPLETIONS OF SMALL SUBCATEGORIES

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Completions of categories and related questions are studied in many papers, namely [1],[8],[10],[13],[14]. Some questions concerning these problems are considered also in the present paper where a few theorems on the existence of a completion with given properties of a small subcategory of a given category are proved.

In the first part of the present paper the categories $\mathbb{M}^+$ and $\mathbb{M}^-$ of all small subcategories of a given category M and all their M^+-functors or M^--functors respectively are defined and considered. It is shown that the category $\mathbb{S}^+$ is not complete, S being the category of all sets and all their mappings; even a directed presheaf need not have a direct limit in $\mathbb{S}^+$. A definition of an $\mathbb{M}^+$-limit of a directed presheaf in $\mathbb{M}^+$ is given so that if every directed presheaf in M has a direct limit, then every directed presheaf in $\mathbb{M}^+$ has an $\mathbb{M}^+$-limit.

In the second part of the present paper some lemmas needed for the constructions of completions are proved. In the third part the following results are proved: Let M be a replete complete category, k its arbitrary small subcategory. Then there exist complete subcategories K_1, K_2, K_3 of M such that k is a full subcategory of K_1, K_2, K_3 and

1) the inclusion functor $I_1 : k \to K_1$ preserves all (direct and inverse) limits (already existing in k);

2) the inclusion functor $\bar{I}_2 : K_2 \to M$ preserves all limits;

3) the inclusion functor $I_3 : k \to K_3$ preserves all direct and the inclusion functor $\bar{I}_3 : K_3 \to M$ all inverse limits.

In the fourth part we define the notion of $M^+(\mathbb{G}, V)$-completion of a subcategory k of a given category M , where $\mathbb{G}$ is a class of diagrams in k , V is a class of diagram schemas. $M^+(\mathbb{G}, V)$ -completion of k is , roughly speaking, the "smallest" subcategory K of M such that every diagram in K with the schema from V has a direct limit in K , k is a full subcategory of K and the inclusion functor $I : k \to K$ preserves direct limit (whenever exists) of every diagram from $\mathbb{G}$ (for exact definition see IV.1). Under some assumptions about the category M , these being satisfied by many familiar categories, the existence of $M^+(\mathbb{G}, V)$ -completion of every small subcategory k of M is proved whenever either $\mathbb{G}$ is an arbitrary class of collections in k and V is an arbitrary class of discrete diagram schemas or $\mathbb{G}$ is an arbitrary set of diagrams in k and V is an arbitrary class of diagram schemas. Dually the notion of $M^-(\mathbb{G}, V)$ -completion is defined and considered.

The present paper is written within the Bernays-Gödel set theory, [5], without any further requirement, unless expressly stated (only in IV.6). The axioms of set-theory are consistently respected and the axiom of choice is assumed.

Conventions, notation and known definitions (cf. [2], [3], [4], [7], [9], [10], [11], [12]):

If K is a category, then K^{o} denotes the class of all its objects, and K^{m} is the class of all its morphisms. If a, b $\in K^{o}$, then K(a,b) is the set of all morphisms of K from a to b. For $\alpha \in K(a, b)$ put $\overleftarrow{\alpha} = a$, $\overrightarrow{\alpha} = b$. If $\alpha \in K(a, b)$, $\beta \in K(b, c)$, then the composition of α and β is denoted by $\beta \cdot \alpha$. We shall assume that all functors are covariant unless we explicitly say otherwise. A functor $\Phi : K \rightarrow H$ is called full whenever $\Phi(K)$ is a full subcategory of H; it is called faithful if its restriction to every set $K(a, b)$ is a one-to-one mapping in the set $H(\Phi(a), \Phi(b))$. For $a \in K^{o}$ denote by e_a the identity-morphism of a. A category K such that K^{m} contains identity-morphisms only will be called a discrete category. A category K such that every K(a,b) contains at most one morphism is also considered as a quasi-ordered class: $a \leqq b \Longleftrightarrow K(a, b) \neq \emptyset$. Let J, K be categories, J small, $\mathcal{F} : \mathcal{J} \rightarrow K$ a functor; $\mathcal{F}$ is termed a diagram in K, J is called a diagram schema; if J is a quasi-ordered set, then $\mathcal{F}$ will be called a presheaf and if $\{\alpha\} = \mathcal{J}(a, b)$, then $\mathcal{F}(\alpha)$ will be denoted by ${}^{b}_{a}\mathcal{F}$; if J is a directed set, then $\mathcal{F}$ will be called a directed presheaf; if the category dual to J is a directed set, then $\mathcal{F}$ will be called an inverse presheaf; if J is a discrete category, then $\mathcal{F}$ will be called a collection.

Let $\mathcal{F} : \mathcal{J} \to K$ be a diagram in K. A couple $\alpha = \langle a ; \{ \chi_i ; i \in \mathcal{J}^\sigma \} \rangle$ will be called a <u>direct</u> (or an <u>inverse</u>) <u>bound</u> of $\mathcal{F}$ whenever $a \in K^\sigma$ and $\{\chi_i ; i \in \mathcal{J}^\sigma\}$ is a natural transformation of $\mathcal{F}$ into the constant functor $\mathcal{K} : \mathcal{J} \to K$ (or of $\mathcal{K}$ into $\mathcal{F}$), where $\mathcal{K}(i) = a$ for all $i \in J^\sigma$, i.e. $\chi_i \in K(\mathcal{F}(i), a)$ (or $\chi_i \in K(a, \mathcal{F}(i))$) and if $\sigma \in \mathcal{J}(i, i')$, then $\chi_i = \chi_{i'} \cdot \mathcal{F}(\sigma)$ (or $\mathcal{F}(\sigma) \cdot \chi_i = \chi_{i'}$, respectively); put $a = (\alpha)$; if $\Phi : K \to H$ is a functor, put $\Phi(\alpha) = \langle \Phi(a) ; \{\Phi(\chi_i) ; i \in \mathcal{J}^\sigma\} \rangle$. If $\alpha = \langle a ; \{\chi_i ; i \in \mathcal{J}^\sigma\}\rangle$, $\alpha' = \langle a' ; \{\chi'_i ; i \in \mathcal{J}^\sigma\} \rangle$ are both direct (or inverse) bounds of $\mathcal{F}$, then every morphism $\beta \in K(a, a')$ such that $\beta \cdot \chi_i = \chi'_i$ (or $\beta \in K(a', a)$ such that $\chi_i \cdot \beta = \chi'_i$, respectively) for all $i \in J^\sigma$ will be called an α-<u>canonical morphism</u> of α'. A direct (or inverse) bound α of $\mathcal{F} : \mathcal{J} \to K$ is called its <u>direct</u> (or <u>inverse</u>) <u>limit</u> whenever every direct (or inverse, respectively) bound α' of $\mathcal{F}$ has exactly one α-canonical morphism (then it is called, for short, canonical). If $\alpha = \langle a ; \{\chi_i ; i \in \mathcal{J}^\sigma\} \rangle$ is a direct (or inverse) limit of $\mathcal{F}$, we denote it by $\alpha = \overrightarrow{\lim}_K \mathcal{F}$ (or $\alpha = \overleftarrow{\lim}_K \mathcal{F}$, respectively). If $\mathcal{F}$ is a collection, then $a = (\overrightarrow{\lim}_K \mathcal{F})$ (or $a = (\overleftarrow{\lim}_K \mathcal{F})$) will be also denoted by $a = \bigvee_{i \in \mathcal{J}^\sigma} \mathcal{F}(i)$ (or $a = \bigwedge_{i \in \mathcal{J}^\sigma} \mathcal{F}(i)$, respectively). A functor $\Phi : K \to H$ is said <u>to preserve direct</u> (or an <u>inverse</u>) <u>limit</u> of a <u>diagram</u> $\mathcal{F} : \mathcal{J} \to K$ whenever, if there exists a direct (or an inverse) limit α of $\mathcal{F}$, then $\Phi(\alpha)$ is a direct (or an inverse, respectively) limit of the diagram $\Phi\mathcal{F} : \mathcal{J} \to H$. Let $\mathbb{G}_d$ (or $\mathbb{G}_i$) be a class of diagrams in K ; a functor $\Phi : K \to H$ is said to be $\overrightarrow{\mathbb{G}_d}$ -preserving (or to be $\overleftarrow{\mathbb{G}_i}$ -preserving) whenever it preserves direct (or inverse) limits of all diagrams from $\mathbb{G}_d$ (or $\mathbb{G}_i$); if $\mathbb{G}_d$ (or $\mathbb{G}_i$) is the class of all diagrams in K , then a $\overrightarrow{\mathbb{G}_d}$ -preserving (or $\overleftarrow{\mathbb{G}_i}$ -preserving) functor is also said to be $\overrightarrow{\text{all}}$-preserving (or $\overleftarrow{\text{all}}$-preserving, respectively); a functor which is both $\overrightarrow{\mathbb{G}_d}$ -preserving and $\overleftarrow{\mathbb{G}_i}$ -preserving is said to be $(\overrightarrow{\mathbb{G}_d}, \overleftarrow{\mathbb{G}_i})$ -preserving. Let V be a class of diagram schemas; if K is a category, denote by K^V the class of all diagrams in K , the schema of which belongs to V ; a category K is called directly (or inversely) V-complete whenever every diagram from K^V has a direct (or an inverse, respectively) limit in K . If V is the class of all diagram schemas, then a directly (or inversely) V-complete category is called directly (or inversely, respectively) complete. A category which is both directly and inversely complete is called complete.

The category of all sets and all their mappings is denoted by S . If q is an ordinal number, then T_q denotes the set of all ordinal numbers less than q .

I. Categories of subcategories.

I.1. Definition. Let M be a category, let h, k be its subcategories, let $I_h : h \to M$, $I_k : k \to M$ be the inclusion functors. A couple $\langle \Phi , \varphi \rangle$ will be called an M^+-functor (of h in k or from h to k) whenever $\Phi : \mathfrak{h} \to \mathfrak{k}$ is a functor and $\varphi = \{\varphi_a ; a \in \mathfrak{h}^\sigma\}$ is a natural transformation from $I_{\mathfrak{h}}$ to $I_{\mathfrak{k}} \cdot \Phi$. A couple $\langle \Phi, \varphi \rangle$ will be called an M^--__functor__ whenever $\Phi : \mathfrak{h} \to \mathfrak{k}$ is a functor and φ is a natural transformation from $I_{\mathfrak{k}} \cdot \Phi$ to $I_{\mathfrak{h}}$.

__Notation__: If $\langle \Phi, \varphi \rangle : \mathfrak{h} \to \mathfrak{k}$, $\langle \Psi, \psi \rangle : \mathfrak{k} \to \mathfrak{l}$ are M^+-functors, put $\langle \Psi, \psi \rangle \cdot \langle \Phi, \varphi \rangle = \langle \Psi \cdot \Phi, \psi \cdot \varphi \rangle$ where $\psi \cdot \varphi = \{\psi_{\Phi(a)} \cdot \varphi_a ; a \in \mathfrak{h}^\sigma\}$. If they are M^--functors, put $\langle \Psi, \psi \rangle \cdot \langle \Phi, \varphi \rangle = \langle \Psi \cdot \Phi, \varphi \cdot \psi \rangle$ where $\varphi \cdot \psi = \{\varphi_a \cdot \psi_{\Phi(a)} ; a \in \mathfrak{h}^\sigma\}$. Evidently all small subcategories of M and all their M^+-functors (or M^--functors) form a category *); denote it by $\mathbb{M}^+$ (or $\mathbb{M}^-$ respectively).

I.2. If M is a complete category, then $\mathbb{M}^+$ may not be complete.

Now we give an __example__:

We show that in the category $\mathbb{S}^+$ (we recall that S denotes the category of all sets and all their mappings) a directed presheaf need not have a direct limit: Let $\mathcal{N}$ be the directed set of all positive integers with its natural order; let $\{a_s ; s = 0, 1, \dots\}$, $\{b_s ; s = 0, 1, \dots\}$ be two one-to-one sequences, a,b be the sets of all its members, let $a \neq b$. Let $\mathcal{F} : \mathcal{N} \to \mathbb{S}^+$ be the following presheaf: $\mathcal{F}(n) = \mathfrak{k}_n$ where $\mathfrak{k}_n^\sigma = \{a, b\}$,

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*) The morphisms of $\mathbb{M}^+$ are precisely the triples $\langle \mathfrak{h}, \langle \Phi, \varphi \rangle, \mathfrak{k} \rangle$ where $\langle \Phi, \varphi \rangle : \mathfrak{h} \to \mathfrak{k}$ is an M^+-functor and analogously for $\mathbb{M}^-$.

$k_n(a,a)=\{e_a\}$, $k_n(b,b)=\{e_b\}$, $k_n(b,a)=\emptyset$, $k_n(a,b)=\{\sigma, f_n\}$

where σ is the constant mapping which maps a on b_0, $f_n(a_s) = b_{max(0,s-n)}$; if $n, m \in \mathcal{N}^{\sigma}$, $n < m$, we define ${}^m_n\mathcal{F} = \langle {}^m_n\Phi, {}^m_n\varphi \rangle$ as follows:

${}^m_n\Phi(a) = a$, ${}^m_n\Phi(b) = b$, ${}^m_n\Phi(\sigma) = \sigma$, ${}^m_n\Phi(f_n) = f_m$;

${}^m_n\varphi_a$ is the identical mapping of a onto itself,

${}^m_n\varphi_b(b_s) = b_{max(0,s-m+n)}$. Now we prove that $\mathcal{F}$ has no direct limit: we define two direct bounds α, β of $\mathcal{F}$ in $\mathfrak{S}^+$:

$\alpha = \langle h; \{\langle {}^n\mathcal{U}, {}^n u \rangle; n \in \mathcal{N}^{\sigma}\} \rangle$ where

$h^{\sigma} = \{a, p\}$, $p = \{b_0\}$, h has exactly one non- identical morphism $\sigma_h : a \to p$ (which is, of course, a constant mapping); ${}^n\mathcal{U} : k_n \to h$ is a functor such that ${}^n\mathcal{U}(a) = a$, ${}^n\mathcal{U}(b) = p$, ${}^n\mathcal{U}(f_n) = {}^n\mathcal{U}(\sigma) = \sigma_h$, ${}^n u_a$ is the identical mapping of a onto itself, ${}^n u_b$ is the constant mapping of b onto p.

$\beta = \langle \ell; \{\langle {}^n W, {}^n w \rangle; n \in \mathcal{N}^{\sigma}\} \rangle$ where

$\ell^{\sigma} = \{a \cup b, b\}$, ℓ has exactly two non-identical morphisms, namely $\sigma_\ell : a \cup b \to b$ which is the constant mapping on b_0 and $g : a \cup b \to b$ such that $g(x) = b_0$ whenever $x \in a$, $g(x) = x$ whenever $x \in b$; ${}^n W : k_n \to \ell$ is defined as follows: ${}^n W(a) = a \cup b$, ${}^n W(b) = b$, ${}^n W(\sigma) = \sigma_\ell$, ${}^n W(f_n) = g$; ${}^n w_a : a \to a \cup b$ is the inclusion mapping, ${}^n w_b : b \to b$ is the constant mapping on b_0 .

Suppose that $\mathcal{F}$ has a direct limit, denote it by $\langle \mathcal{k}; \{\langle {}^nV, {}^n v \rangle : n \in \mathcal{N}^\sigma\}\rangle$. Since $\langle {}^nV, {}^n v\rangle = \langle {}^mV, {}^m v\rangle \cdot \langle {}^m_n\Phi, {}^m_n\varphi\rangle$ whenever $n \leqq m$, there is ${}^n v_b = {}^m v_b \cdot {}^m_n\varphi_b$ and consequently ${}^n v_b(b)$ is a one-point set. Denote $c = {}^nV(a)$, $c' = {}^n v_a(a)$. If $c' = c$, then ${}^nV(\sigma) = {}^nV(f_n)$ because ${}^nV(\sigma)\cdot {}^n v_a = {}^n v_b \cdot \sigma = {}^n v_b \cdot f_n = {}^nV(f_n)\cdot {}^n v_a$. But then β would have no canonical morphism because ${}^nW(\sigma) \neq {}^nW(f_n)$. Consequently $c' \subsetneqq c$. Choose $x \in c - c'$. Let $\langle \Psi, \psi\rangle$ be the canonical morphism of α ; then $\psi_c(x)$ may be an arbitrary element of $\Psi(c) = {}^n\mathcal{U}(a) = a$; i.e. there exist many canonical morphisms of α .

I.3. <u>Definition</u>. Let M be a category, let h,k be its subcategories, let $\langle \Phi, \varphi\rangle : \mathcal{h} \to \mathcal{k}$, $\langle \Psi, \psi \rangle : \mathcal{h} \to \mathcal{k}$ be M^+-functors. A collection $\{\beta_a ; a \in \mathcal{h}^\sigma\}$ is called a <u>natural transformation from</u> $\langle \Phi, \varphi\rangle$ to $\langle \Psi, \psi\rangle$ if it is a natural transformation from Φ to Ψ and $\beta_a \cdot \varphi_a = \psi_a$ for all $a \in h^\sigma$. It is a natural equivalence if all β_a are isomorphisms of k . Then $\langle \Phi, \varphi\rangle$ and $\langle \Psi, \psi\rangle$ are said to be naturally equivalent.

<u>Definition</u>. Let ℓ be a subcategory of h . An M^+-functor $\langle \Phi, \varphi\rangle : \mathcal{h} \to \mathcal{k}$ will be called identical on ℓ if $\Phi(\alpha) = \alpha$ whenever $\alpha \in \ell^\sigma \cup \ell^m$ and all φ_a, $a \in \ell^\sigma$, are identity-morphisms. It will be called

identity-M^+-functor if $\ell = h = k$. $\langle \Phi, \varphi \rangle$ is called an M^+-isofunctor if Φ is an isofunctor and all φ_a are isomorphisms of M. Two subcategories k, h of M are said to be M^+-isomorphic whenever there exists an M^+-isofunctor of k onto h.

Definition. Let M be a category, let h, k be its subcategories. h and k are said to be M^+-equivalent if there exists M^+-functors $\langle \Phi, \varphi \rangle : h \to k$, $\langle \Psi, \psi \rangle : k \to h$ such that $\langle \Psi, \psi \rangle \cdot \langle \Phi, \varphi \rangle$ and $\langle \Phi, \varphi \rangle \cdot \langle \Psi, \psi \rangle$ are naturally equivalent with identity-M^+-functors.

Proposition. Let M be a category, h, k its subcategories, $\langle \Phi, \varphi \rangle : h \to k$, $\langle \Psi, \psi \rangle : k \to h$ M^+-functors such that $\langle \Psi, \psi \rangle \cdot \langle \Phi, \varphi \rangle$ and $\langle \Phi, \varphi \rangle \cdot \langle \Psi, \psi \rangle$ are equivalent with identity-M^+-functors. Then all φ_a, $a \in h^\sigma$, and all ψ_b, $b \in k^\sigma$, are isomorphisms of M, Φ and Ψ are faithful full functors, and the skeletons of h and k are M^+-isomorphic.

Proof: This is easy.

Note: The definitions of a natural transformation of M^--functors, of an identity-M^--functor, of an M^--isofunctor, of M^--isomorphic and M^--equivalent subcategories of M are obtained by replacing M by its dual category. Evidently two subcategories of M are M^+-isomorphic if and only if they are M^--isomorphic; they are M^+-equivalent if and only if they are M^--equivalent and this happens if and only if they are equivalent in the sense of [8]. T M^+-isomorphic or M^+-equivalent subcategories will be called simply M-isomorphic or M-equivalent.

I.4. Definition. An M^+-functor $\langle \Phi, \varphi \rangle : h \to k$ will be called an epi-M^+-functor if $\rho_1 \varphi_a \neq \rho_2 \varphi_a$ whenever $a \in h^\sigma$, $c \in k^\sigma$, $\rho_1, \rho_2 \in k(\Phi(a), c)$, $\rho_1 \neq \rho_2$. Mono-M^--functor is defined dually.

Note: We recall that the Mac Lane's representation-isofunctor R_ℓ of a small category ℓ is defined as follows: $R_\ell(a) = \bigcup_{c \in \ell^\sigma} \ell(c, a)$; if $\mu \in \ell(a, b)$, then $R_\ell(\mu) : R_\ell(a) \to R_\ell(b)$ is the mapping such that $[R_\ell(\mu)](\nu) = \mu \cdot \nu$. It is known [3] that R_ℓ is an isofunctor of ℓ onto a small subcategory $\bar{\ell}$ of S. But R_ℓ s are not only representations of small categories. They define in a natural way the functor $R: C \to S^+$ of the category C of all small categories into S^+ if we put $R(\ell) = \bar{\ell}$; if $\Phi : \ell_1 \to \ell_2$ is a functor, we put $R(\Phi) = \langle \Psi, \psi \rangle$ where $\Psi = R_{\ell_2} \cdot \Phi \cdot R_{\ell_1}^{-1}$, $\psi = \{\psi_A ; A \in \bar{\ell}_1^\sigma\}$ where, for $A = R_{\ell_1}(a)$, $\psi_A : \bigcup_{c \in \ell_1^\sigma} \ell_1(c, a) \to \bigcup_{\alpha \in \ell_2^\sigma} \ell_2(d, \Phi(a))$ is a mapping such that $\psi_A(\nu) = \Phi(\nu)$.

Proposition: $R: C \to S^+$ is a faithful functor. If $\Phi \in C^m$, then $R(\Phi)$ is an epi-S^+-functor.

Proof: If ℓ_1, ℓ_2 are small categories, $\Phi : \ell_1 \to \ell_2$ is a functor, $R(\Phi) = \langle \Psi, \psi \rangle$, $A = R_{\ell_1}(a) \in \bar{\ell}_1^\sigma$, μ, $\mu' \in \ell_2(\Phi(a), d)$, then $R_{\ell_2}(\mu) \cdot \psi_A \neq R_{\ell_2}(\mu') \cdot \psi_A$ because these mappings have distinct values at the argument $e_a \in A$. R is evidently faithful.

Note: The functor $D: C \to S^-$ with dual properties is obtained as follows: for $\ell \in C^\sigma$ define the isofunctor

$D_\ell : \ell \to \widetilde{\ell}$ onto a small subcategory of S such that $D_\ell(a) = \exp(\bigcup_{c \in \ell^\sigma} \ell(a, c))$; if $\mu \in \ell(a, b)$, then $D_\ell(\mu)$ is the mapping such that, if $\rho \in D_\ell(a)$. $[D_\ell(\mu)](\rho)$ is the set of all $\nu \in \bigcup_{c \in \ell^\sigma} \ell(b, c)$ for which $\nu \cdot \mu \in \rho$ (thus D_ℓ is the composition of the Mac Lane's representation-isofunctor of the dual category with the restriction of the functor P^- defined in [6]). Put $D(\ell) = \widetilde{\ell}$; if $\Phi : \ell_1 \to \ell_2$, $\Phi \in C^m$, put $D(\Phi) = \langle \Psi, \psi \rangle$ where $\Psi = D_{\ell_2} \cdot \Phi \cdot D_{\ell_1}^{-1}$, $\psi = \{\psi_A ;\ A \in \widetilde{\ell}_1^{\sigma}\}$ where, if $A = D_{\ell_1}(a)$, then $\psi_A : \exp \bigcup_{\alpha \in \ell_2^\sigma} \ell_2(\Phi(a,d) \to \exp \bigcup_{c \in \ell_1^\sigma} \ell_1(a, c)$ is the mapping such that $\psi_A(\rho)$ is the set of all $\nu \in \bigcup_{c \in \ell_1^\sigma} \ell(a, c)$ for which $\Phi(\nu) \in \rho$. One may easily prove that $D : C \to S^-$ is a faithful functor. If $\Phi \in C^m$, then $D(\Phi)$ is a mono-S^--functor.

I.5. Definition: Let $\mathcal{F} : (T, \leqq) \to \mathbb{M}^+$ (or $\mathcal{F} : (T, \leqq) \to \mathbb{M}^-$) be a directed presheaf, $\mathcal{F}(t) = k_t$. Its direct bound $\beta = \langle k; \{\langle {}^tW, {}^tw \rangle;\ t \in T\} \rangle$ will be called admissible if it satisfies the following condition: if $t_0 \in T$, $\alpha, \beta \in k_{t_0}(a, b)$, ${}^{t_0}W(\alpha) \neq {}^{t_0}W(\beta)$, then there exists $t_1 \in T$, $t_1 \leqq t_0$ such that

$${}^{t_0}W(\alpha) \cdot {}^{t_1}w_{\bar{a}} \neq {}^{t_0}W(\beta) \cdot {}^{t}w_{\bar{a}} \qquad \text{where } \bar{a} = {}^{t^1}_{t_0}\mathcal{F}(a)$$

$$(\text{or } {}^{t_1}w_{\bar{b}} \cdot {}^{t_0}W(\alpha) \neq {}^{t_1}w_{\bar{b}} \cdot {}^{t_0}W(\beta) \qquad \text{where}$$

$$\bar{b} = {}^{t_1}_{t_0}\mathcal{F}(b), \qquad \text{respectively}).$$

An admissible direct bound α of $\mathcal{F}$ is called its $\mathbb{M}^+$-limit (or $\mathbb{M}^-$-limit, respectively) if every admissible direct bound of $\mathcal{F}$ has exactly one α-canonical morphism.

Theorem: Let M be a replete category in which every directed presheaf has a direct limit (or every inverse presheaf has an inverse limit). Then every directed presheaf in $\mathbb{M}^+$ (or in $\mathbb{M}^-$) has an $\mathbb{M}^+$-limit (or an $\mathbb{M}^-$-limit, respectively).

Proof: Let $\mathcal{F} : \langle T, \preceq \rangle \to \mathbb{M}^+$ be a directed presheaf. Put $k_t = \mathcal{F}(t)$; if $t \preceq t'$, put ${}_t^{t'}\mathcal{F} = \langle {}_t^{t'}\Phi, {}_t^{t'}\varphi \rangle$. We define the relation R on the set $P = \bigcup_{t \in T} \{t\} \times k_t^{\sigma}$ as follows:

$\langle t, a \rangle R \langle t', a' \rangle \Longleftrightarrow t \preceq t'$ and ${}_t^{t'}\Phi(a) = a'$. We define the relation S on the set $Q = \bigcup_{t \in T} \{t\} \times k_t^{m}$ as follows: $\langle t, \alpha \rangle S \langle t', \alpha' \rangle \Longleftrightarrow t \preceq t'$ and ${}_t^{t'}\Phi(\alpha) = \alpha'$. Let R^* and S^* be the smallest equivalence on P and Q containing R and S respectively. Put

$\mathbb{P} = P/_{R^*}$, $\mathbb{Q} = Q/_{S^*}$. Every element $p \in \mathbb{P}$ is directed by the restriction R_p of R. Now we define presheaves $\mathcal{F}_p : \langle p, R_p \rangle \to M$ such that $\mathcal{F}_p(\langle t, a \rangle) = a$; if $\langle t, a \rangle R_p \langle t', a' \rangle$, then ${}_{\langle t,a \rangle}^{\langle t',a' \rangle}\mathcal{F}_p = {}_t^{t'}\varphi_a : a \to a'$.

If $p \in \mathbb{P}$, put $\langle \ell_p ; \{ {}_s\lambda_p ; s \in p \} \rangle = \overrightarrow{\lim}_M \mathcal{F}_p$, where we choose $\ell_p \neq \ell_{p'}$ whenever $p \neq p'$. If $q \in Q$, $\langle t, \alpha \rangle \in q$, $\alpha \in k_t(a, a')$, $s = \langle t, a \rangle \in p$, $s' = \langle t, a' \rangle \in p'$, then there exists exactly one mapping $V(\alpha) : \ell_p \to \ell_{p'}$ such that ${}_{s'}\lambda_{p'} \cdot \alpha = V(\alpha) \cdot {}_s\lambda_p$. If also $\langle \bar{t}, \bar{\alpha} \rangle \in q$, then evidently $V(\alpha) = V(\bar{\alpha})$. Now let k be the category for which $k^\sigma = \{\ell_p ; p \in P\}$; if $\ell_p, \ell_{p'} \in k^\sigma$, then $k(\ell_p, \ell_{p'}) = \{V(\alpha); \alpha \in k_t(a,a'), \langle t,a\rangle \in p, \langle t,a'\rangle \in p'\}$.

If $t \in T$, put ${}^tV(a) = \ell_p$ whenever $\langle t, a\rangle \in p$,

${}^tV(\alpha) = V(\alpha)$, ${}^tv_a = {}_s\lambda_p$ whenever $s = \langle t, a\rangle \in p$;

then, as it may be easily proved $\langle {}^tV, {}^tv\rangle : k_t \to k$ is an M^+-functor and $\langle k; \{\langle {}^tV, {}^tv\rangle ; t \in T\}\rangle$ is an admissible direct bound of $\mathcal{F}$. We shall prove that it is the $\mathbb{M}^+$-limit of $\mathcal{F}$. Let $\Gamma = \langle h; \{\langle {}^t\Psi, {}^t\psi\rangle ; t \in T\}\rangle$ be an admissible direct bound of $\mathcal{F}$. If $p \in P$, $\langle t, a\rangle \in p$, $\langle t', a'\rangle \in p$, then ${}^t\Psi(a) = {}^{t'}\Psi(a') = \tilde{p}$, and $\langle \tilde{p}; \{{}^t\Psi_a ; \langle t, a\rangle \in p\}\rangle$ is a direct bound of $\mathcal{F}_p$ (in M). Denote by $g_p : \ell_p \to \tilde{p}$ its canonical morphism (i.e. $g_p \cdot {}^tv_a = {}^t\psi_a$). If $\beta \in k(\ell_p, \ell_{p'})$, then there exists $t_0 \in T$, $\alpha \in k^m_{t_0}$ such that ${}^{t_0}V(\alpha) = \beta$. Denote ${}^{t_0}\Psi(\alpha)$ by γ.

1) First, let us prove

(*)
$$\gamma \cdot g_p = g_{p'} \cdot \beta .$$

Let $t_0 \preceq t$, and put $\alpha_t = {}^t_{t_0}\Phi(\alpha)$, $a = \overleftarrow{\alpha_t}$, $a' = \overrightarrow{\alpha_t}$. Then ${}^tv_{a'} \cdot \alpha_t = \beta \cdot {}^tv_a$ and also ${}^t\psi_{a'} \cdot \alpha_t = \gamma \cdot {}^t\psi_a$. Using the equalities $g_p \cdot {}^tv_a = {}^t\psi_a$, $g_{p'} \, {}^tv_{a'} = {}^t\psi_{a'}$ there is $(g_{p'} \cdot \beta) \cdot {}^tv_a = (\gamma \cdot g_p) \cdot {}^tv_a$

and therefore (*) holds.

2) Secondly, we shall prove: if also $\beta = {}^{\bar{t}}V(\bar{\alpha})$ for some $\bar{\alpha} \in k^m_{\bar{t}}$, then necessarily ${}^{\bar{t}}\Psi(\bar{\alpha}) = \gamma$. But this

easily follows from (*) and from the fact that Γ is admissible.

Now we define the M^+-functor $\langle F, f \rangle : k \to h$ as follows: $F(\ell_p) = \tilde{p}$, $F(\beta) = \gamma$, $f_{\ell_p} = g_p$. It is easy to see that $\langle F, f \rangle$ is the unique M^+-functor such that $\langle F, f \rangle \cdot \langle {}^tV, {}^tv \rangle = \langle {}^t\Psi, {}^t\psi \rangle$.

The assertion of the theorem concerning $|M|^-$ is obtained replacing M by its dual category.

Note: From the construction of $|M|^+$-limit it is easy to see that the $|M|^+$-limit (under the same assumptions on M, of course) has the following properties:

1) If all $k_t = \mathcal{F}(t)$ contain a category $\tilde{k}$ as a subcategory and all $\langle {}^{t'}_t\Phi, {}^{t'}_t\varphi \rangle = {}^{t'}_t\mathcal{F}$ are identical on $\tilde{k}$, then the $|M|^+$-limit $\langle k; \{\langle {}^tV, {}^tv \rangle; t \in T\} \rangle$ can be chosen so that k contains $\tilde{k}$ and all $\langle {}^tV, {}^tv \rangle$ are identical on $\tilde{k}$; if $\tilde{k}$ is a full subcategory of all k_t, then it is also a full subcategory of k.

2) If $\Gamma = \langle h; \{\langle {}^t\Psi, {}^t\psi \rangle; t \in T\} \rangle$ is a direct bound of $\mathcal{F}$ such that all $\langle {}^t\Psi, {}^t\psi \rangle$ are epi-M^+-functors, then it is an admissible direct bound and its canonical M^+-functor is an epi-M^+-functor.

3) If $z \in k^o$ (or $z \in k^m$), then there exist $t \in T$ and $z' \in k^o_t$ (or $z' \in k^m_t$) such that $z = {}^tV(z')$. If $t_o \in T$, $a, a' \in k^o_{t_o}$, ${}^{t_o}V(a) = {}^{t_o}V(a')$, then necessarily there exists $t \geqq t_o$ such that ${}^t_{t_o}\Phi(a) = {}^t_{t_o}\Phi(a')$. But the analogous proposition for morphisms of k_{t_o} is generally not true as it is shown in Example I.2 (the direct bound α is the $|M|^+$-limit of $\mathcal{F}$). In the following

lemma we shall prove that this proposition holds in a very special case.

All these properties of $\mathbb{M}^+$-limits will be used in the following parts of the present paper without explicit mention.

I.7. We shall call a couple $\langle M, | \, | \rangle$ a concrete category whenever M is a category and $| \, | : M \to S$ is a faithful functor. Let m be a positive cardinal number. We shall say that a directed set $\langle T, \preceq \rangle$ is m-inaccessible if it has no last element and every set $T_0 \subset T$ with $\operatorname{card} T_0 < m$ has an upper bound in T.

Lemma: Let $\langle M, | \, | \rangle$ be a concrete category, let M have direct (or inverse) limits of directed (or inverse, respectively) presheaves, let $| \, |$ preserve these limits. Let $\langle T, \preceq \rangle$ be an m-inaccessible directed set, let $\mathcal{F} : \langle T, \preceq \rangle \to \mathbb{M}^+$ (or $\mathcal{F}, \langle T, \preceq \rangle \to \mathbb{M}^-$) be a presheaf; put $k_t = \mathcal{F}(t)$; if $t \preceq t'$, denote ${}_t^{t'}\mathcal{F} = \langle {}_t^{t'}\Phi, {}_t^{t'}\varphi \rangle$, $\langle k; \{ \langle {}^tV, {}^tv \rangle ; t \in T \} \rangle = \mathbb{M}^+ \text{-} \lim \mathcal{F}$ (or ... = $\mathbb{M}^-$-lim $\mathcal{F}$). Let $t_0 \in T$, $a, a' \in k_{t_0}^{\sigma}$, $\alpha, \beta \in k_{t_0}(a, a')$, ${}^{t_0}V(\alpha) = {}^{t_0}V(\beta)$. If $\operatorname{card} |a| < m$ and all mappings $| {}_{t_0}^{t}\varphi_a | : |a| \to | {}_{t_0}^{t}\Phi(a) |$ are onto (or all mappings $| {}_{t_0}^{t}\varphi_{a'} | : | {}_{t_0}^{t}\Phi(a') | \to |a'|$ are one-to-one, respectively), then ${}_{t_0}^{t}\Phi(\alpha) = {}_{t_0}^{t}\Phi(\beta)$ for some $t \succeq t_0$.

Proof: 1) First we shall prove the direct case: If z, $z' \in |a'|$, $| {}^{t_0}v_{a'} | (z) = | {}^{t_0}v_{a'} | (z')$, then there exists

$t \leqq t_0$ such that $| {}_{t_0}^{t}\varphi_{a'} |(x) = | {}_{t_0}^{t}\varphi_{a'} |(x')$. Since ${}^{t_0}v_{a'} \cdot \alpha = {}^{t_0}V(\alpha) \cdot {}^{t_0}v_a = {}^{t_0}V(\beta) \cdot {}^{t_0}v_a = {}^{t_0}v_{a'} \cdot \beta$ for every $x \in |a|$ there exists $t_x \leqq t_0$ such that $| {}_{t_0}^{t_x}\Phi(\alpha) \cdot {}_{t_0}^{t_x}\varphi_a |(x) = | {}_{t_0}^{t_x}\Phi(\beta) \cdot {}_{t_0}^{t_x}\varphi_a |(x)$. Let $\bar{t} \in T$, $\bar{t} \leqq t_x$ for all $x \in |a|$. Then $| {}_{t_0}^{\bar{t}}\Phi(\alpha) \cdot {}_{t_0}^{\bar{t}}\varphi_a | = | {}_{t_0}^{\bar{t}}\Phi(\beta) \cdot {}_{t_0}^{\bar{t}}\varphi_a |$ and $| {}_{t_0}^{\bar{t}}\varphi_a |$ is onto.

2) Now we prove the inverse case: Put $b = | {}^{t_0}v_a |(| {}^{t_0}V(a) |)$; then $b \subset |a|$. denote by $i : b \rightarrow |a|$ the inclusion mapping. Since $\alpha \cdot {}^{t_0}v_a = {}^{t_0}v_{a'} \cdot {}^{t_0}V(\alpha) = {}^{t_0}v_{a'} \cdot {}^{t_0}V(\beta) = \beta \cdot {}^{t_0}v_a$, we have $|\alpha| \cdot i = |\beta| \cdot i$. For every $x \in |a| - b$ there exists $t_x \leqq t_0$ such that $x \notin | {}_{t_0}^{t_x}\varphi |(| {}_{t_0}^{t_x}\Phi(a) |)$. Let $\bar{t} \in T$, $\bar{t} \leqq t_x$ for all $x \in |a| - b$. Then $| {}_{t_0}^{\bar{t}}\varphi_a |(| {}_{t_0}^{\bar{t}}\Phi(a) |) \subset b$, and therefore $| {}_{t_0}^{\bar{t}}\varphi_{a'} \cdot {}_{t_0}^{\bar{t}}\Phi(\alpha) | = | \alpha \cdot {}_{t_0}^{\bar{t}}\varphi_a | = | \beta \cdot {}_{t_0}^{\bar{t}}\varphi_a | = | {}_{t_0}^{\bar{t}}\varphi_{a'} \cdot {}_{t_0}^{\bar{t}}\Phi(\beta) |$, and $| {}_{t_0}^{\bar{t}}\varphi_{a'} |$ is one-to-one.

II. <u>Auxiliary lemmas</u>.

II.1. <u>Convention</u>. Let M be a category, $a \in M^{\sigma}$. A set L will be called a <u>star from</u> a <u>in</u> M (or a <u>star to</u> a <u>in</u> M) whenever L is a set of couples $\langle \mu, \mu' \rangle$ of morphisms of M such that $\overrightarrow{\mu} = \overrightarrow{\mu'}$, $a = \overleftarrow{\mu} = \overleftarrow{\mu'}$ (or $a = \overrightarrow{\mu} = \overrightarrow{\mu'}$, $\overleftarrow{\mu} = \overleftarrow{\mu'}$ respectively). Let L be a

star from a (or to a); a morphism ν such that $\mu \cdot \nu = \mu' \cdot \nu$ (or $\nu \cdot \mu = \nu \cdot \mu'$) whenever $\langle \mu, \mu' \rangle \in L$ and every $\bar{\nu}$, for which also $\mu \cdot \bar{\nu} = \mu' \cdot \bar{\nu}$ (or $\bar{\nu} \cdot \mu = \bar{\nu} \cdot \mu'$) for all $\langle \mu, \mu' \rangle \in L$, uniquely factors through ν, will be called a kernel (or cokernel respectively) of L.
(Thus a kernel of a star is a strict monomorphism in the sense of [8]).

<u>Lemma</u>: Let M be a category, $a \in M^{\sigma}$, L a star from a, ν its kernel. Let $\sigma \in M(a,a)$ such that $\langle \mu \cdot \sigma, \mu' \cdot \sigma \rangle \in L$ whenever $\langle \mu, \mu' \rangle \in L$. Then there exists exactly one $\bar{\sigma} \in M(\overleftarrow{\nu}, \overleftarrow{\nu})$ such that $\sigma \cdot \nu = \nu \cdot \bar{\sigma}$.

<u>Proof</u>: Since $\mu \cdot (\sigma \cdot \nu) = \mu' \cdot (\sigma \cdot \nu)$ for all $\langle \mu, \mu' \rangle \in L$, $\sigma \cdot \nu$ uniquely factors through ν.

II.2. <u>Lemma</u>: Let M be a replete inversely complete category. Let h be its subcategory, $a \in h^{\sigma}$; let k be a full subcategory of h such that $h^{\sigma} - k^{\sigma} = \{a\}$; let L be a star from a in h such that:

1) if $\langle \mu, \mu' \rangle \in L$, $\sigma \in h(a,a)$, then $\langle \mu \cdot \sigma, \mu' \cdot \sigma \rangle \in L$;
2) if $f \in h(c,a)$, $c \in k^{\sigma}$, then $\mu \cdot f = \mu' \cdot f$.

Then there exists a subcategory h' of M and an M^{-}-functor $\langle \Psi, \psi \rangle : h \to h'$ such that

1) k is a full subcategory of h', $\langle \Psi, \psi \rangle$ is identical on k, $(h')^{\sigma} - k^{\sigma} = \{\Psi(a)\}$;
2) if $\langle \mu, \mu' \rangle \in L$, then $\Psi(\mu) = \Psi(\mu')$;

3) if $\langle \Phi, \varphi \rangle : \mathfrak{h} \to H$ is a mono-M^--functor such that $\Phi(\mu) = \Phi(\mu')$ whenever $\langle \mu, \mu' \rangle \in L$, then there exists exactly one M^--functor $\langle \Phi', \varphi' \rangle : h' \to H$ such that $\langle \Phi, \varphi \rangle = \langle \Phi', \varphi' \rangle \cdot \langle \Psi, \psi \rangle$. $\langle \Phi', \varphi' \rangle$ is also a mono-M^--functor.

Proof: I. We shall construct $\mathfrak{h}'$ with the required properties. Choose a kernel $i \in M(a', a)$ of L so that $a' \notin k^\sigma$. For every $f \in \mathfrak{h}(c, a)$, $c \in \mathfrak{k}^\sigma$, there exists exactly one $f' \in M(c, a')$ such that $f = i \cdot f'$. For every $\sigma \in \mathfrak{h}(a, a)$ there exists exactly one $\sigma' \in M(a', a')$ such that $\sigma \cdot i = i \cdot \sigma'$. Let h' be the category such that $(\mathfrak{h}')^\sigma = \mathfrak{k}^\sigma \cup \{a'\}$, $\mathfrak{k}$ is a full subcategory of h', $\mathfrak{h}'(a', a')$ is the semigroup of all σ'; if $c \in \mathfrak{k}^\sigma$, then $\mathfrak{h}'(c, a')$ is the set of all f' with $f \in h(c,a)$, $h'(a',c)$ is the set of all $\nu \cdot i$ with $\nu \in \mathfrak{h}(a, c)$. The definition of $\langle \Psi, \psi \rangle$ is evident. Of course, we put $\psi_a = i$.

II. If $\langle \Phi, \varphi \rangle : \mathfrak{h} \to H$ is a mono-M^--functor such that $\Phi(\mu) = \Phi(\mu')$ whenever $\langle \mu, \mu' \rangle \in L$, then also $\mu \cdot \varphi_a = \varphi_{\bar{\xi}} \cdot \Phi(\mu) = \varphi_{\bar{\xi}} \cdot \Phi(\mu') = \mu' \cdot \varphi_a$ and there exists exactly one $\varphi_{a'} \in M(\Phi(a), a')$ such that $\varphi_a = i \cdot \varphi_{a'}$. Now it is evident how to define $\langle \Phi', \varphi' \rangle$. It is only necessary to prove that: $\sigma_1' = \sigma_2' \Rightarrow \Phi(\sigma_1) = \Phi(\sigma_2)$; $f_1' = f_2' \Rightarrow \Phi(f_1) = \Phi(f_2)$; $\nu_1 \cdot i = \nu_2 \cdot i \Rightarrow \Phi(\nu_1) = \Phi(\nu_2)$.

But this assertion can be easily proved using the fact that $\langle \Phi, \varphi \rangle$ is a mono-M^--functor. Evidently, $\langle \Phi', \varphi' \rangle$ is also a mono-M^--functor.

Note: In what follows, the category h' will be denoted by $h/_{L}$, and the M^--functor $\langle \Psi, \psi \rangle$ will be called an L-projection.

II.3. Definition. A faithful functor $|\,| : M \to S$ will be called **inversely power-preserving** if it satisfies the following conditions:

1) if $\mu \in M(a, b)$ is a monomorphism, then card $|a| \leqq$ card $|b|$;

2) if $\{a_i \,;\, i \in J\}$ is a collection in M, $a = \bigwedge_{i \in J} a_i$, then card $|a| \leqq \prod_{i \in J}$ card $|a_i|$.

Note: Evidently forgetful functors of many familiar categories of sets with a structure and all their mappings naturally induced by this structure are, in most cases, inversely power-preserving.

II.4. Convention: In the following two lemmas we assume that a replete inversely complete category M is given, k and h are both small subcategories of M, k is a full subcategory of h, $h^\sigma - k^\sigma = \{a\}$, $\langle I, \iota \rangle : k \to h$ is the inclusion M^--functor.

Lemma: Let $\mathcal{Y} : \mathcal{J} \to k$ be a diagram in k which has an inverse limit $\alpha_{\mathcal{Y}} = \langle \alpha ; \{\lambda_j \,;\, j \in \mathcal{J}^\sigma\} \rangle$ in k. Let $\beta = \langle a ; \{\chi_j \,;\, j \in \mathcal{J}^\sigma\} \rangle$ be an inverse bound of $I\mathcal{Y}$. Then there exists a small subcategory h' of M and an M^--functor $\langle \Psi, \psi \rangle : h \to h'$ such that:

1) k is a full subcategory of h', $\langle \Psi, \psi \rangle$ is identical on k, $h'^{\sigma} - k^{\sigma} = \{a'\}$, $\Psi(a) = a'$; the restriction of Ψ to the set h(c,a) is a one-to-one mapping onto the set $h'(c,a')$ for all $c \in k^{\sigma}$.

2) There exists $\sigma_{\mathcal{Y}}$-canonical morphism μ of $\Psi(\beta)$ in h'.

3) If $\langle \Phi, \varphi \rangle : h \to H$ is a mono-M^--functor such that $\Phi(\sigma_{\mathcal{Y}})$ is an inverse limit of $\Phi I \mathcal{Y}$ in H, then there exists exactly one M^--functor $\langle \Phi', \varphi' \rangle : h' \to H$ such that $\langle \Phi', \varphi' \rangle \cdot \langle \Psi, \psi \rangle = \langle \Phi, \varphi \rangle$ and $\Phi'(\mu)$ is the $\Phi(\sigma_{\mathcal{Y}})$-canonical morphism of $\Phi(\beta)$ in H.

$\langle \Phi' \varphi' \rangle$ is a mono-M^--functor.

Moreover if $|\;| : M \to S$ is an inversely power-preserving functor and m is a cardinal number such that $m > \operatorname{card} |a| \times (\operatorname{card} |d|)^{\operatorname{card} h(a,a)}$, then $m > \operatorname{card} |a'|$.

<u>Proof</u>: I. Write $\Sigma = h(a,a)$. For every $\sigma \in \Sigma$ put $d_\sigma = d$. Put $b = \bigwedge_{\sigma \in \Sigma} d_\sigma$, $r = a \wedge b$ in M; let $\pi_a \in M(r, a)$, $\pi_b \in M(r, b)$, $\pi_\sigma \in M(b, d_\sigma)$ be projections. If $\rho \in \Sigma$, denote by $\rho^r \in M(r, r)$ the morphism such that $\rho \cdot \pi_a = \pi_a \cdot \rho^r$, $\pi_{\sigma \cdot \rho} \cdot \pi_b = \pi_\sigma \cdot \pi_b \cdot \rho^r$.
If $f \in h(c, a)$, $c \in k^{\sigma}$, $\sigma \in \Sigma$, denote by $\bar{f}^{\sigma}$ the $\sigma_{\mathcal{Y}}$-canonical morphism of the inverse bound

$\langle c; \{\chi_j \cdot \sigma \cdot f;\ j \in J^{\sigma}\} \rangle$ of $\mathcal{Y}$ in k. Denote by $f^r \in M(c, r)$ the morphism such that $\pi_a \cdot f^r = f$,

$\pi_\sigma \cdot \pi_b \cdot f^k = \bar{f}^\sigma$. It is easy to prove that

1) $(\sigma \cdot f)^k = \sigma^k \cdot f^k$;

2) $(\sigma_2 \cdot \sigma_1)^k = \sigma_2^k \cdot \sigma_1^k$;

3) $(f \cdot \varkappa)^k = f^k \cdot \varkappa$ where $\varkappa \in k^m$.

II. Let $\langle \Phi, \varphi \rangle : h \longrightarrow H$ be a mono-M^--functor satisfying the requirements of the Lemma. Denote by μ' the canonical morphism of $\Phi(\beta)$ in H. Let $\varphi^k \in M(\Phi(a), k)$ be the morphism such that $\pi_a \cdot \varphi^k = \varphi_a$, $\pi_\sigma \cdot \pi_b \cdot \varphi^k = \varphi_d \cdot \mu' \cdot \Phi(\vartheta)$. It is easy to prove that

1) if $\rho \in \Sigma$, then $\rho^k \cdot \varphi^k = \varphi^k \cdot \Phi(\rho)$;

2) if $f \in h(c, a)$, $c \in k^\sigma$, then $f^k \cdot \varphi_c = \varphi^k \cdot \Phi(f)$;

3) $\pi_{e_a} \cdot \pi_b \cdot \varphi^k = \varphi_d \cdot \mu'$.

III. Let L be the set of all couples $\langle \chi_j \cdot \sigma \cdot \pi_a ,$ $\lambda_j \cdot \pi_\sigma \cdot \pi_b \rangle$, where $\sigma \in \Sigma, j \in \mathcal{J}^\sigma$ and of all $\langle f^k \cdot \vartheta \cdot \pi_a , \sigma^k \rangle$ with $\sigma \in \Sigma, f \in h(c,a), \vartheta \in h(a, c), c \neq a, \sigma = f \cdot \vartheta$. It is easy to prove (using 1)2)3) of I. and 1)2) of II. of the present proof) that

1) if $\langle \nu, \nu' \rangle \in L$, $\rho \in \Sigma$, then $\langle \nu \cdot \rho^k , \nu' \cdot \rho^k \rangle \in L$;

2) if $f \in h(c, a)$, $c \neq a$, then $\nu \cdot f^k = \nu' \cdot f^k$ whenever $\langle \nu, \nu' \rangle \in L$;

3) $\nu \cdot \varphi^k = \nu' \cdot \varphi^k$ whenever $\langle \nu, \nu' \rangle \in L$.

Let i be a kernel of L such that $a' = \overleftarrow{i} \notin k^\sigma$. Denote by $\varphi' \in M(a', a')$, $f' \in M(c, a')$, $\varphi'_{a'} \in M(\Phi(a), a')$ the morphisms for which $i \cdot \rho' = \rho^{k} \cdot i$, $i \cdot f' = f^{k}$, $i \cdot \varphi'_{a'} = \varphi^{k}$.

IV. Now it is evident how to define h'. Put $\mu = \pi_{e_a} \cdot \pi_b \cdot i$. Put $h'^{\sigma} = k^{\sigma} \cup \{a'\}$, k is to be a full subcategory of h'; if $c \in k^{\sigma}$, then $h'(c, a')$ is to be the set of all f' where $f \in h(c, a)$, $h'(a', c)$ is to be the set of all $\vartheta \cdot \pi_a \cdot i$ where $\vartheta \in h(a, c)$ and of all $\varkappa \cdot \mu$ where $\varkappa \in k(d, c)$; $h'(a', a')$ is the smallest semigroup containing all ρ' where $\rho \in \Sigma$ and all $f' \cdot \mu$ where $f \in h(d, a)$; $\Psi(a) = a'$, $\psi_a = \pi_a \cdot i$, $\Psi(f) = f'$, $\Psi(\rho) = \rho'$, $\Psi(\vartheta) = \vartheta \cdot \psi_a$. It is only necessary to prove that

1) $(\sigma_2 \cdot \sigma_1)' = \sigma_2' \cdot \sigma_1'$ whenever $\sigma_2, \sigma_1 \in \Sigma$;

2) $(\sigma \cdot f)' = \sigma' \cdot f'$ whenever $\sigma \in \Sigma$, $f \in h(c, a)$, $c \neq a$;

3) $(f \cdot \varkappa)' = f' \cdot \varkappa$ whenever $f \in h(c, a)$, $\varkappa \in k(c, c')$;

4) $\vartheta \cdot \psi_a \cdot f' = \vartheta \cdot f$ whenever $f \in h(c, a)$, $\vartheta \in h(a, c')$, $c, c' \in k^{\sigma}$;

5) $\sigma' = f' \cdot (\vartheta \cdot \psi_a)$ whenever $\sigma = f \cdot \vartheta$, $\sigma \in \Sigma$, $f \in h(c, a)$, $c \neq a$.

But the proof is easily accomplished by using 1)2)3) of I. of the present proof and the fact that i is a monomorphism.

V. Now it is also evident how we shall define $\langle \Phi', \varphi' \rangle$. $\langle \Phi', \varphi' \rangle$ is to be equal to $\langle \Phi, \varphi \rangle$ on k, $\Phi'(a') = \Phi(a)$, $\varphi'_{a'}$ is already defined, $\Phi'(\mu) = \mu'$, $\Phi'(f') = \Phi(f)$ whenever $f \in h(c, a)$, $c \neq a$, $\Phi'(\sigma') = \Phi(\sigma)$

whenever $\sigma \in \Sigma$ and extend Φ' on the whole h'. It is only necessary to prove that

1) $f' \cdot \varphi_c = \varphi'_{a'} \cdot \Phi(f)$ whenever $f \in h(c,a)$, $c \neq a$;

2) $\varphi_d \cdot \mu' = \mu \cdot \varphi'_{a'}$;

3) $\mu' \cdot \Phi(f) = \Phi(\varkappa)$ whenever $\mu \cdot f' = \varkappa$, $f \in h(c,a)$, $c \neq a$;

4) $\varphi'_{a'} \cdot \Phi(\sigma) = \sigma' \cdot \varphi'_{a'}$ whenever $\sigma \in \Sigma$;

5) $\Phi(\sigma_1) = \Phi(\sigma_2)$ whenever $\sigma_1, \sigma_2 \in \Sigma$, $\sigma'_1 = \sigma'_2$;

6) $\Phi(\vartheta_1) = \Phi(\vartheta_2)$ whenever $\vartheta_1, \vartheta_2 \in h(a,c)$, $c \neq a$, $\vartheta_1 \cdot \psi_a = \vartheta_2 \cdot \psi_a$;

7) $\Phi(\eta) = \Phi(\vartheta) \cdot \mu'$ whenever $\eta \in h(a,c)$, $\vartheta \in h(d,c)$, $\eta \cdot \psi_a = \vartheta \cdot \mu$;

8) $\Phi(f) \cdot \mu' \cdot \Phi(\sigma) = \Phi(\rho)$ whenever $f \in h(d,a)$, $\sigma, \rho \in \Sigma$, $f' \cdot \mu \cdot \sigma' = \rho'$;

9) $\Phi(f_1) \cdot \mu' \cdot \Phi(\sigma_1) = \Phi(f_2) \cdot \mu' \cdot \Phi(\sigma_2)$ whenever $f_1, f_2 \in h(d,a)$, $\sigma_1, \sigma_2 \in \Sigma$, $f'_1 \cdot \mu \cdot \sigma'_1 = f'_2 \cdot \mu \cdot \sigma'_2$.

But these statements may be easily proved by using the definition of μ, $\varphi'_{a'}$ and the fact that $\langle \Phi, \varphi \rangle$ is a mono-M^--functor. It is easy to see that $\langle \Phi', \varphi' \rangle$ is also a mono-M^--functor.

II.5. <u>Notation</u>: If Σ is a semigroup with an identity element e, and τ is an element, $\tau \notin \Sigma$, denote by $\Sigma(\tau)$ the semigroup with the following properties:

a) Σ is a subsemigroup of $\Sigma(\tau)$, e is the identity element of $\Sigma(\tau)$, $\tau \in \Sigma(\tau)$;

b) $g : \Sigma \to \Xi$ being a semigroup-homomorphism, $g(e)$ being the identity element of Ξ, $\wp \in \Xi$, there exists exactly one homomorphism $g' : \Sigma(\tau) \to \Xi$ such that $g'(\tau) = \wp$ and $g = g' \cdot i$, where by $i : \Sigma \to \Sigma(\tau)$ is denoted the inclusion homomorphism.

Evidently every $\sigma \in \Sigma(\tau)$ may be uniquely written in the form $\sigma = \sigma_n \cdot \sigma_{n-1} \cdot \ldots \cdot \sigma_2 \cdot \sigma_1$, where $\sigma_1 = e$, $\sigma_2, \ldots, \sigma_n \in \Sigma \cup \{\tau\}$ and if $\sigma_i \in \Sigma$, then $\sigma_{i+1} = \tau$ $(i = 2, \ldots, n-1)$. This expression will be called the <u>standard decomposition</u> of σ, n will be denoted by $n(\sigma)$.

II.6. <u>Lemma:</u> Let $\mathcal{F} : \mathcal{J} \to \mathcal{k}$ be a diagram, let $\gamma = \langle a; \{\lambda_i; i \in \mathcal{J}^\sigma\}\rangle$ be an inverse bound of $I\mathcal{F}$ such that every inverse bound of $\mathcal{F}$ in k has exactly one γ-canonical morphism in h. Let $\beta = \langle a; \{\chi_i; i \in \mathcal{J}^\sigma\}\rangle$ be an inverse bound of $I\mathcal{F}$. Then there exists a small subcategory h′ of M and an M^--functor $\langle \Psi, \psi \rangle : \mathcal{h} \to \mathcal{h}'$ such that:

1) k is a full subcategory of h′, $\langle \Psi, \psi \rangle$ is identical on k, $\mathcal{h}'^\sigma - \mathcal{k}^\sigma = \{a'\}$, $a' = \Psi(a)$, the restriction of Ψ to the set $\mathcal{h}(c, a)$, $c \in \mathcal{k}^\sigma$, is a one-to-one mapping onto the set $h'(c, a')$;

2) There exists a $[\Psi(\gamma)]$-canonical morphism μ of $\Psi(\beta)$ in h′.

3) If $\langle \Phi, \varphi \rangle : \mathcal{h} \to H$ is a mono-M^--functor such that $\Phi(\gamma)$ is an inverse limit of $\Phi I \mathcal{F}$

then there exists exactly one M^--functor $\langle \Phi', \varphi' \rangle : \mathfrak{h}' \to H$ such that $\langle \Phi', \varphi' \rangle \cdot \langle \Psi, \psi \rangle = \langle \Phi, \varphi \rangle$ and $\Phi'(\mu) = \mu'$, μ' being the canonical morphism of $\Phi(\beta)$ in H. $\langle \Phi', \varphi' \rangle$ is also a mono-M^--functor.

Moreover, if $| \, | : M \to S$ is an inversely power-preserving functor and $\mathfrak{m}$ is a cardinal such that

$\mathfrak{m} > (\mathrm{card}\, |a|)^{\max(\aleph_0, \mathrm{card}\, \mathfrak{h}(a,a))}$, then $\mathfrak{m} > \mathrm{card}\, |a'|$.

Proof: I. Put $\Sigma = \mathfrak{h}(a, a)$; for $\tau \notin \Sigma$ put $\Sigma' = \Sigma(\tau)$. For every $\rho \in \Sigma'$ put $a_\rho = a$; put $\varkappa = \bigwedge_{\rho \in \Sigma'} a_\rho$, and let $\pi_\rho : \varkappa \to a_\rho$ be projections. For every $\sigma \in \Sigma'$ denote by $\sigma^\varkappa \in M(\varkappa, \varkappa)$ the morphism for which $\pi_\rho \cdot \sigma^\varkappa = \pi_{\rho \cdot \sigma}$ for all $\rho \in \Sigma'$. If $f \in \mathfrak{h}(c, a)$, $c \neq a$, put $\bar{f}^\sigma = \sigma \cdot f$ whenever $\sigma \in \Sigma$; if $\sigma = \tau$, denote by $\bar{f}^\sigma$ the γ-canonical morphism of $\langle c; \{\chi_i \cdot f; \ i \in J^\sigma\} \rangle$ in h; if $\rho \in \Sigma'$, $\sigma_n \cdot \ldots \cdot \sigma_1$ is its standard decomposition, put $\bar{f}^\rho = \overline{(\bar{f}^{\sigma_{n-1} \cdot \ldots \cdot \sigma_1})^{\sigma_n}}$. Evidently, if $\rho_1, \rho_2 \in \Sigma'$, then $\overline{\bar{f}^{\rho_1}}^{\rho_2} = \bar{f}^{\rho_2 \cdot \rho_1}$. Denote by $f^\varkappa \in M(c, \varkappa)$ the morphism such that $\pi_\rho \cdot f^\varkappa = \bar{f}^\rho$. Then:

1) $\rho_1, \rho_2 \in \Sigma' \Rightarrow \rho_2^\varkappa \cdot \rho_1^\varkappa = (\rho_2 \cdot \rho_1)^\varkappa$. The proof is easy.

2) $\sigma \in \Sigma, f \in \mathfrak{h}(c, a) \Rightarrow (\sigma \cdot f)^\varkappa = \sigma^\varkappa \cdot f^\varkappa$. Proof: if $\rho \in \Sigma'$, then $\pi_\rho \cdot (\sigma \cdot f)^\varkappa = \overline{\sigma \cdot f}^\rho$, $\pi_\rho \cdot \sigma^\varkappa \cdot f^\varkappa = \bar{f}^{\rho \cdot \sigma}$. To prove $\overline{\sigma \cdot f}^\rho = \bar{f}^{\rho \cdot \sigma}$ use induction on $n(\rho)$.

3) $f \in \mathfrak{h}(c, a), \varkappa \in \mathfrak{k}(c', c) \Rightarrow (f \cdot \varkappa)^\varkappa = f^\varkappa \cdot \varkappa$. The proof is analogous to 2).

4) $f \in h(c,a), c \neq a, \rho \in \Sigma' \Rightarrow (\bar{f}^{\rho})^{k} = \rho^{k} \cdot f^{k}$. The proof is easy.

II. Let $\langle \Phi, \varphi \rangle : h \longrightarrow H$ be a mono-M^{-}-functor satisfying the requirements of the Lemma. Denote by μ' the $\Phi(\gamma)$ -canonical morphism of $\Phi(\beta)$. Put $\tilde{\Phi}(\sigma) = \Phi(\sigma)$ whenever $\sigma \in \Sigma$; if $\rho \in \Sigma'$ and $\sigma_n \cdot \ldots \cdot \sigma_1$ is its standard decomposition, put $\tilde{\Phi}(\rho) = \Phi(\sigma_n) \cdot \tilde{\Phi}(\sigma_{n-1} \cdot \ldots \cdot \sigma_1)$ whenever $\sigma_n \in \Sigma$, $\tilde{\Phi}(\rho) = \mu' \cdot \tilde{\Phi}(\sigma_{n-1} \cdot \ldots \cdot \sigma_1)$ whenever $\sigma_n = \tau$. Denote by $\varphi^{k} \in M(\Phi(a), k)$ the morphism such that $\pi_{\rho} \cdot \varphi^{k} = \varphi_a \cdot \tilde{\Phi}(\rho)$. Then:

1) $\sigma \in \Sigma' \Rightarrow \sigma^{k} \cdot \varphi^{k} = \varphi^{k} \cdot \tilde{\Phi}(\sigma)$. Proof: if $\rho \in \Sigma'$, then $\pi_{\rho} \cdot \sigma^{k} \cdot \varphi^{k} = \pi_{\rho \cdot \sigma} \cdot \varphi^{k} = \varphi_a \cdot \tilde{\Phi}(\rho \cdot \sigma) = \varphi_a \cdot \tilde{\Phi}(\rho) \cdot \tilde{\Phi}(\sigma) = \pi_{\rho} \cdot \varphi^{k} \cdot \tilde{\Phi}(\sigma)$.

2) $\rho_1, \rho_2 \in \Sigma', \rho_1^{k} = \rho_2^{k} \Rightarrow \tilde{\Phi}(\rho_1) = \tilde{\Phi}(\rho_2)$. Proof: $\varphi_a \cdot \tilde{\Phi}(\rho_1) = \pi_{e_a} \cdot \varphi^{k} \cdot \tilde{\Phi}(\rho_1) = \pi_{e_a} \cdot \rho_1^{k} \cdot \varphi^{k} = \pi_{e_a} \cdot \rho_2^{k} \cdot \varphi^{k} = \pi_{e_a} \cdot \varphi^{k} \cdot \tilde{\Phi}(\rho_2) = \varphi_a \cdot \tilde{\Phi}(\rho_2)$.

3) $f \in h(c,a), c \neq a \Rightarrow f^{k} \cdot \varphi_c = \varphi^{k} \cdot \Phi(f)$. Proof: First prove that $\Phi(\bar{f}^{\rho}) = \tilde{\Phi}(\rho) \cdot \Phi(f)$ for every $\rho \in \Sigma'$, using the induction on $n(\rho)$. Then $\pi_{\rho} \cdot f^{k} \cdot \varphi_a = \bar{f}^{\rho} \cdot \varphi_c = \varphi_a \cdot \Phi(\bar{f}^{\rho}) = \varphi_a \cdot \tilde{\Phi}(\rho) \cdot \Phi(f) = \pi_{\rho} \cdot \varphi^{k} \cdot \Phi(f)$.

4) $\tau^{k} \cdot \varphi^{k} = \varphi^{k} \cdot \mu'$. The proof is easy.

III. Let L be the set of all couples $\langle \chi_i \cdot \pi_{\rho}, \lambda_i \cdot \pi_{\tau \cdot \rho} \rangle$ where $i \in J^{\sigma}$, $\rho \in \Sigma'$, and of all couples $\langle f_0^{\varkappa} \cdot \vartheta \cdot \pi_{\wp}, \sigma_0^{\varkappa} \cdot \varphi^{\varkappa} \rangle$ where $\wp \in \Sigma'$, $\sigma_0 \in \Sigma$, $\sigma_0 = f_0 \cdot \vartheta$, $f_0 \in h(c,a)$, $c \neq a$. It is easy to prove (using 1)2)3)4) of I. and 1) of II.) that:

1) $\wp \in \Sigma'$, $\langle \nu, \nu' \rangle \in L \Rightarrow \langle \nu \cdot \wp^{\varkappa}, \nu' \cdot \wp^{\varkappa} \rangle \in L$.

2) $f \in h(c,a)$, $c \neq a$, $\langle \nu, \nu' \rangle \in L \Rightarrow \nu \cdot f^{\varkappa} = \nu' \cdot f^{\varkappa}$. Proof: Evidently, $\chi_i \cdot \pi_{\wp} \cdot f^{\varkappa} = \lambda_i \cdot \pi_{\tau \cdot \wp} \cdot f^{\varkappa}$ whenever $\wp \in \Sigma'$. Let $\wp \in \Sigma'$, $\sigma_0 \in \Sigma$, $\sigma_0 = f_0 \cdot \vartheta$, $f_0 \in h(c_0, a)$, $c_0 \neq a$; then $f_0^{\varkappa} \cdot (\vartheta \cdot \pi_{\wp} \cdot f^{\varkappa}) = (f_0 \cdot \vartheta \cdot \pi_{\wp} \cdot f^{\varkappa})^{\varkappa} = (\sigma_0 \cdot \bar{f}^{\wp})^{\varkappa} = \sigma_0^{\varkappa} \cdot (\bar{f}^{\wp})^{\varkappa} = \sigma_0^{\varkappa} \cdot \wp^{\varkappa} \cdot f^{\varkappa}$.

3) $\langle \nu, \nu' \rangle \in L \Rightarrow \nu \cdot \varphi^{\varkappa} = \nu' \cdot \varphi^{\varkappa}$.

Let i be a kernel of L such that $a' = \overleftarrow{i} \notin k^{\sigma}$. Denote by $\wp' \in M(a', a')$, $f' \in M(c, a')$, $\tilde{\varphi} \in M(\Phi(a), a')$ the morphisms for which $i \cdot \wp' = \wp^{\varkappa} \cdot i$, $i \cdot f' = f^{\varkappa}$, $i \cdot \tilde{\varphi} = \varphi^{\varkappa}$.

IV. Now it is evident how to define the category h' and $\langle \Psi, \psi \rangle$. Put $\mu = \tau'$. Put $h'^{\sigma} = k^{\sigma} \cup \{a'\}$, k is a full subcategory of h'; h'(a', a') is the set of all $\wp'$ where $\wp \in \Sigma'$; if $c \in k^{\sigma}$, then h'(c,a') is the set of all f' where $f \in h(c,a)$, $h'(a', c)$ is the set of all $\vartheta \cdot \pi_{\wp} \cdot i$ where $\wp \in \Sigma'$, $\vartheta \in h(a,c)$; $\Psi(a) = a'$, $\psi_a = \pi_{e_a} \cdot i$, $\Psi(f) = f'$, $\Psi(\vartheta) = \vartheta \cdot \psi_a$, $\Psi(\sigma) = \sigma'$.

It is only necessary to prove that

1) $(\sigma_2 \cdot \sigma_1)' = \sigma_2' \cdot \sigma_1'$ whenever $\sigma_2, \sigma_1 \in \Sigma$;

2) $(\sigma \cdot f)' = \sigma' \cdot f'$ whenever $\sigma \in \Sigma$, $f \in h(c,a)$, $c \neq a$;

3) $(f \cdot \varkappa)' = f' \cdot \varkappa$ whenever $f \in h(c,a)$, $\varkappa \in k(c',c)$;

4) $\vartheta \cdot \psi_a \cdot f' = \vartheta \cdot f$ whenever $f \in h(c, a), \vartheta \in$ $\in h(a, c'), c, c' \in k^{\sigma}$;

5) $\sigma' = f' \cdot (\vartheta \cdot \psi_a)$ whenever $\sigma = f \cdot \vartheta, \sigma \in \Sigma,$ $f \in h(c, a), \vartheta \in h(a, c), c \neq a.$

But these statements may be easily proved by using the fact that i is a monomorphism of M .

V. Now it is also evident how to define $\langle \Phi', \varphi' \rangle$. Put $\Phi'(a') = \Phi(a), \varphi'_{a'} = \tilde{\varphi}, \langle \Phi', \varphi' \rangle$ is to be equal to $\langle \Phi, \varphi \rangle$ on k, $\Phi'(\sigma') = \tilde{\Phi}(\sigma)$ whenever $\sigma \in$ $\in \Sigma', \Phi'(f') = \Phi(f)$ whenever $f \in h(c, a), c \neq a,$

$\Phi'(\vartheta \cdot \psi_a) = \Phi(\vartheta)$ whenever $\vartheta \in h(a, c), c \neq a;$

further, extend Φ' to the whole h'. It is only necessary to prove that:

1) $f' \cdot \varphi_c = \tilde{\varphi} \cdot \Phi(f)$ whenever $f \in h(c, a), c \neq a$;

2) $\tilde{\varphi} \cdot \mu' = \mu \cdot \tilde{\varphi}$;

3) $\tilde{\varphi} \cdot \Phi'(\rho') = \rho' \cdot \tilde{\varphi}$ whenever $\rho' \in \Sigma'$;

4) $\tilde{\Phi}(\sigma_1) = \tilde{\Phi}(\sigma_2)$ whenever $\sigma_1, \sigma_2 \in \Sigma', \sigma_1' = \sigma_2'$;

5) $\Phi(\vartheta_1) \cdot \Phi'(\varrho_1') = \Phi(\vartheta_2) \cdot \Phi'(\varrho_2')$ whenever $\vartheta_1, \vartheta_2 \in$ $\in h(a, c), c \neq a, \varrho_1, \varrho_2 \in \Sigma', \vartheta_1 \cdot \psi_a \cdot \varrho_1' = \vartheta_2 \cdot \psi_a \cdot \varrho_2'$.

But these statements may be easily proved by using the fact that $\langle \Phi, \varphi \rangle$ is a mono-M^--functor. It is easy to see that $\langle \Phi', \varphi' \rangle$ is also a mono-M^--functor.

Note: It is easy to see that Lemmas dual to Lemmas II.3 and II.5 hold, too. For the proof it is sufficient to replace M by its dual category (with the exception of the proposition concerning the functor $|\ \ |$).

III. Construction of completions of a small subcategory of a given category.

III.1. We recall some definitions and propositions given in [14].

Definition. Let $\mathcal{F} : \mathcal{J} \longrightarrow k$ be a diagram, let $\langle a ; \{ \lambda_i ; i \in \mathcal{J}^{\sigma} \} \rangle$ be its inverse limit. The set $T_{\mathcal{F}}$ of all triples $\langle \lambda_i , \mathcal{F}(\sigma), \lambda_{i'} \rangle$, where $\sigma \in \mathcal{J}(i, i')$, will be called the inverse substance of $\mathcal{F}$ in k. Two diagrams in k which both have an inverse limit in k are said to be inversely equivalent if they have the same inverse substance.

Note: The inverse substance $T_{\mathcal{F}}$, of course, depends on the choice of an inverse limit of $\mathcal{F}$. But if $T_{\mathcal{F}}$ and $\overline{T}_{\mathcal{F}}$ are two inverse substances of $\mathcal{F}$, then there exists an isomorphism $\wp$ of k such that $\langle \alpha, \mu, \alpha' \rangle \in T_{\mathcal{F}} \Longleftrightarrow$

$\Longleftrightarrow \langle \alpha \cdot \wp, \mu, \alpha' \cdot \wp \rangle \in \overline{T}_{\mathcal{F}}$. Thus two diagrams are inversely equivalent iff they have the same inverse substances. The inverse equivalence is a reflexive symmetric transitive relation on the class $\mathbb{D}$ of all diagrams in k which have an inverse limit in k. If $\mathbb{G}$ is a class of diagrams in k, denote by $\mathbb{V}$ some choice-class of $\mathbb{G} \cap \mathbb{D}$ (i.e., no two different diagrams from $\mathbb{V}$ being inversely equivalent and every diagram from $\mathbb{G} \cap \mathbb{D}$ being inversely equivalent with some diagram from $\mathbb{V}$) and call it the <u>inversely substantial class of diagrams from</u> $\mathbb{G}$. If k is small, then $\mathbb{V}$ is a set.

III.2. <u>Definition</u>. Let k be a full subcategory of h , let $I : \mathscr{k} \to \mathscr{h}$ be the inclusion functor, let $\mathcal{F} : \mathcal{J} \to \mathscr{k}$ be a diagram, and let $\langle a ; \{ \lambda_i ; i \in \mathcal{J}^{\sigma} \} \rangle$ be its inverse limit(in k). An inverse bound $\langle b ; \{ \chi_i ; i \in \mathcal{J}^{\sigma} \} \rangle$ of $I\mathcal{F}$ in h will be called an inverse bound of the inverse substance of $\mathcal{F}$ (in k) if $\chi_i = \chi_{i'}$ whenever $\lambda_i = \lambda_{i'}$.

<u>Lemma:</u> Let k be a full subcategory of h , and let $I : \mathscr{k} \to \mathscr{h}$ be the inclusion functor. Let diagrams $\mathcal{F}, \mathcal{G}$ be inversely equivalent in k . Let every inverse bound of $I\mathcal{G}$ in h be an inverse bound of the inverse substance in k . If I preserves the inverse limit of $\mathcal{F}$, then it also preserves the inverse limit of $\mathcal{G}$.

<u>Proof:</u> Cf. Lemma I.5 of [14].

The following lemma is well known:

<u>Lemma:</u> Let k be a full subcategory of a category h , let $I : \mathscr{k} \to \mathscr{h}$ be the inclusion functor. Suppose that fpr every $c \in h^{\sigma}$ there exists a diagram $\mathcal{F}_c$ in k such that $c = | \overleftarrow{\lim}_{\mathscr{h}} I \mathcal{F}_c |$. Then I preserves direct limits of all diagrams in k .

<u>Proof:</u> [8],[10], Lemma I.7 of [14].

III.3. <u>Convention</u>: Let M be a category, and let $\mathbf{Z}$ be a class of morphisms of M . We shall denote by $\text{gen}_M \mathbf{Z}$ the smallest subcategory H of M such that $H^m \supset \mathbf{Z}$.

<u>Lemma</u>: Let M be a replete inversely complete category. Let k be a small subcategory of M , $\mathcal{F} : \mathcal{J} \to k$ a diagram. Then there exists a small subcategory K of M such that k is a full subcategory of K , the inclusion functor $I : k \to K$ preserves all direct and inverse limits already existing in k and $I\mathcal{F}$ has an inverse limit in K .

<u>Proof</u>: I. Put $\gamma = \langle a_0 ; \{\lambda_i ; i \in \mathcal{J}^\sigma\}\rangle = \overleftarrow{\lim}_M \bar{I}\mathcal{F}$ where $\bar{I} : k \to M$ is the inclusion functor and a_0 is chosen so that $a_0 \notin k^\sigma$. Denote by $\mathbb{A}$ the set of all inverse bounds of $\mathcal{F}$ in k . For every $\alpha = \langle b_\alpha ; \{\eta_i^\alpha ; i \in \mathcal{J}^\sigma\}\rangle \in \mathbb{A}$ denote by f_α its γ-canonical morphism in M . Put $k_0 = \text{gen}_M (k^m \cup \{\lambda_i ; i \in \mathcal{J}^\sigma\} \cup \{f_\alpha ; \alpha \in \mathbb{A}\})$.

Then evidently k is a full subcategory of k_0 ; denote by ${}^0I : k \to k_0$ the inclusion functor.

II. Denote by $\mathbb{D}$ the class of all diagrams in k which have an inverse limit in k . Put $\delta_{\mathcal{G}} = \gamma$ whenever $\mathcal{G} = \mathcal{F}$, $\delta_{\mathcal{G}} = \overleftarrow{\lim}_k \mathcal{G}$ whenever $\mathcal{G} \in \mathbb{D}$. Let $\delta_{\mathcal{G}} = \langle d ; \{\delta_j ; j \in \mathcal{J}^\sigma\}\rangle$. Let $\mathbb{V}$ be an inversely substantial set of diagrams from $\mathbb{D}$. Let p be the smallest ordinal number such that card p is a regular cardinal number and card p > card $\mathcal{J}^m$ for $\mathcal{G} \in \mathbb{V} \cup \{\mathcal{F}\}$, $\mathcal{G} : \mathcal{J} \to k$. We shall define a presheaf $\mathcal{T} : \langle T_p , \subseteq \rangle \to |M|^-$ using the transfinite induction.

III. Let $q \in T_n$ and let a presheaf $\mathcal{T}_q : \langle T_q, \leqq \rangle \to \mathbb{M}^-$ be defined as follows:

1) $\mathcal{T}_q(0) = k_0$;

let $\mathcal{T}_q(v) = k_v$, ${}^{v'}_{v}\mathcal{T}_q = \langle {}^{v'}_{v}\Psi ; {}^{v'}_{v}\psi \rangle$, $a_v = {}^{v}_{0}\Psi(a_0)$;

2) k_v contains k as a full subcategory, $k_v^\sigma - k^\sigma = \{a_v\}$, ${}^{v'}_{v}\mathcal{T}_q$ is identical on k and ${}^{v'}_{v}\Psi$ is one-to-one mapping of $\mathcal{A}^v$ onto $\mathcal{A}^{v'}$ where $\mathcal{A}^v = \bigcup_{c \in k^\sigma} k_v(c, a_v)$;

3) if $v < v' < q$, then every ${}^{v'}_{v}\Psi(m)$ where m is an inverse bound of ${}^{v}_{0}\Psi \circ I\,\mathcal{Y}$ in k_v, $\mathcal{Y} \in V \cup \{\mathcal{F}\}$, has a ${}^{v'}_{c}\Psi(\sigma_{\mathcal{Y}})$ -canonical morphism in $k_{v'}$;

4) if $v < q$ is non-isolated ordinal, then $\langle k_v ; \{ {}^{v}_{u}\mathcal{T}_q ; u \in T_v \} \rangle = \mathbb{M}^- \text{-} \lim \mathcal{T}_v$, where $\mathcal{T}_v : \langle T_v, \leqq \rangle \to \mathbb{M}^-$ is the restriction of $\mathcal{T}_q$; if v is a positive isolated ordinal, $\nu, \nu' \in k_v^m$ and $\nu . {}^{v}_{0}\Psi(f_\alpha) = \nu' . {}^{v}_{0}\Psi(f_\alpha)$ for all $\alpha \in A$, then $\nu = \nu'$.

IV. Now we shall define k_q and $\langle {}^{q}_{v}\Psi, {}^{q}_{v}\psi \rangle$. If q is non-isolated, put $\langle k_q ; \{ \langle {}^{q}_{v}\Psi, {}^{q}_{v}\psi \rangle ; v \in T_q \} \rangle = \mathbb{M}^- \text{-} \lim \mathcal{T}_q$ where we choose k_q such that it contains k and all $\langle {}^{q}_{v}\Psi, {}^{q}_{v}\psi \rangle$ are identical on k. Every ${}^{q}_{v}\Psi$ is a one-to-one mapping of $\mathcal{A}^v$ onto $\mathcal{A}^q = \bigcup_{c \in k^\sigma} k_q(c, a_q)$, $a_q = {}^{q}_{v}\Psi(a_v)$.

For, if $\alpha \neq \alpha'$, then $\chi_i^{\alpha} \neq \chi_i^{\alpha'}$ for some $i \in \mathcal{J}^{\sigma}$;

$\overset{x}{_o}\Psi(f_{\alpha}) = \overset{x}{_o}\Psi(f_{\alpha'})$ implies $\chi_i^{\alpha} = \overset{x}{_o}\Psi(\chi_i^{\alpha}) =$

$= \overset{x}{_o}\Psi(f_{\alpha}) \cdot \overset{x}{_o}\Psi(\lambda_i) = \overset{x}{_o}\Psi(f_{\alpha'}) \cdot \overset{x}{_o}\Psi(\lambda_i) = \overset{x}{_o}\Psi(\chi_i^{\alpha'}) = \chi_i^{\alpha'}$.

V. Let q be an isolated ordinal, $q = x + 1$. Let $\mathbb{P}$ be the set of all inverse bounds of all $\overset{x}{_o}\Psi \circ I\, \mathcal{Y}$, $\mathcal{Y} \in \mathcal{V} \cup \{\mathcal{F}\}$ which have no $\overset{x}{_o}\Psi(\sigma_{\mathcal{Y}})$-canonical morphism. Let $p : \mathbb{P} \to T_{\bar{p}}$ be a one-to-one mapping onto the set of all isolated positive ordinals from $T_{\bar{p}}$. We shall define a presheaf $\bar{\mathcal{T}}_{\bar{p}} : \langle T_{\bar{p}}, \leqq \rangle \to \mathbb{M}^-$ using the transfinite induction. Let $\bar{q} \in T_{\bar{p}}$ and let the presheaf $\bar{\mathcal{T}}_{\bar{q}} : \langle T_{\bar{q}}, \leqq \rangle \to \mathbb{M}^-$ be defined as follows:

$\bar{1}$) $\bar{\mathcal{T}}_{\bar{q}}(0) = k_x$;

let $\bar{\mathcal{T}}_{\bar{q}}(v) = \bar{k}_v$, $\overset{v'}{_v}\bar{\mathcal{T}}_{\bar{q}} = \langle \overset{v'}{_v}\bar{\Psi}, \overset{v'}{_v}\bar{\psi} \rangle$, $\bar{a}_v = \overset{v}{_o}\bar{\Psi}(a_x)$;

$\bar{2}$) $\bar{k}_v$ contains k as a full subcategory, $\bar{k}_v^{\sigma} - k^{\sigma} = \{\bar{a}_v\}$,

$\overset{v'}{_v}\bar{\mathcal{T}}_{\bar{q}}$ is identical on k, $\overset{v'}{_v}\bar{\Psi}$ maps $\bar{A}^v$ one-to-one onto $\bar{A}^{v'}$ where $\bar{A}^v = \bigcup_{c \in k^{\sigma}} \bar{k}_v(c, \bar{a}_v)$;

$\bar{3}$) if $v < \bar{q}$ is non-isolated, then $\langle \bar{k}_v ; \{\overset{v}{_u}\bar{\mathcal{T}}_{\bar{q}} ; u \in T_v\}\rangle =$

$= \mathbb{M}^- \text{-} \lim \bar{\mathcal{T}}_v$, where $\bar{\mathcal{T}}_v : \langle T_v, \leqq \rangle \to \mathbb{M}^-$ is the restriction of $\bar{\mathcal{T}}_{\bar{q}}$;

if $v < \bar{q}$ is isolated, then the inverse bound $\overset{v}{_o}\bar{\Psi}(p^{-1}(v))$ has a $[\overset{v}{_o}\bar{\Psi}\ \overset{x}{_o}\Psi(\sigma_{\mathcal{Y}})]$-canonical morphism in $\bar{k}_v$.

Now we shall define $\bar{k}_{\bar{q}}$ and $\langle {}^{\bar{q}}_{v}\bar{\Psi}, {}^{\bar{q}}_{v}\bar{\psi} \rangle$, $v \in T_{\bar{q}}$. If $\bar{q}$ is non-isolated, the definition is evident. Let $\bar{q} = \bar{x} + 1$. If $p^{-1}(\bar{q})$ is an inverse bound of ${}^{x}_{o}\Psi \circ I\,\mathcal{F}$, then use Lemma II.5; if it is an inverse bound of some ${}^{x}_{o}\Psi \circ I\,\mathcal{G}$, $\mathcal{G} \in \mathbb{V}$, then use Lemma II.3. Using the transfinite induction we have defined the presheaf $\bar{\mathcal{T}}_{\bar{p}} : \langle T_{\bar{p}}, \leqq \rangle \to \mathbb{M}^{-}$. Put $\langle h ; \{ \langle {}^{u}\Gamma, {}^{u}\gamma \rangle ; u \in T_{\bar{p}} \} \rangle = \mathbb{M}^{-}\text{-}\lim \bar{\mathcal{T}}_{\bar{p}}$, where h is chosen so that it contains k and all $\langle {}^{u}\Gamma, {}^{u}\gamma \rangle$ are identical on k. Put $\bar{a} = {}^{u}\Gamma(\bar{a}_u)$. Let L be the set of all couples $\langle \nu, \nu' \rangle$ of morphisms of h such that $\overrightarrow{\nu} = \bar{a}$ and $\nu \cdot [{}^{o}\Gamma\, {}^{x}_{o}\Psi(f_\alpha)] = \nu' \cdot [{}^{o}\Gamma\, {}^{x}_{o}\Psi(f_\alpha)]$ for all $\alpha \in \mathbb{A}$. If $\rho \in h(\bar{a}, \bar{a})$, then for every $\alpha \in A$ there exists $\alpha' \in \mathbb{A}$ such that $\rho \cdot [{}^{o}\Gamma\, {}^{x}_{o}\Psi(f_\alpha)] = {}^{o}\Gamma\, {}^{x}_{o}\Psi(f_{\alpha'})$. Consequently, if $\langle \nu, \nu' \rangle \in L$, then $\langle \nu \cdot \rho, \nu' \cdot \rho \rangle \in L$. Put $k_q = h/_L$, $\langle {}^{q}_{x}\Psi, {}^{q}_{x}\psi \rangle = \langle \Xi, \xi \rangle \cdot \langle {}^{o}\Gamma, {}^{o}\gamma \rangle$ where $\langle \Xi, \xi \rangle : h \to k_q$ denotes the L-projection M^{-}-functor.

VI. Using the transfinite induction we have defined the presheaf $\mathcal{T} : \langle T_p, \leqq \rangle \to \mathbb{M}^{-}$; if $q, q' \in T_p$, let $\mathcal{T}(q) = k_q$, ${}^{q'}_{q}\mathcal{T} = \langle {}^{q'}_{q}\Psi, {}^{q'}_{q}\psi \rangle$, $a_q = {}^{q}_{o}\Psi(a_o)$, $A^q = \bigcup_{c \in k^o} k_q(c, a_q)$; put $\langle K ; \{ \langle {}^{q}\theta, {}^{q}\vartheta \rangle ; q \in T_p \} \rangle = \mathbb{M}^{-}\text{-}\lim \mathcal{T}$ where K is chosen so that k is a full subcategory of K and all

$\langle {}^{q}\Theta, {}^{q}\vartheta \rangle$ are identical on k . Evidently every

${}^{q}\Theta$ is a one-to-one mapping of $\mathcal{A}^{q}$ onto the set

$\mathcal{A} = \bigcup_{c \in k^{\sigma}} K(c, a)$ (cf. IV. of the present proof).
Put $a = {}^{o}\Theta(a_o)$, $I = {}^{o}\Theta\, {}^{o}I$, $g_{\alpha} = {}^{o}\Theta(f_{\alpha})$: We shall prove that K and I have all required properties. First of all, we shall prove (*) and (* *) .

(*) If

$$\nu, \nu' \in K(a, c), \quad \nu \cdot g_{\alpha} = \nu' \cdot g_{\alpha}$$

for all $\alpha \in \mathbb{A}$, then $\nu = \nu'$.

Proof: Choose $q \in T_n$, $\mu, \mu' \in k_q(a_q, c)$ whenever $c \in k^{\sigma}$, $\mu, \mu' \in k_q(a_q; a_q)$ whenever $c = a_q$ such that ${}^{q}\Theta(\mu) = \nu$, ${}^{q}\Theta(\mu') = \nu'$. If for some $\alpha \in \mathbb{A}$ there is $\mu \cdot {}^{q}_{o}\Psi(f_{\alpha}) \neq \mu' \, {}^{q}_{o}\Psi(f_{\alpha})$, then ${}^{q}\Theta(\mu \cdot {}^{q}_{o}\Psi(f_{\alpha})) \neq {}^{q}\Theta(\mu' \cdot {}^{q}_{o}\Psi(f_{\alpha}))$ since ${}^{q}\Theta$ is one-to-one on $k^{m} \cup \mathcal{A}^{q}$ and then $\nu \cdot g_{\alpha} \neq \nu' \cdot g_{\alpha}$. Consequently if $\nu \cdot g_{\alpha} = \nu' \cdot g_{\alpha}$ for all $\alpha \in \mathbb{A}$, then also $\mu \cdot {}^{q}_{o}\Psi(f_{\alpha}) = \mu' \cdot {}^{q}_{o}\Psi(f_{\alpha})$ for all α and then ${}^{q+1}_{q}\Psi(\mu) = {}^{q+1}_{q}\Psi(\mu')$.

(* *) If

$$q \in T_n \text{ is isolated, then } {}^{q}\Theta \text{ is one-to-one.}$$

Proof: All ${}^{q}\Theta$ are one-to-one on $k^{m} \cup \mathcal{A}^{q}$. Let q be isolated, $\mu, \mu' \in k^{q}$, $a_q = \vec{\mu} = \vec{\mu}'$ and ${}^{q}\Theta(\mu) = {}^{q}\Theta(\mu')$. Then $\mu \cdot {}^{q}_{o}\Psi(f_{\alpha}) = \mu' \cdot {}^{q}_{o}\Psi(f_{\alpha})$ for all $\alpha \in \mathbb{A}$. For, if $\mu \cdot {}^{q}_{o}\Psi(f_{\alpha}) \neq \mu' \cdot {}^{q}_{o}\Psi(f_{\alpha})$ for some $\alpha \in \mathbb{A}$, then, since ${}^{q}\Theta$ is one-to-one on

$\mathfrak{k}^m \cup \mathcal{A}^{\varkappa}$, there is ${}^{\varkappa}\theta(\mu \cdot {}^{\varkappa}_{0}\Psi(f_\alpha)) \neq {}^{\varkappa}\theta(\mu' \cdot {}^{\varkappa}_{0}\Psi(f_\alpha))$,

i.e. ${}^{\varkappa}\theta(\mu) \cdot g_\alpha \neq {}^{\varkappa}\theta(\mu') \cdot g_\alpha$ Now we use 4) of III. of the present proof.

VII. Using (*) one may easily prove:

1) If $\mathcal{Y} \in \mathbb{D}$, then every inverse bound $\langle a ; \{ \xi_j ; j \in \mathcal{J}^\sigma \} \rangle$ of I $\mathcal{Y}$ is an inverse bound of the inverse substance of $\mathcal{Y}$ in k . For, if $\sigma_j = \sigma_{j'}$ for some j, j' $\in \mathcal{J}^\sigma$, then necessarily $\xi_j \cdot g_\alpha = \xi_{j'} \cdot g_\alpha$ for all $\alpha \in \mathbb{A}$.

2) If $\mathcal{Y} \in \mathbb{D} \cup \{ \mathcal{F} \}$, then every inverse bound m of I $\mathcal{Y}$ has at most one $[{}^{0}\theta(\sigma_{\mathcal{Y}})]$ -canonical monomorphism in K . If $(m) \in \mathfrak{k}^\sigma$ then this is evident. Let $(m) = a$ and let ν , ν' both be $[{}^{0}\theta(\sigma_{\mathcal{Y}})]$ - canonical morphism of m . Then ${}^{0}\theta(\sigma_j) \cdot \nu \cdot g_\alpha =$ $= {}^{0}\theta(\sigma_j) \cdot \nu' \cdot g_\alpha$ for all $j \in \mathcal{J}^\sigma, \alpha \in \mathbb{A}$, and therefore $\nu = \nu'$.

VIII. The proof will be finished by proving that every inverse bound $n = \langle a ; \{ \xi_j ; j \in \mathcal{J}^\sigma \} \rangle$ of I $\mathcal{Y}$, where $\mathcal{Y} \in \mathbb{V} \cup \{ \mathcal{F} \}$, has at least one $[{}^{0}\theta(\sigma_{\mathcal{Y}})]$ -canonical morphism in K . Therefore it is sufficient to find $\bar{\varkappa} \in T_p$ and an inverse bound m of ${}^{\bar{\varkappa}}_{0}\Psi \circ I\,\mathcal{Y}$ in $k_{\bar{\varkappa}}$ such that ${}^{\bar{\varkappa}}\theta(m) = n$. For, ${}^{\bar{\varkappa}+1}_{\bar{\varkappa}}\Psi(m)$ has a $[{}^{\bar{\varkappa}+1}_{0}\Psi(\sigma_{\mathcal{Y}})]$ -canonical morphism μ in $k_{\bar{\varkappa}+1}$ and then evidently ${}^{\bar{\varkappa}+1}\theta(\mu)$ is a $[{}^{0}\theta(\sigma_{\mathcal{Y}})]$ -canonical morphism of n in K . Now we find such $\bar{\varkappa} \in T_p$

and m. Put $c_j = \mathcal{D}(j)$. For every $j \in \mathcal{J}^o$ choose $q_j \in T_p$ and $\bar{u}_j \in k_{q_j}(a_{q_j}, c_j)$ such that ${}^{q_j}\theta(\bar{u}_j) = \xi_j$. Put $q = \sup_{j \in \mathcal{J}^o} q_j$, $u_j = {}^{q}_{q_j}\Psi(\bar{u}_j)$.

Since $(**)$ holds there is ${}^{q+1}_{q}\Psi(u_{j'}) = \mathcal{D}(\sigma) \cdot {}^{q+1}_{q}\Psi(u_j)$ whenever $\sigma \in \mathcal{J}(j, j')$. Put $\bar{q} = q + 1$, $v_j = {}^{q+1}_{q}\Psi(u_j)$, $n = \langle a_{\bar{q}} ; \{v_j ; j \in \mathcal{J}^o\} \rangle$.

Then n has the required properties.

III.4. Theorem: Let M be a replete complete category, k its small subcategory. Then there exists a complete subcategory K of M such that k is a full subcategory of K and the inclusion functor $I : k \to K$ preserves direct and inverse limits of all diagrams already existing in k.

Proof: Using Lemma III.3 and its dual one may easily construct a small subcategory k_m of M, for every cardinal number m, so that

1) $k_0 = k$;
2) if $m \leq n$, then k_m is a full subcategory of k_n and the inclusion functor $I^n_m : k_m \to k_n$ is $(\overrightarrow{all}, \overleftarrow{all})$-preserving;
3) if $\mathcal{D} : \mathcal{J} \to k_m$ is a diagram such that card $\mathcal{J}^m \leq m$, then $I^n_m \mathcal{D}$ has a direct and an inverse limit in k_n whenever $n > m$.

Put $K = \bigcup_m k_m$.

III.5. Note: Using Theorem III.4 one may easily prove: Let

$\langle \mathcal{k}, || \rangle$ be a small concrete category. Then there exists a concrete category $\langle K, ||^* \rangle$ such that K is complete, k is a full subcategory of K, the inclusion functor I : $: \mathcal{k} \to K$ is $(\overrightarrow{all}, \overleftarrow{all})$-preserving and $|| = ||^* \cdot I$.

III.6. Convention: Let M be a category, let k be its small subcategory, let $\langle \tilde{I}, \tilde{\iota} \rangle : \mathcal{k} \to M$ be the inclusion M^--functor. Let $\mathcal{F} : \mathcal{J} \to \mathcal{k}$ be a diagram which has an inverse limit in k. We shall say that $\overleftarrow{lim}_{\mathcal{k}} \mathcal{F}$ is absolute whenever $\overleftarrow{lim}_{\mathcal{k}} \mathcal{F} = \overleftarrow{lim}_{M} I \mathcal{F}$.

By faithful M^--functor we mean an M^--functor $\langle \Phi, \varphi \rangle$ such that Φ is faithful and all φ_a are isomorphisms of M.

Lemma: Let M be a replete inversely complete category, let k be its small subcategory, let $\langle \tilde{I}, \tilde{\iota} \rangle : \mathcal{k} \to M$ be the inclusion M^--functor, and let $\mathcal{F} : \mathcal{J} \to \mathcal{k}$ be a diagram. Then there exists a small subcategory h of M such that k is a full subcategory of h (denote by $\langle I, \iota \rangle :$ $: \mathcal{k} \to \mathcal{h}$, $\langle \bar{I}, \bar{\iota} \rangle : \mathcal{h} \to M$ the inclusion M^--functors) and

1) there exists $\overleftarrow{lim}_{\mathcal{h}} I \mathcal{F}$ and it is absolute; if $\mathcal{G} : \mathcal{J} \to \mathcal{k}$ has an absolute inverse limit in k, then I preserves it; I is $\overleftarrow{all}$-preserving;
2) if $\langle \Phi, \varphi \rangle : \mathcal{k} \to \mathcal{h}'$ is a faithful M^--functor which preserves absolute inverse limits and $\Phi \mathcal{F}$ has an absolute inverse limit in h', then there exists a faithful M^-- functor $\langle \Phi', \varphi' \rangle : \mathcal{h} \to \mathcal{h}'$ such that $\langle \Phi', \varphi' \rangle \cdot \langle I, \iota \rangle = \langle \Phi, \varphi \rangle$.

Proof: Denote by $\mathcal{V}$ the class of all diagrams in $\mathcal{k}$

which have an absolute inverse limit in k . Choose $\gamma =$ $= \langle a; \{\lambda_i ; i \in \mathcal{J}^{\sigma}\} \rangle = \overleftarrow{lim}_M \tilde{I} \mathcal{F}$ so that $a \notin k^{\sigma}$. For every inverse bound α of $\mathcal{F}$ in k denote by f_{α} its γ -canonical morphism. Put $k_0 = gen_M (k^m \cup \{\lambda_i ; i \in \mathcal{J}^{\sigma}\} \cup \{f_{\alpha} ; \alpha\})$. Let $L = \bigcup_{c \in k_0^{\sigma}} M(a, c)$. Let p be the smallest ordinal number such that card p is a regular cardinal number and card p > card L . We shall construct h using the transfinite induction. If $q \in T_p$, put $\ell_q = gen (\bigcup_{q' < q} k_{q'}^m)$, denote by $I_q : k \to \ell_q$ the inclusion functor; let $S_q \subset L$ be the set of all canonical morphisms of all inverse bounds m of all $I_q \mathcal{Y}$ in ℓ_q, where $\mathcal{Y} \in \mathcal{V} \cup \mathcal{F}$, such that $(m) = a$; put $k_q = gen_M (\ell_q^m \cup S_q)$. Put $h = gen_M (\bigcup_{q \in T_p} k_q^m)$. Then evidently h has all required properties.

Theorem: Let M be a replete complete category, let k be its small subcategory. Then there exists a complete subcategory K of M such that

1) k is a full subcategory of K and the inclusion functor $I : K \to M$ is $(\overrightarrow{all}, \overleftarrow{all})$-preserving;
2) if H is a complete subcategory of M such that k is a full subcategory of H and the inclusion functor $I' : H \to M$ is $(\overrightarrow{all}, \overleftarrow{all})$-preserving, then there exists a faithful M^{-}-functor of K into H which is identical on k .

Proof: The theorem may be easily proved by a suitable iteration of Lemma III.6 and its dual.

III.7. Note: 1) One may easily prove theorems analogous to Theorem III.4, III.6 but concerning either direct or inverse limits only.

2) Using a suitable iteration of Lemma III.3 and the lemma dual to Lemma III.6 one may easily prove the following theorem:

Theorem: Let M be a replete complete category, k its small subcategory. Then there exists a complete subcategory K of M such that k is a full subcategory of K, the inclusion functor $I : \mathcal{k} \to K$ preserves all inverse limits and the inclusion functor $\widetilde{I} : K \to M$ preserves all direct limits.

IV. $M^-(\mathbb{G}, V)$ and $M^+(\mathbb{G}, V)$-completions and some theorems concerning their existence.

IV.1. Definition: Let M be a category, k its subcategory, $\mathbb{C}$ a class of diagrams in k, V a class of diagram schemas. A subcategory K of M will be called a $M^-(\mathbb{G}, V)$-completion of k whenever

1) k is a full subcategory of K, K is $\overleftarrow{V}$-complete, the inclusion M^--functor $\langle I, \iota \rangle : \mathcal{k} \to K$ is $\overleftarrow{\mathbb{G}}$-preserving;

2) if $\langle \Phi, \varphi \rangle : \mathcal{k} \to H$ is a $\overleftarrow{\mathbb{G}}$-preserving mono-$M^-$-functor into a $\overleftarrow{V}$-complete subcategory H of M, then there exists a $\overleftarrow{K^V}$-preserving M^--functor $\langle \Phi', \varphi' \rangle : K \to H$, unique up to natural equivalence, such that $\langle \Phi, \varphi \rangle = \langle \Phi', \varphi' \rangle \cdot \langle I, \iota \rangle$. Moreover $\langle \Phi', \varphi' \rangle$ is also a mono-M^--functor.

Note: Evidently if K_1 and K_2 are both $M^-(\mathbb{G}, V)$-

completions of k, then they are M-equivalent.

<u>Note</u>: The definition of the dual notion of $M^+(\mathbb{G}, V)$-completion is evident. We obtain it by replacing the category M by its dual category.

IV.2. <u>Lemma a)</u>. Let S be a set and let z be the smallest ordinal such that card z is a regular cardinal and card $z >$ card S. Let be given, for every $q \in T_z$, a decomposition $\mathcal{D}_q$ of S such that $\mathcal{D}_q$ is a refinement of $\mathcal{D}_{q'}$ whenever $q \leqq q'$. Then there exists $x \in T_z$ such that $\mathcal{D}_q = \mathcal{D}_x$ for all $q \geqq x$.

<u>Proof:</u> For every $D \in \mathcal{D}_0$, $q \in T_z$, denote by A_D^q the element of $\mathcal{D}_q$ for which $D \subset A_D^q$. Since $\{ A_D^q ;$ $q \in T_z \}$ is a monotone system of subsets of S, there exists $x_D \in T_z$ such that $A_D^q = A_D^{x_D}$ whenever $q \geqq x_D$. Put $x = \sup_{D \in \mathcal{D}_0} x_D$.

<u>Lemma b)</u>: Let M be a category, and let h be its small subcategory. Let z be the smallest ordinal such that card z is a regular cardinal and card $z >$ card h^m. Let $\mathcal{S} : \langle T_z, \leqq \rangle \to |M|^-$ be a presheaf, and let $\mathcal{S}(q) = h_q$. Let

1) $\mathcal{S}(0) = h$;
2) if q is non-isolated, then $\langle h_q ; \{ {}^q_{q'}\mathcal{S} ; q' \in T_q \} \rangle =$ $= |M|^- - \lim \mathcal{S}_q$, where $\mathcal{S}_q : \langle T_q, \leqq \rangle \to |M|^-$ is a restriction of $\mathcal{S}$;
3) if $q = q' + 1$, then $h_q = h_{q'}/L_q$ and ${}^q_{q'}\mathcal{S}$ is the L_q-projection M^--functor where L_q is a star from $a_q \in h^o_{q'}$

in $h_{q'}$.

Then there exists $x \in T_x$ such that $\mu = \mu'$ for $\langle \mu, \mu' \rangle \in L_q$, $q \geq x$.

<u>Proof</u>: If $q' \leq q$, let ${}^{q}_{q'}\mathcal{Y} = \langle {}^{q}_{q'}\Xi, {}^{q}_{q'}\xi \rangle$. Evidently ${}^{q}_{q'}\Xi$ is onto. Denote by the same symbol ${}^{q}_{q'}\Xi$ the induced mapping of the set $h^{m}_{q'}$ onto the set h^{m}_{q}. If $q \in T_x$, denote by $\mathcal{D}_q$ the decomposition of h^{m}_{o} induced by the mapping ${}^{q}_{o}\Xi$ and use Lemma a).

IV.2. <u>Lemma</u>: Let M be a replete inversely complete category, let k be a small subcategory of M. Let $\mathbb{G}$ be a class of collections in k, $\mathcal{F}$ a collection in k. Then there exists a small subcategory K of M such that

1) k is a full subcategory of K, the inclusion M^{-}-functor $\langle I, \iota \rangle : k \to K$ is $(\overrightarrow{\mathrm{all}}, \overleftarrow{\mathbb{G}})$-preserving and $I\mathcal{F}$ has a product in K;

2) if $\langle \Phi, \varphi \rangle : k \to H$ is a $\overleftarrow{\mathbb{G}}$-preserving mono-$M^{-}$-functor and $\Phi\mathcal{F}$ has a product in H, then there exists an M^{-}-functor $\langle \Phi', \varphi' \rangle : K \to H$, unique up to natural equivalence, such that $\langle \Phi', \varphi' \rangle \cdot \langle I, \iota \rangle = \langle \Phi, \varphi \rangle$ and Φ' preserves the product of $I\mathcal{F}$. $\langle \Phi', \varphi' \rangle$ is a mono-M^{-}-functor.

<u>Proof</u>: I. Be given k, $\langle \Phi, \varphi \rangle : k \to H$, $\mathbb{G}$, $\mathcal{F} : \mathcal{J} \to k$ with the properties of the Lemma. Denote by $\overline{I} : k \to M$ the inclusion functor. Put $s_i = \mathcal{F}(i)$, $i \in \mathcal{J}^{\sigma}$. Choose $\gamma = \langle a_\iota ; \{\pi_i ; i \in \mathcal{J}^{\sigma}\} \rangle = \overleftarrow{\lim}_M \overline{I}\mathcal{F}$; for every inverse bound α of $\mathcal{F}$ in k denote by f_α its γ-canonical morphism in M. Put $k_o = gen_M(k^m \cup \{\pi_i ; i \in \mathcal{J}^{\sigma}\} \cup \{f_\alpha ; \alpha\})$.

Evidently k is a full subcategory of k_o , denote by

$\langle {}^\circ I, {}^\circ\iota \rangle : \mathcal{k} \to \mathcal{k}_o$ the inclusion M^--functor.

II. Let $\langle A ; \{ P_i ; i \in \mathcal{J}^\sigma \} \rangle = \overleftarrow{\lim}_H \Phi \mathcal{F}$, let $\bar{\varphi} \in M(A, a_o)$ be the γ -canonical morphism of $\langle A ; \{ \varphi_{s_i} \cdot P_i ; i \in \mathcal{J}^\sigma \} \rangle$ in M . It is easy to see that $\langle \Phi , \varphi \rangle$ may be essentially uniquely extended to $\langle {}^\circ\Phi , {}^\circ\varphi \rangle :$ $: \mathcal{k}_c \to H$. Of course, we put ${}^\circ\varphi_{a_o} = \bar{\varphi}$. It may be easily proved that $\langle {}^\circ\Phi , {}^\circ\varphi \rangle$ is a mono-M^--functor.

III. Let $\mathbb{D}$ be the class of all collections in k which have a product in k . Put $\sigma_{\mathcal{Y}} = \gamma$ whenever $\mathcal{Y} = \mathcal{F}$, $\sigma_{\mathcal{Y}} = \overleftarrow{\lim}_{\mathcal{k}} \mathcal{Y}$ whenever $\mathcal{Y} \in \mathbb{D}$. Let $\sigma_{\mathcal{Y}} = \langle d ; \{ \sigma_j , j \in \mathcal{J}^\sigma \} \rangle$. Let $\mathbb{V}$ be an inversely substantial set from $\mathbb{D} \cap \mathbb{G}$. Let p be the smallest positive ordinal such that card p is a regular cardinal number and card p > card $\mathcal{J}^m$ whenever $\mathcal{Y} : \mathcal{J} \to \mathcal{k}$, $\mathcal{Y} \in \mathbb{V} \cup \{ \mathcal{F} \}$. Using transfinite induction and Lemmas II.4, II.6 one may construct the presheaf $\mathcal{T} : \langle T_n , \leq \rangle \to |\mathbb{M}|^-$ and its admissible inverse bound $F = \langle H ; \{ \langle {}^q\Phi , {}^q\varphi \rangle ; q \in T_n \} \rangle$ (we may suppose that H is small) such that (put $\mathcal{T}(q) =$ $= \mathcal{k}_q$, ${}^{q'}_{q}\mathcal{T} = \langle {}^{q'}_{q}\Psi , {}^{q'}_{q}\psi \rangle$):

1) $\mathcal{T}(0) = \mathcal{k}_o$; $\mathcal{k}_q$ contains k as a full subcategory, $\mathcal{k}^\sigma_q - \mathcal{k}^\sigma = \{ a_q \}$, $a_q = {}^q_o\Psi(a_o)$; ${}^{q'}_{q}\mathcal{T}$ is identical on k , ${}^{q'}_{q}\Psi$ is a one-to-one mapping of $\mathcal{A}^q$

onto the set $\mathcal{A}^{q'}$ where $\mathcal{A}^{q} = \bigcup_{c \in k^{\sigma}} k_q(c, a_q)$.

2) if $q < q'$ and m is an inverse bound of some

$${}^{q}_{o}\Psi \circ I\mathcal{Y}, \quad \mathcal{Y} \in \mathcal{V} \cup \{\mathcal{F}\} \text{ in } k_q \text{, then } {}^{q'}_{q}\Psi(m)$$

has at least one $[{}^{q'}_{o}\Psi(\sigma_{\mathcal{Y}})]$ -canonical morphism in k_q.

3) All $\langle {}^{q}\Phi, {}^{q}\varphi \rangle$ are mono-M^--functors.

(The construction is analogous to that given in the proof of Lemma III.3.)

Put $\langle \bar{K}; \{\langle {}^{q}\theta, {}^{q}\vartheta \rangle; q \in T_n\} \rangle = |M|^- - \lim \mathcal{T}$, $\bar{a} = {}^{o}\theta(a_o)$,

$$\langle \bar{I}, \bar{\iota} \rangle = \langle {}^{o}\theta, {}^{o}\vartheta \rangle \cdot \langle {}^{o}I, {}^{o}\iota \rangle. \text{ Then } \langle \bar{I}, \bar{\iota} \rangle : k \to \bar{K}$$

is the inclusion M^--functor onto a full subcategory of $\bar{K}$. Denote by $\langle \bar{\Phi}, \bar{\varphi} \rangle$ the canonical morphism of F in $|M|^-$. Then $\langle \bar{\Phi}, \bar{\varphi} \rangle$ is a mono-M^--functor. Every inverse bound m of $\bar{I}\mathcal{Y}$ in $\bar{K}$, $\mathcal{Y} \in \mathcal{V} \cup \{\mathcal{F}\}$ has, in $\bar{K}$, at least one $[{}^{o}\theta(\sigma_{\mathcal{Y}})]$ -canonical morphism (the proof of this assertion is analogous to VIII. of the proof of Lemma III.3); if $m = \bar{I}(n)$, where n is an inverse bound of $\mathcal{Y}$ in k, then it has exactly one ${}^{o}\theta(\sigma_{\mathcal{Y}})$ -canonical morphism in $\bar{K}$.

IV. Let $g_\alpha = {}^{o}\theta(f_\alpha)$. Let L be the set of all couples $\langle \mu, \mu' \rangle$ such that $\mu = \chi_{j_o}$, $\mu' = \chi_{j'_o}$, where $\langle \bar{a}; \{\chi_j; j \in \mathcal{J}^{\sigma}\} \rangle$ is an inverse bound of some $\bar{I}\mathcal{Y}$, $\mathcal{Y} \in \mathbb{G} \cap \mathbb{D}$, and $\sigma_{j_o} = \sigma_{j'_o}$ (we recall that $\sigma_{\mathcal{Y}} = \overleftarrow{\lim}_{k} \mathcal{Y} = \langle d; \{\sigma_j; j \in \mathcal{J}^{\sigma}\} \rangle$). Evidently, if

$$\sigma \in \bar{K}(\bar{a}, \bar{a}), \langle \mu, \mu' \rangle \in L, \text{ then } \langle \mu \cdot \sigma, \mu' \cdot \sigma \rangle \in L.$$

If $\langle \mu, \mu' \rangle \in L$, then $\overline{\Phi}(\mu) = \overline{\Phi}(\mu')$ and $\mu \cdot g_\alpha = \mu' \cdot g_\alpha$ for all α. Put $K_0 = \overline{K}/L$, denote by $\langle R, \rho \rangle : \overline{K} \to K_0$ the L-projection M^--functor, $A_0 = R(\bar{a})$. Then every inverse bound of every $R\, \overline{I}\, \mathcal{Y}$, $\mathcal{Y} \in \mathbb{G} \cap \mathbb{D}$ in K_0, is an inverse bound of its inverse substance in k. Let $\langle {}^\circ\overline{\Phi}, {}^\circ\overline{\varphi} \rangle : K_0 \to H$ be the M^--functor such that $\langle \overline{\Phi}, \overline{\varphi} \rangle = \langle {}^\circ\overline{\Phi}, {}^\circ\overline{\varphi} \rangle \cdot \langle R, \rho \rangle$. $\langle {}^\circ\overline{\Phi}, {}^\circ\overline{\varphi} \rangle$ is a mono-M^--functor.

V. Let z be the smallest ordinal number such that card z is a regular cardinal number and card z > card K_0^m. Using Lemma II.2 and the transfinite induction it is possible to construct a presheaf $\mathcal{L} : \langle T_z, \leqq \rangle \to |M|^-$ and its admissible inverse bound $\overline{F} = \langle H; \{ \langle {}^q\overline{\Phi}, {}^q\overline{\varphi} \rangle ; q \in T_z \} \rangle$ such that (put $\mathcal{L}(q) = K_q$, ${}^{q'}_{q}\mathcal{L} = \langle {}^{q'}_{q}\Lambda, {}^{q'}_{q}\lambda \rangle$):

1) $\mathcal{L}(0) = K_0$; all K_q contain k as a full subcategory; $K_q^\sigma - k^\sigma = \{A_q\}$, $A_q = {}^q_0\Lambda(A_0)$; ${}^q_{q'}\mathcal{L}$ are identicatl on k; ${}^q_{q'}\Lambda$ are onto;
2) if q is non-isolated, then $\langle K_q; \{ {}^q_{q'}\mathcal{L}; q' \in T_q \} \rangle = |M|^- \text{-}\lim \mathcal{L}_q$ where $\mathcal{L}_q : \langle T_q, \leqq \rangle \to M^-$ is the restriction of $\mathcal{L}$; if $q = q' + 1$, then $K_q = K_{q'}/L_q$ and ${}^q_{q'}\mathcal{L}$ is the L_q - projection M^--functor where L_q is the set of all couples $\langle \nu, \nu' \rangle$ such that ν and ν' are both $[\, {}^{q'}_{0}\Lambda\, R\, {}^\circ\theta\, (\sigma_{\mathcal{Y}})\,]$ -canonical

morphisms of an inverse bound m of ${}^{\varkappa'}_{\circ}\Lambda\, R\, \bar{I}\, \mathcal{Y}$,

where $\mathcal{Y} \in V \cup \{\mathcal{F}\}$, $|m| = A_{\varkappa'}$;

3) all $\langle {}^{\varkappa}\bar{\Phi}, {}^{\varkappa}\bar{\varphi} \rangle$ are mono-M^{-}-functors.

Using Lemma VI.1.b) we can show that there exists $x \in T_{\varkappa}$ such that L_x consists of couples $\langle \nu, \nu' \rangle$ for which $\nu = \nu'$. It is easy to see that $K = K_x$, $\langle \Phi', \varphi' \rangle = \langle {}^{x}\bar{\Phi}, {}^{x}\bar{\varphi} \rangle$, $\langle I, \iota \rangle = \langle {}^{x}_{\circ}\Lambda, {}^{x}_{\circ}\lambda \rangle \cdot \langle R, \rho \rangle \cdot \langle \bar{I}, \bar{\iota} \rangle$ have all required properties.

IV.3. Using Lemma IV.2 and the transfinite induction one may easily prove:

__Theorem__: Let M be a replete inversely complete category, let V be a class of discrete diagram schemas. Let k be a small subcategory of M, let $\mathbb{G}$ be a class of collections in k. Then there exists an $M^{-}(\mathbb{G}, V)$ -completion K of k. The inclusion functor $I: k \to K$ is $\overrightarrow{\text{all}}$-preserving. If V is a set, then we may choose K small.

__Note:__ If we replace the category M in the Theorem by its dual category $\tilde{M}$, we obtain the dual theorem concerning $\tilde{M}^{+}(\mathbb{G}, V)$ -completions.

IV.4. __Lemma__: Let $(M, |\ |)$ be a concrete category. Let M be replete directly complete, let $|\ |$ preserve direct limits of direct presheaves. Let k be a small subcategory of M, let $\mathbb{G}$ be a set of diagrams in k, and let $\mathcal{F}$ be a diagram in k. Then there exists a small subcategory K of M such that

1) k is a full subcategory of K, the inclusion M^{+}-functor $\langle I, \iota \rangle : k \to K$ is $(\overrightarrow{\mathbb{G}}, \overleftarrow{\text{all}})$ -preserving; $I\mathcal{F}$ has a direct limit in K;

2) If $\langle \Phi, \varphi \rangle : \mathcal{k} \to H$ is a $\overrightarrow{\mathbb{G}}$-preserving epi-$M^+$-functor and $\Phi \mathcal{F}$ has a direct limit in H, then there exists an M^+-functor $\langle \Phi', \varphi' \rangle : K \to H$ unique up to natural equivalence such that $\langle \Phi, \varphi \rangle = \langle \Phi', \varphi' \rangle \cdot \langle I, \iota \rangle$ and $\langle \Phi', \varphi' \rangle$ preserves the direct limit of $I \mathcal{F}$. $\langle \Phi', \varphi' \rangle$ is an epi-M^+-functor.

Note: The proof of the Lemma is based on the same principle as the proof of Lemma IV.2. But the identifications which we have made at the end of the proof of Lemma IV.2 would now lead to the appearance of new direct bounds. We shall now sketch the proof of Lemma IV.4 and show where it differs from that of Lemma IV.2.

Proof: I. Be given $\mathcal{k}, \mathbb{G}, \mathcal{F} : \mathcal{J} \to \mathcal{k}$, $\langle \Phi, \varphi \rangle : \mathcal{k} \to H$ with the properties of our Lemma; put $\mathcal{F}(i) = s_i$; denote by $\bar{I} : \mathcal{k} \to M$ the inclusion functor. Choose $\gamma = \langle a_0 ; \{ v_i ; i \in \mathcal{J}^\sigma \} \rangle = \overrightarrow{\lim}_M \bar{I} \mathcal{F}$. For every direct bound α of $\mathcal{F}$ in k denote by f_α its γ-canonical morphism. Put $\mathcal{k}_0 = gen_M(\mathcal{k}^m \cup \{ v_i ; i \in \mathcal{J}^\sigma \} \cup \{ f_\alpha ; \alpha \})$. Evidently k is a full subcategory of k_0, denote by $\langle {}^\circ I, {}^\circ\iota \rangle : \mathcal{k} \to \mathcal{k}_0$ the inclusion M^+-functor. $\langle \Phi, \varphi \rangle$ may be essentially extended on $\langle {}^\circ\Phi, {}^\circ\varphi \rangle : \mathcal{k}_0 \to H$ so that ${}^\circ\Phi(\gamma) = \overrightarrow{\lim}_H \Phi \mathcal{F}$; of course, ${}^\circ\varphi a_0$ is the γ-canonical morphism in M of the direct bound $\langle A ; \{ w_i \cdot \varphi_{s_i} ; i \in \mathcal{J}^\sigma \} \rangle$ of $\bar{I} \mathcal{F}$, where $\langle A ; \{ w_i ; i \in \mathcal{J}^\sigma \} \rangle = \overrightarrow{\lim}_H \Phi \mathcal{F}$. Evidently $\langle {}^\circ\Phi, {}^\circ\varphi \rangle$ is an epi-M^+-functor.

II. Let p be the smallest ordinal number such that card p is a regular cardinal number, card p > card $\mathcal{J}^m$ whenever $\mathcal{G}: \mathcal{J} \to \mathcal{k}$, $\mathcal{G} \in \mathbb{G} \cup \mathcal{F}$ and card p > > card |c| whenever $c \in k^{\sigma}_{\sigma}$. Denote by $\overline{\mathbb{G}}$ the set of all $\mathcal{G} \in \mathbb{G}$ for which $\overrightarrow{lim}_{\mathcal{k}} \mathcal{G}$ exists. Put

$\delta_{\mathcal{G}} = \overrightarrow{lim}_{\mathcal{k}} \mathcal{G}$ whenever $\mathcal{G} \in \mathbb{G}$, $\delta_{\mathcal{G}} = \gamma$ whenever $\mathcal{G} = \mathcal{F}$. Let $\delta_{\mathcal{G}} = \langle d ; \{ \delta_j ; j \in \mathcal{J}^{\sigma} \} \rangle$. Using transfinite induction and Lemmas II.2, II.4, II.6, IV.2.b) one may construct a presheaf $\mathcal{T} : \langle T_n, \leq \rangle \to |M|^+$ and its admissible direct bound $F = \langle H ; \{ \langle {}^q\Phi , {}^q\varphi \rangle ;$

$q \in T_n \} \rangle$ such that:

(put $\mathcal{T}(q) = \mathcal{k}_q$, ${}^{q'}_{q}\mathcal{T} = \langle {}^{q'}_{q}\Psi, {}^{q'}_{q}\psi \rangle$)

1) $\mathcal{T}(0) = \mathcal{k}_0$; $\mathcal{k}_q$ contains k as a full subcategory, $\mathcal{k}^{\sigma}_q - \mathcal{k}^{\sigma} = \{ a_q \}$, $a_q = {}^{q}_{0}\Psi(a_0)$; ${}^{q'}_{q}\mathcal{T}$ is identical on k, ${}^{q'}_{q}\Psi$ is a one-to-one mapping of the set $\mathcal{A}^q$ onto $\mathcal{A}^{q'}$ where $\mathcal{A}^q = \bigcup_{c \in \mathcal{k}^{\sigma}} \mathcal{k}_q(a_q, c)$;

2) if $q < q'$, then every ${}^{q'}_{q}\Psi(m)$ where m is a direct bound of some ${}^{q}_{0}\Psi \circ I \mathcal{G}$, $\mathcal{G} \in \overline{\mathbb{G}} \cup \{ \mathcal{F} \}$,

has <u>exactly one</u> ${}^{q'}_{0}\Psi(\delta_{\mathcal{G}})$ -canonical morphism in $k_{q'}$;

3) all $\langle {}^q\Phi, {}^q\varphi \rangle$ are epi-M^+-functors.

(If all $\mathcal{k}_{q'}$, $q' < q$, are defined, one may define k_q using the construction [with small modifications] given in III. - V. of the proof of Lemma IV.2 .)

Put $\langle K ; \{ {}^q\theta, {}^q\vartheta \rangle ; q \in T_n \} \rangle = |M|^+ - lim\ \mathcal{T}$, where

K is chosen so that K contains k and all $\langle {}^q\theta, {}^q\vartheta \rangle$ are identical on k. Put $a = {}^{0}\theta(a_0)$. Put $\langle I, \iota \rangle =$

$= \langle {}^{\circ}\theta, {}^{\circ}\vartheta \rangle \cdot \langle {}^{\circ}I, {}^{\circ}\iota \rangle$. Denote by $\langle \Phi', \varphi' \rangle$ the canonical morphism of F.

III. Now we shall prove that K has all required properties. Let $m = \langle a; \{ x_j ; j \in \mathcal{J}^{\sigma} \} \rangle$ be a direct bound of some $I\mathcal{G}$, $\mathcal{G} \in \overline{\mathbb{G}} \cup \{ \mathcal{F} \}$. Put $t_j = \mathcal{G}(j)$. For every $j \in \mathcal{J}^{\sigma}$ choose $q_j \in T_p$ and $\xi_j \in k^m_{q_j}$ so that ${}^{q_j}\theta(\xi_j) = x_j$. Since card p > card $|t_j|$ for all $j \in \mathcal{J}^{\sigma}$ and all $|{}^{q}\vartheta_{t_j}|$ are identical mappings of the set $|t_j|$ onto itself, one may use Lemma I.7; then for every $\sigma \in \mathcal{J}(j, j')$ there exists $q_\sigma \in T_p$, $q_\sigma \geq q_j$, $q_\sigma \geq q_{j'}$, such that ${}^{q_\sigma}_{q_j}\Psi(\xi_j) = {}^{q_\sigma}_{q_{j'}}\Psi(\xi_{j'}) \cdot \mathcal{G}(\sigma)$. Put

$\bar{q} = \sup_{\sigma \in \mathcal{J}^m} q_\sigma$, $x'_j = {}^{\bar{q}}_{q_j}\Psi(\xi_j)$. Then $m' = \langle a_{\bar{q}} ; \{ x'_j ; j \in \mathcal{J}^{\sigma} \} \rangle$ is the direct bound of ${}^{\bar{q}}_{0}\Psi \circ I\mathcal{G}$ for which ${}^{\bar{q}}\theta(m') = m$ and therefore m has at least one ${}^{\circ}\theta(\sigma_{\mathcal{G}})$-canonical morphism in K. Let μ and μ' be both ${}^{\circ}\theta(\sigma_{\mathcal{G}})$-canonical morphisms of it in K, i.e.

$\mu \cdot {}^{\circ}\theta(\sigma_j) = \mu' \cdot {}^{\circ}\theta(\sigma_j)$ for all $j \in \mathcal{J}^{\sigma}$. Then there exists $q \in T_p$ and $\nu, \nu' \in k^m_q$ such that $\mu = {}^{q}\theta(\nu)$, $\mu' = {}^{q}\theta(\nu')$. Since ${}^{q}\theta(\nu \cdot {}^{q}_{0}\Psi(\sigma_j)) = \mu \cdot {}^{\circ}\theta(\sigma_j) = \mu' \cdot {}^{\circ}\theta(\sigma_j) = {}^{q}\theta(\nu' \cdot {}^{q}_{0}\Psi(\sigma_j))$, we get, using Lemma I.7 again, that there exists $\bar{q}_j \geq q$ such that

${}^{\bar{q}_j}_{q}\Psi(\nu) \cdot {}^{\bar{q}_j}_{q}\Psi(\sigma_j) = {}^{\bar{q}_j}_{q}\Psi(\nu') \cdot {}^{\bar{q}_j}_{0}\Psi(\sigma_j)$; put $\tilde{q} = \sup_{j \in \mathcal{J}^{\sigma}} \bar{q}_j$, $\tilde{\mu} = {}^{\tilde{q}}_{q}\Psi(\nu)$, $\tilde{\mu}' = {}^{\tilde{q}}_{q}\Psi(\nu')$. Then $\tilde{\mu}$ and $\tilde{\mu}'$

are both $[\, {}^{\tilde{\mathfrak{z}}}_{0}\Psi(\sigma_{\mathscr{Y}})\,]$ -canonical morphisms of the direct bound $\langle a_{\tilde{\mathfrak{z}}} ; \{ \tilde{\mu} \cdot {}^{\tilde{\mathfrak{z}}}_{0}\Psi(\sigma_j) ; j \in \mathcal{J}^{\sigma} \} \rangle$ of ${}^{\tilde{\mathfrak{z}}}_{0}\Psi \circ I \mathscr{Y}$ in $k_{\tilde{\mathfrak{z}}}$ and thus ${}^{\tilde{\mathfrak{z}}+1}_{\tilde{\mathfrak{z}}}\Psi(\tilde{\mu}) = {}^{\tilde{\mathfrak{z}}+1}_{\tilde{\mathfrak{z}}}\Psi(\tilde{\mu}')$; consequently $\mu = \mu'$.

IV.5. <u>Theorem</u>: Let $(M , |\,|)$ be a concrete category, let M be replete directly complete, let $|\,|$ preserve direct limits of directed presheaves. Then for every small subcategory k of M there exists its $M^{+}(\mathfrak{G}, V)$ -completion K , where $\mathfrak{G}$ is an arbitrary <u>set</u> of diagrams in k , V is an arbitrary class of diagram schemas. Moreover, the inclusion functor $I : k \to K$ is $\overleftarrow{\text{all}}$-preserving. If V is a set, then we may choose K small.

<u>Proof</u>: Use Lemma IV.4 and the transfinite induction.

<u>Note</u>: If in the Theorem the category M is replaced by its dual category $\tilde{M}$, we obtain the theorem concerning $\tilde{M}^{-}(\mathfrak{G}, V)$ -completions of small subcategories of $\tilde{M}$. But in this dual theorem the assumptions about $\tilde{M}$ does not seem to be natural, namely the existence of a contravariant faithful functor $\Gamma : \tilde{M} \to S$ which turns inverse limits of inverse presheaves into direct limits. In what follows we shall prove the theorem concerning $M^{-}(\mathfrak{G}, V)$ completions in which the assumptions about M are satisfied evidently by many familiar categories, but this being done under a strong assumption about the set-theory.

IV.6. We recall that a cardinal number $\aleph_{\tau}$ is called strongly inaccessible if it is an uncountable regular cardinal

such that

$$\aleph_\alpha < \aleph_\tau \Longrightarrow 2^{\aleph_\alpha} < \aleph_\tau .$$

Convention: In the following we assume that for every cardinal number $\mathfrak{m}$ there exists a strongly inaccessible cardinal greater than $\mathfrak{m}$.

Theorem: Let $(M, | |)$ be a concrete category, let M be replete inversely complete, let $| |$ preserve inverse limits of inverse presheaves and be inversely power-preserving. Then for every small subcategory k of M there exists its $M^{-}(\mathbb{G}, V)$ -completion K , where $\mathbb{G}$ is an arbitrary set of diagrams in k , V is an arbitrary class of diagram schemas. Moreover, the inclusion functor $I : k \to K$ is $\overrightarrow{\text{all}}$-preserving. If V is a set, then we may choose K small.

Proof: Let $(M, | |)$ satisfy the assumptions of the Theorem, let k be a small subcategory of M , let $\mathbb{G}$ be a set of diagrams in k , let $\mathcal{F} : \mathcal{J} \to k$ be a diagram. It is sufficient to show that it is possible to join an inverse limit of $\mathcal{F}$ to k so that all requirements are satisfied. The proof is analogous to the dualization of the proof of Lemma IV.4. We must only set p to be the smallest ordinal number such that card p is a strongly inaccessible cardinal and card p > card $\mathcal{J}^m$ whenever $\mathcal{G} : \mathcal{J} \to k$, $\mathcal{G} \in \mathbb{G} \cup \{\mathcal{F}\}$, card p > card $|a_0|$, where $a_0 = (\varprojlim_M \bar{I}\mathcal{F})$

($\bar{I} : k \to M$ is the inclusion functor), card p > card $|c|$ for all $c \in k^\sigma$, card p > card k^m , so that we may use the inverse case of Lemma I.7 at the end of the proof. Choosing p

in such a way, we have card p > card $|a_q|$ for all $q \in T_p$. (This follows from the assertions about an inversely power-preserving functor in Lemmas II.4, II.6 and the fact that $| \; | : M \to S$ preserves inverse limits of inverse pre-sheaves and it is inversely power-preserving.)

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(Received February 9,1967)