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Jiří Durdil

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ON THE DIFFERENTIABILITY OF URYSOHN AND NEMYCKIJ OPERATORS

Jiří DURDIL, Praha

In this paper we deal with the differentiability of Urysohn and Nemyckij operators in spaces L_p and C. Some results in those topics have been obtained by M.M. Vajnberg, M.A. Krasnoselskij, J.B. Rutickij, Wang Shen-Wang, S. Aširov, L.V. Kantorovič (cf. [1] - [7],[12],[13]). Very recently there appeared a paper of P.P. Zabrejko [8] where the differentiability of Urysohn operators is discussed under more general conditions. Our assumption on the increase of $K(s, t, u)$ is more restrict than in [8], but our results are much stronger than the ones of the above mentioned works.

In section 2 we discuss the existence of continuous bounded (on every bounded set) Fréchet derivative $F': L_p \to \to [L_p \to L_q]$ of an Urysohn operator F in the space $L_p (p \geq 1)$. Thus, theorems 1,2 give sufficient conditions under which Urysohn operators are Lipschitzian. Theorems 3, 4 deal with the first and second derivatives and their properties (Lipschitzian condition of the first Fréchet derivative F', continuity and boundedness properties of the second Fréchet derivative F'' (or FG -derivative)). Similar results are derived for Urysohn operators acting in the space of continuous functions (section 3). Section 4 concerns with the differentiability of Nemyckij operators in L_p

$(p \geq 2)$. First of all we generalize some results of M.M. Vajnberg [2] and then we present several global and local theorems concerning the first and second continuous and bounded Fréchet derivatives of Nemyckij operators acting in L_p $(p > 2)$. We conclude this paper with the second differentials of these operators in L_2 .

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1. Notations and definitions

First of all we shall introduce some well-known notations and definitions. A function $f: [x, y] \to f(x, y)$, where x is fixed and y is variable, is denoted by $f(x, \cdot)$; hence, we shall use the notation which occured in [9].

Let X, Y be real Banach spaces, $x_0 \in X$, F a mapping from a neighbourhood $U(x_0)$ of the point x_0 into Y. We shall use the symbols $DF(x_0, \cdot)$, $dF(x_0, \cdot)$ to denote the linear Gâteaux and Fréchet differentials of the mapping F at the point x_0 , respectively. A mapping $x \to \to DF(x, \cdot)$, where $x \in M \subset X$, is denoted by DF and is called the linear Gâteaux differential on the set M . We introduce the similar notation dF for the Fréchet differential on a set in X . Often we shall use the symbol F' instead of DF or dF ; if $F': M \to [X \to Y]$ ($[X \to Y]$ denotes the space of all linear continuous mappings from X into Y), we call it the derivative of F on the set M .

Suppose F has the Fréchet differential dF in a neighbourhood $U(x_0)$ of the point $x_0 \in X$. If there exists

$$(1.1) \qquad \lim_{\tau \to 0} \frac{1}{\tau} \cdot [dF(x_0 + \tau k, h) - dF(x_0, h)] = G(h, k)$$

uniformly with respect to $h \in X$, $\|h\|_X \leq 1$, then the mapping $G : X \times X \to Y$ is called FG -differential of F at the point x_0. If G is bilinear (i.e. G is linear in each variable if the other is fixed), we denote this mapping by $F''(x_0)$. If $G \in [X \times X \to Y]$ (i.e. bilinear and bounded), G is called the FG -derivative of F at the point x_0.

Being $G : X \times X \to Y$ such bilinear mapping that the formula (1.1) holds uniformly with respect to $h, k \in X$, $\|h\|_X \leq 1$, $\|k\|_X \leq 1$, G is called the FF -differential of F at x_0. If $G \in [X \times X \to Y]$, it is said to be the FF -derivative of F at x_0 (or the second Fréchet derivative of F at x_0, too). It is easy to see that $G \in$ $\in [X \times X \to Y]$ is the FF -derivative of F at x_0 if and only if the Fréchet derivative F' of F exists on a neighbourhood of x_0 and

$$\lim_{\|k\|_X \to 0} \frac{1}{\|k\|_X} \cdot \| F'(x_0 + k) - F'(x_0) - G(\cdot, k) \|_{[X \to Y]} = 0 .$$

Throughout this paper, the terms number, function, measure are always meant a real number, a real almost everywhere finite function and a real Lebesgue measure, respectively. The symbol G denotes a bounded closed subset of E_n (E_n is the Euclidean n -space).

In next we shall use the following lemma which is a slight generalization of the theorem 106 [10].

Lemma 1. Let X be a metric space, $G \subset E_n$, $\alpha_0 \in X$, f a function defined on $X \times X \times G$. Suppose the following conditions are satisfied:

(a) $\lim_{\alpha \to \alpha_0} f(\alpha, \beta, t) = \varphi(\beta, t)$ exists for almost every $t \in G$ and every $\beta \in X$.

(b) $f(\alpha, \beta, \cdot)$ is measurable in G for every fixed $\alpha, \beta \in X$, $\alpha \neq \alpha_0$.

(c) There is a function g on $X \times G$ such that $|f(\alpha, \beta, t)| \leq g(\beta, t)$ for every $\alpha, \beta \in X$, $\alpha \neq \alpha_0$ and almost every $t \in G$; suppose that $g(\beta, \cdot) \in L(G)$ for each $\beta \in X$.

Then for every $\beta \in X$

$$(1.2) \quad \lim_{\alpha \to \alpha_0} \int_G f(\alpha, \beta, t)\, dt = \int_G \varphi(\beta, t)\, dt .$$

Moreover, if (a) holds uniformly with respect to $\beta \in B$ (B is a subset of X) and if the condition "There is a function $g \in L(G)$ such that $|f(\alpha, \beta, t)| \leq g(t)$ for every $\alpha \in X, \alpha \neq \alpha_0, \beta \in B$ and almost every $t \in G$" is valid instead of (c), then (1.2) holds uniformly with respect to $\beta \in B$. If the function g in the condition (c) is such that $g(\beta, \cdot) \in L_p(G)$ $(p \geq 1)$ for every fixed $\beta \in X$, then also $f(\alpha, \beta, \cdot) \in L_p(G)$ for every $\alpha, \beta \in X$, $\alpha \neq \alpha_0$, $\varphi(\beta, \cdot) \in L_p(G)$ for every $\beta \in X$ and $\lim_{\alpha \to \alpha_0} \| f(\alpha, \beta, \cdot)\|_{L_p} = \|\varphi(\beta, \cdot)\|_{L_p}$ for every $\beta \in X$.

<u>Definition 1</u>. Let K be a function of three variables defined on $G \times G \times E_1$. Let $K(\cdot, t, u)$ ($t \in G$ and $u \in E_1$ are fixed) be a measurable function on G for almost every $t \in G$ and every $u \in E_1$, $K(s, \cdot, u)$ a measurable function on G for almost every $s \in G$ and every $u \in E_1$ and $K(s, t, \cdot)$ a continuous one on E_1 for almost every $s, t \in G$. If these conditions are satisfied, we shall say that K is a U_L-function on $G \times G \times E_1$.

Lemma 2. Let K be a U_L-function on $G \times G \times E_1$, let x be a measurable function on G. Suppose that k_s ($s \in G$) is such function defined on G that $k_s(t) = K(s, t, x(t))$ for almost every $t \in G$. Then k_s is a measurable function on G for almost every $s \in G$.

Definition 2. Let K be a function on $G \times G \times J$ where $J \subset E_1$. Suppose that $K(\cdot, t, u)$ is continuous on G for almost every $t \in G$ and every $u \in J$, $K(s, \cdot\, u)$ is measurable on G for every $s \in G$ and $u \in J$ and $K(s, t, \cdot)$ is continuous on J for every $s \in G$ and almost every $t \in G$.

Then K is said to be a U_L-function on $G \times G \times J$.

2. The differentiability of the Urysohn operator in the spaces $L_p(G)$, $p \geq 1$.

Theorem 1. Let $K^{\bullet}$ be a U_L-function on $G \times G \times E_1$, F an operator of Urysohn generated by the function K. Let there exist $K'_u(s, t, u)$ for every $u \in E_1$ and almost every $s, t \in G$ and be such that K'_u is a U_L-function on $G \times G \times E_1$. Let $p > 1$, $q \geq 1$ and

$$(2.1) \qquad \int_G |K(\cdot, t, 0)|\, dt \in L_q(G).$$

Suppose that there exists an integer n, numbers λ_i $(i = 1, \dots, n)$ and functions M_0 and M_i $(i = 1, \dots, n)$ defined on G and $G \times G$, respectively, such that

$$(2.2) \quad M_0 \in L_q(G), \ \left(\int_G |M_i(\cdot, t)|^{\frac{p}{p-\lambda_i}} dt\right)^{\frac{p-\lambda_i}{p}} \in L_q(G), \ 1 \leq \lambda_i < p.$$

If

$$(2.3)\qquad |K'_u(s,t,u)| \le \sum_{i=1}^{n} M_i(s,t)\cdot|\;|^{\lambda_i-1} + M_0(s)\cdot|u|^{p-1}$$

for almost every $s, t \in G$ and every $u \in E_1$, then F is the mapping of $L_p(G)$, into $L_q(G)$, Lipschitzian on every bounded set in $L_p(G)$ and having a continuous and bounded (i.e. bounded on every bounded set in $L_p(G)$) Fréchet derivative F' on all the space $L_p(G)$. Moreover,

$$(2.4)\quad F'(x)h(s) = \int_G K'_u(s,t,x(t))\,h(t)\,dt \qquad (x, h \in L_p(G))$$

for almost every $s \in G$.

<u>Proof</u>. (a) Set $G(x,h) = \int_G K'_u(\cdot,t,x(t))\,h(t)\,dt$,

$\omega(x,h) = F(x+h) - Fx - G(x,h)$ for $x, h \in L_p(G)$. There exists a function ϑ defined on $G \times G$ such that

$$K(s,t,u) = K(s,t,0) + K'_u(s,t,\vartheta(s,t,\vartheta(s,t))u,\ 0<\vartheta(s,t)<1$$

for almost every $s, t \in G$ and every $u \in E_1$. Using the Hölder and Minkowski inequalities, lemma 2 and formula (2.3)

$$(2.5)\quad |Fx(s)| \le \int_G |K(s,t,0)|\,dt + \sum_{i=1}^{n}\Big(\int_G |M_i(s,t)|^{\frac{p}{p-\lambda_i}}dt\Big)^{\frac{p-\lambda_i}{p}}\cdot$$

$$\cdot\,\|x\|_{L_p}^{\lambda_i-1}\cdot\|x\|_{L_p} + |M_0(s)|\cdot\|x\|_{L_p}^{p-1}\cdot\|x\|_{L_p}\,,$$

$$(2.6)\quad |G(x,h)(s)| \le \sum_{i=1}^{n}\Big(\int_G |M_i(s,t)|^{\frac{p}{p-\lambda_i}}dt\Big)^{\frac{p-\lambda_i}{p}}\cdot\|x\|_{L_p}^{\lambda_i-1}\cdot\|h\|_{L_p} +$$

$$+\,|M_0(s)|\cdot\|x\|_{L_p}^{p-1}\cdot\|h\|_{L_p}$$

(the exponents of the Hölder inequalities are $\frac{p}{p-\lambda_i}$, $\frac{p}{\lambda_i-1}$, p and $\frac{p}{p-1}$, p, respectively). According to (2.1),(2.2), (2.5),(2.6), lemma 2 and the inequality of Minkowski, it follows that F maps $L_p(G)$ into $L_q(G)$, G is a mapping of $L_p(G) \times L_p(G)$ into $L_q(G)$ and $G(x,\cdot)$

$(x \in L_p(G)$ is fixed) is linear and bounded.

(b) Set $K'_u(s,\cdot,x(\cdot)) = k_s x \quad (x \in L_p(G))$ for arbitrary $s \in G$. By the theorem of M.A. Krasnoselskij ([3], chapter I, § 2), k_s is a continuous and bounded (on every bounded set) operation from $L_p(G)$ into $L_{\frac{p}{p-1}}(G)$ for almost every $s \in G$. Hence, we can write for almost every $s \in G$

$$(2.7)\quad \frac{1}{\|h\|_{L_p}} \cdot |\omega(x,h)(s)| = \frac{1}{\|h\|_{L_p}} \cdot \Big| \int_G [K(s,t,x(t)+h(t)) - K(s,t,x(t)) - K'_u(s,t,x(t))\cdot h(t)]\,dt \Big| =$$

$$= \frac{1}{\|h\|_{L_p}} \cdot \Big| \int_G [K'_u(s,t,x(t)+\vartheta(s,t)h(t)) - K'_u(s,t,x(t))]\cdot h(t)\,dt \Big| \le$$

$$\le \frac{1}{\|h\|_{L_p}} \cdot \Big(\int_G |K'_u(s,t,x(t)+\vartheta(s,t)h(t)) - K'_u(s,t,x(t))|^{\frac{p}{p-1}} dt \Big)^{\frac{p-1}{p}} \cdot$$

$$\cdot \Big(\int_G |h(t)|^p dt \Big)^{\frac{1}{p}} = \| k_s(x+\vartheta(s,\cdot)h) - k_s x \|_{L_{\frac{p}{p-1}}} \to 0$$

whenever $\|h\|_{L_p} \to 0$,

where $x, h \in L_p(G)$ and $0 < \vartheta(s,t) < 1$ for almost every $s, t \in G$. According to (2.3) we obtain for almost every $s \in G$ and $h \in L_p(G)$, $\|h\|_{L_p} \le 1$

$$(2.8)\quad \frac{1}{\|h\|_{L_p}} \cdot |\omega(x,h)(s)| \le \Big(\int_G |K'_u(s,t,x(t)+\vartheta(s,t)h(t)) - K'_u(s,t,x(t))|^{\frac{p}{p-1}} dt \Big)^{\frac{p-1}{p}} \le$$

$$\le \Big(\int_G |K'_u(s,t,x(t)+\vartheta(s,t)h(t))|^{\frac{p}{p-1}} dt \Big)^{\frac{p-1}{p}} + \Big(\int_G |K'_u(s,t,x(t))|^{\frac{p}{p-1}} dt \Big)^{\frac{p-1}{p}} \le$$

$$\le \sum_{i=1}^{n} \Big(\int_G |M_i(s,t)\cdot|x(t)+\vartheta(s,t)h(t)|^{\lambda_i-1}|^{\frac{p}{p-1}} dt \Big)^{\frac{p-1}{p}} +$$

$$+|M_0(s)|\cdot\Big(\int_G |x(t)+\vartheta(s,t)h(t)|^p dt\Big)^{\frac{p-1}{p}}+$$

$$+\sum_{i=1}^{m}\Big(\int_G \big|M_i(s,t)\cdot|x(t)|^{\lambda_i-1}\big|^{\frac{p}{p-1}}dt\Big)^{\frac{p-1}{p}}+|M_0(s)|\cdot\Big(\int_G|x(t)|^p dt\Big)^{\frac{p-1}{p}} \le$$

$$\le \sum_{i=1}^{m}\Big(\int_G |M_i(s,t)|^{\frac{p}{p-\lambda_i}}dt\Big)^{\frac{p-\lambda_i}{p}}\cdot\Big[\Big(\int_G|x(t)|^p dt\Big)^{\frac1p}+\Big(\int_G|\vartheta(s,t)h(t)|^p dt\Big)^{\frac1p}\Big]^{\lambda_i-1}+$$

$$+|M_0(s)|\cdot\Big[\Big(\int_G|x(t)|^p dt\Big)^{\frac1p}+\Big(\int_G|\vartheta(s,t)h(t)|^p dt\Big)^{\frac1p}\Big]^{p-1}+$$

$$+\sum_{i=1}^{m}\Big(\int_G |M_i(s,t)|^{\frac{p}{p-\lambda_i}}dt\Big)^{\frac{p-\lambda_i}{p}}\cdot\Big(\int_G|x(t)|^p dt\Big)^{\frac{\lambda_i-1}{p}}+$$

$$+|M_0(s)|\cdot\Big(\int_G|x(t)|^p dt\Big)^{\frac{p-1}{p}}\le$$

$$\le \sum_{i=1}^{m}\Big(\int_G |M_i(s,t)|^{\frac{p}{p-\lambda_i}}dt\Big)^{\frac{p-\lambda_i}{p}}\cdot\Big[(\|x\|_{L_p}+1)^{\lambda_i-1}+$$

$$+\|x\|_{L_p}^{\lambda_i-1}\Big]+|M_0(s)|\cdot\Big[(\|x\|_{L_p}+1)^{p-1}+$$

$$+\|x\|_{L_p}^{p-1}\Big]\ ;$$

the last expression (as a function of s) belongs to $L_q(G)$ by (2.2).

According to lemma 1, it follows from (2.7) and (2.8)

$$\frac{1}{\|h\|_{L_p}}\cdot\|\omega(x,h)\|_{L_q}=\Big(\int_G\Big[\frac{1}{\|h\|_{L_p}}\cdot|\omega(x,h)(s)|\Big]^q ds\Big)^{\frac1q}\to 0 \text{ if } \|h\|_{L_p}\to 0.$$

This proves the existence of the Fréchet derivative $F'(x)$ at every point $x \in L_p(G)$. Furthermore, $F'(x)h=G(x,h)$ holds for all $h \in L_p(G)$.

(c) Let $x_0, x, h \in L_p(G)$; then for almost every $s \in G$

$$|[F'(x_0+x)h-F'(x_0)h](s)|\le\Big(\int_G|K'_u(s,t,x_0(t)+x(t))-$$

$$-K'_u(s,t,x_0(t))|^{\frac{p}{p-1}}dt\Big)^{\frac{p-1}{p}}\cdot\Big(\int_G|h(t)|^p dt\Big)^{\frac1p}=$$

$$=\|k_s(x_0+x)-k_s x_0\|_{L_{\frac{p}{p-1}}}\cdot\|h\|_{L_p}\to 0 \text{ if } \|x\|_{L_p}\to 0$$

uniformly with respect to $h \in L_p(G)$, $\|h\|_{L_p} \leq 1$. We can find (similarly as in (b)) an integrable in $L_q(G)$ majorant which does not depend on x and h, $\|x\|_{L_p} \leq 1$, $\|h\|_{L_p} \leq 1$. According to lemma 1, we get

$$\| F'(x_0 + x) - F'(x_0) \|_{[L_p \to L_q]} = \sup_{\|h\|_{L_p} \leq 1} \Big(\int_G | F'(x_0 + x) h(s) - F'(x_0) h(s) |^q ds \Big)^{\frac{1}{q}} \longrightarrow 0 \quad \text{whenever } \|x\|_{L_p} \longrightarrow 0 .$$

This proves the continuity of the mapping $F' : L_p \to [L_p \to L_q]$ at an arbitrary point $x_0 \in L_p(G)$.

(d) In view of (2.6)

$$\| F'(x) \|_{[L_p \to L_q]} = \sup_{\|h\|_{L_p} \leq 1} \| G(x, h) \|_{L_q} \leq$$
$$\leq \sum_{i=1}^{n} \Big[\int_G \Big(\int_G | M_i(s,t) |^{\frac{p}{p-\alpha_i}} dt \Big)^{\frac{p-\alpha_i}{p}} ds \Big]^{\frac{1}{q}} \cdot \|x\|_{L_p}^{\alpha_i - 1} +$$
$$+ \Big(\int_G | M_0(s) |^q ds \Big)^{\frac{1}{q}} \cdot \|x\|_{L_p}^{p-1} .$$

Therefore, F' is a mapping from $L_p(G)$ into $[L_p(G) \to L_q(G)]$, bounded on every bounded set in $L_p(G)$. Hence, F is Lipschitzian on every bounded set in $L_p(G)$, which completes the proof.

<u>Theorem 2</u>. Let K be a U_L-function on $G \times G \times E_1$, F the Urysohn operator generated by the function K. Let there exist $K'_u(s, t, u)$ for almost every $s, t \in G$ and every $u \in E_1$ and be such that K'_u is a U_L-function on $G \times G \times E_1$. Suppose $q \geq 1$, $\int_G | K(\cdot, t, 0) | dt \in L_q(G)$ and assume that there exists a function $M \in L_q(G)$ such that

$$(2.9) \qquad | K'_u(s, t, u) | \leq M(s)$$

for almost every $s, t \in G$ and every $u \in E_1$. Then F is the Lipschitzian operation from $L_1(G)$ into $L_q(G)$

having the bounded (on all the space $L_1(G)$) Gâteaux derivative F' on $L_1(G)$. Furthermore, the formula (2.4) is valid.

Proof. (a) Set $G(x,h) = \int_G K'_u(\cdot, t, x(t))h(t)dt$, $\omega(x,h) = F(x+h) - Fx - G(x,h)$ for $x, h \in L(G)$. It is easy to verify that F maps $L(G)$ into $L_q(G)$, G maps $L(G) \times L(G)$ into $L_q(G)$. Since

(2.10) $$\| G(x,h)\|_{L_r} \leq \left(\int_G |M(s)|^q ds\right)^{\frac{1}{q}} \cdot \|h\|_L ,$$

$G(x, \cdot)$ ($x \in L(G)$ is fixed) is a linear bounded mapping from $L(G)$ into $L_q(G)$.

(b) Let x, $h \in L(G)$ be fixed points, $\tau \neq 0$ an arbitrary number. Then there exists a function ϑ defined on $G \times G$ such that

$$\left|\frac{1}{\tau}\omega(x,\tau h)(s)\right| \leq \int_G |K'_u(s,t,x(t)+\tau\vartheta(s,t)h(t)) -$$
$$- K'_u(s,t,x(t))| \cdot |h(t)|\, dt$$

for almost every $s \in G$ and $0 < \vartheta(s,t) < 1$ for almost every $s, t \in G$. From the continuity of $K'_u(s,t,\cdot)$ it follows that

$$|K'_u(s,t,x(t)+\tau\vartheta(s,t)h(t)) - K'_u(s,t,x(t))| \cdot |h(t)| \xrightarrow{\tau\to 0} 0$$

for almost every $s, t \in G$. The inequality (2.9) implies

(2.11) $$|K'_u(s,t,x(t)+\tau\vartheta(s,t)h(t)) - K'_u(s,t,x(t))| \cdot$$
$$\cdot |h(t)| \leq 2M(s) \cdot |h(t)|$$

for almost every $s, t \in G$. According to lemma 1,

$$\left|\frac{1}{\tau} \cdot \omega(x,\tau h)(s)\right| \to 0 \quad \text{whenever} \quad \tau \to 0$$

for almost every $s \in G$. From (2.11) it follows for almost every $s \in G$ that

$$\left| \frac{1}{\tau} \cdot \omega(x, \tau h)(s) \right| \leq 2\, M(s) \cdot \| h \|_L \, .$$

By lemma 1,

$$\left\| \frac{1}{\tau} \cdot \omega(x, \tau h) \right\|_{L_q} \longrightarrow 0 \text{ whenever } \tau \longrightarrow 0 \, ;$$

we have proved that $G(x, \cdot)$ is the Gâteaux derivative of the mapping F at the point $x \in L(G)$.

(c) Set $F'(x) = G(x, \cdot) \in [L(G) \to L_q(G)]$ for every $x \in L(G)$. From (2.10) we have that

$$\| F'(x) \|_{[L \to L_q]} \leq \| M \|_{L_q} \, ;$$

hence, the Gâteaux derivative F' is bounded on all the space $L(G)$. Thus F is Lipschitzian on $L(G)$.

Theorem 3. Let K be a U_L-function on $G \times G \times E_1$, F the Urysohn operator generated by the function K. Let there exist $K''_{u^2}(s, t, u)$ for almost every $s, t \in G$ and every $u \in E_1$. Suppose that K'_u and K''_{u^2} are U_L-functions on $G \times G \times E_1$, $p > 2$, $q \geq 1$,

$$\int_G | K(\cdot, t, 0) | \, dt \in L_q(G), \quad \left(\int_G | K'_u(\cdot, t, 0) |^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \in L_q(G) \, .$$

If there exist an integer n, numbers λ_i $(i = 1, \ldots, n)$, a function M_0 on G and functions M_i on $G \times G$ $(i = 1, \ldots, n)$ such that

$$M_0 \in L_q(G), \quad \left(\int_G | M_i(\cdot, t) |^{\frac{p}{p - \lambda_i}} dt \right)^{\frac{p - \lambda_i}{p}} \in L_q(G), \quad 2 \leq \lambda_i < p$$

and

$$| K''_{u^2}(s, t, u) | \leq \sum_{i=1}^{n} M_i(s, t) \cdot |u|^{\lambda_i - 2} + M_0(s) \cdot |u|^{p-2}$$

for almost every $s, t \in G$ and every $u \in E_1$, then the following assertions are valid:

(α) F is a mapping of $L_p(G)$ into $L_q(G)$, Lipschitzian on every bounded set in $L_p(G)$.

(β) F has the Fréchet derivative F' on all the space $L_p(G)$; F' is Lipschitzian on every bounded set.

(γ) F possesses the continuous bounded (on every bounded set) FF-derivative F'' on all $L_p(G)$.

(δ) If $x, h, k \in L_p(G)$ then

$$F'(x)h(s) = \int_G K'_u(s,t,x(t))\, h(t)\, dt ,$$

$$F''(x)(h,k)(s) = \int_G K''_{u^2}(s,t,x(t))\, h(t)\, k(t)\, dt$$

for almost every $s \in G$.

<u>Proof</u>. (a) Let K be a function satisfying the conditions of our theorem. It is easy to verify (by developing of $K(s,t,\cdot)$ at the point $[s,t,0]$) that K satisfies all the conditions of the theorem 1 ; hence, the assertion (α) and the existence of the Fréchet derivative F' on $L_p(G)$ are guaranteed.

(b) Set $H(x,h,k)(s) = \int_G K''_{u^2}(s,t,x(t))\, h(t)\, k(t)\, dt$. Then H maps $L_p(G) \times L_p(G) \times L_p(G)$ into $L_q(G)$ and $H(x,\cdot,\cdot)$ ($x \in L_p(G)$ is fixed) is a bounded bilinear operation. Set $\omega_2(x,k) = F'(x+k) - F'(x) - H(x,\cdot,k) \in [L_p(G) \to L_q(G)]$, $k_s x = K''_{u^2}(s,\cdot,x(\cdot))$ ($s \in G$). By the theorem of M.A. Krasnoselskij ([3],chapt.I,§ 2), k_s is a continuous bounded (on every bounded set) operation from $L_p(G)$ into $L_{\frac{p}{p-2}}(G)$ for almost every $s \in G$. Therefore for almost every $s \in G$ the term

$$\frac{1}{\|k\|_{L_p}} \cdot |\omega_2(x,k)h(s)| = \frac{1}{\|k\|_{L_p}} \cdot \Big|\int_G [K''_{u^2}(s,t,x(t)+\vartheta(s,t)k(t)) -$$

$$- K''_{u^2}(s,t,x(t))]\, h(t)\, k(t)\, dt\Big| \leq$$

$$\leq \|k_s(x+\vartheta(s,\cdot)k) - k_s x\|_{L_{\frac{p}{p-2}}} \cdot \|h\|_{L_p} \qquad (0 < \vartheta(s,t) < 1)$$

vanishes uniformly with respect to $h \in L_p(G)$, $\|h\|_{L_p} \le 1$ as $\|k\|_{L_p} \to 0$. If $\|h\|_{L_p} \le 1$, $\|k\|_{L_p} \le 1$ then (similarly as in (2.8))

$$\frac{1}{\|k\|_{L_p}} \cdot |\omega_2(x,k)h(s)| \le \sum_{i=1}^{n} \left(\int_G |M_i(s,t)|^{\frac{p}{p-r_i}} dt\right)^{\frac{p-r_i}{p}} \cdot [\,\|x\|_{L_p} +$$

$$+1)^{r_i-2} + \|x\|_{L_p}^{r_i-2}] + |M_0(s)| \cdot [(\|x\|_{L_p}+1)^{p-2} + \|x\|_{L_p}^{p-2}] \,.$$

Hence according to lemma 1,

$$\frac{1}{\|k\|_{L_p}} \cdot \|\omega_2(x,k)\|_{[L_p \to L_q]} = \sup_{\|h\|_{L_p} \le 1} \frac{1}{\|k\|_{L_p}} \cdot \|\omega_2(x,k)h\|_{L_q} \to 0$$

whenever $\|k\|_{L_p} \to 0$. This proves the existence of an FF -derivative F'' of F on $L_p(G)$.

(c) The proof that F'' is a continuous and bounded (on every bounded set) mapping from $L_p(G)$ into $[L_p(G) \to L_q(G)]$ is similar to those of (c),(d) of the theorem 1. Hence, F' is Lipschitzian on every bounded set in $L_p(G)$ which completes the proof.

<u>Theorem 4</u>. Let K be a U_L -function on $G \times G \times E_1$, F the Urysohn operator generated by the function K. Let there exist $K''_{u^2}(s,t,u)$ for almost every $s,t \in G$ and every $u \in E_1$; suppose that K'_u and K''_{u^2} are U_L -functions on $G \times G \times E_1$, $q \ge 1$, $\int_G |K(\cdot,t,0)|dt \in L_q(G)$, $(\int_G |K'_u(\cdot,t,0)|^2 dt)^{\frac{1}{2}} \in L_q(G)$. Assume there is a function $M \in L_q(G)$ such that

$$|K''_{u^2}(s,t,u)| \le M(s)$$

for almost every $s,t \in G$ and every $u \in E_1$. Then F is the Lipschitzian operation from $L_2(G)$ into $L_q(G)$ having the Lipschitzian Fréchet derivative F' and the bounded (on all the space $L_2(G)$) FG -derivative F'' on the space $L_2(G)$. Moreover, the assertion (σ) of the theorem 3 is valid for every $x, h, k \in L_2(G)$.

Proof. (a) Since K satisfies the conditions of theorem 1, F is a mapping of $L_2(G)$ into $L_q(G)$ having the Fréchet derivative F' on all the space $L_2(G)$. The relation (2.4) is valid for every $x, h \in L_2(G)$.

(b) Set $H(x,h,k) = \int_G K''_{u^2}(s,t,x(t))h(t)k(t)\,dt$,

$\omega_2(x,k) = F'(x+k) - F'(x) - H(x,\cdot,k)$. By the Hölder inequality

$$|H(x,h,k)(s)| \le |M(s)| \cdot \|h\|_{L_2} \cdot \|k\|_{L_2}$$

for almost every $s \in G$. Hence $H(x,\cdot,\cdot) \in [L_2(G) \times L_2(G) \to \to L_q(G)]$ and $\omega_2(x,k) \in [L_2(G) \to L_q(G)]$, evidently.

Let $x, h, k \in L_2(G)$ be fixed points, $\tau \neq 0$ an arbitrary number. For almost every $s, t \in G$

$$|K''_{u^2}(s,t,x(t)+\tau\vartheta(s,t)k(t)) - K''_{u^2}(s,t,x(t))| \cdot |k(t)| \to 0 \text{ if } \tau \to 0,$$

$$|K''_{u^2}(s,t,x(t)+\tau\vartheta(s,t)k(t)) - K''_{u^2}(s,t,x(t))| \cdot |k(t)| \le 2M(s) \cdot |k(t)|.$$

Hence by lemma 1, for almost every $s \in G$

$$\left|\frac{1}{\tau}\omega_2(x,\tau k)h(s)\right| \le \left(\int_G |K''_{u^2}(s,t,x(t)+\tau\vartheta(s,t)k(t)) - \right.$$

$$\left. - K''_{u^2}(s,t,x(t))|^2 \cdot |k(t)|^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_G |h(t)|^2 dt\right)^{\frac{1}{2}} \xrightarrow{\tau \to 0} 0$$

uniformly with respect to $h \in L_2(G)$, $\|h\|_{L_2} \le 1$. Since

$$\left|\frac{1}{\tau}\omega_2(x,\tau k)h(s)\right| \le 2M(s)\|k\|_{L_2}$$

for almost every $s \in G$ and $\|h\|_{L_2} \le 1$, the relation

$$\left\|\frac{1}{\tau}\omega_2(x,\tau k)\right\|_{[L_2 \to L_q]} =$$

$$= \sup_{\|h\|_{L_2} \le 1} \left(\int_G \left|\frac{1}{\tau}\omega_2(x,\tau k)h(s)\right|^q ds\right)^{\frac{1}{q}} \to 0$$

holds whenever $\tau \to 0$. Hence $H(x,\cdot,\cdot)$ is the FG-derivative of F at the point $x \in L_2(G)$; denote it by $F''(x)$. The inequality

$$\| H(x,h,k)\|_{L_2} \leq \left(\int_G |M(s)|^2 ds\right)^{\frac{1}{2}} \cdot \|h\|_{L_2} \cdot \|k\|_{L_2}$$

implies that F'' maps $L_2(G)$ onto a bounded set in $[L_2(G) \times L_2(G) \to L_2(G)]$. Hence, both F and F' are Lipschitzian mappings in $L_2(G)$. This concludes the proof.

3. The differentiability of the Urysohn operator in the space $C(G)$

Theorem 5. Let K be a U_C -function on $G \times G \times E_1$, F the Urysohn operator generated by the function K. Let there exist $K'_u(s,t,u)$ for every $s \in G$, $u \in E_1$ and almost every $t \in G$ and be such that K'_u is a U_C -function on $G \times G \times E_1$. Assume that there exist function H_1, $L_1 \in L(G)$ and functions H_2, L_2 on E_1, which are bounded on every bounded interval, such that

$$|K(s,t,u)| \leq H_1(t) \cdot H_2(u), \ |K'_u(s,t,u)| \leq L_1(t) \cdot L_2(u)$$

for every $s \in G, u \in E_1$ and almost every $t \in G$. Then F maps $C(G)$ into $C(G)$ is Lipschitzian on every bounded set and possesses the bounded (on every bounded set) Fréchet derivative F' on all the space $C(G)$. If $x, h \in C(G)$ then

$$F'(x)h(s) = \int_G K'_u(s,t,x(t))\, h(t)\, dt \tag{3.1}$$

for every $s \in G$.

Proof. (a) Let $x \in C(G)$. Then $K(s,\cdot,x(\cdot))$ is a measurable function on G for every $s \in G$ (see [1], theorem 18.3) and

$$|K(s,t,x(t))| \leq H_1(t) \cdot \max_{u \in \langle -\|x\|, +\|x\| \rangle} H_2(u) \tag{3.2}$$

for every $s \in G$ and almost every $t \in G$. Hence $\int_G K(s,t,x(t))dt$ converges for all $s \in G$. Since $K(\cdot,t,x(t))$ is a continuous function on G for almost every $t \in G$, by the lemma 1 we see $\int_G K(\cdot,t,x(t))\,dt$ is a continuous one on G. Hence $Fx \in C(G)$.

It can be proved similarly that the functions $K'_u(s,\cdot,x(\cdot))$, $K'_u(s,\cdot,x(\cdot))\cdot h$ are meaasurable in G for all $s \in G$, $\int_G K'_u(s,t,x(t))\,dt$ and $\int_G K'_u(s,t,x(t))h(t)\,dt$ converge for all $s \in G$ and both are continuous functions of s in G.

Hence, the operator G defined by

$$G(x,h)(s) = \int_G K'_u(s,t,x(t))h(t)dt \quad (x,h \in C(G), s \in G)$$

is a mapping of $C(G)$ into $C(G)$. The mapping $G(x,\cdot)$ $(x \in C(G))$ is linear and bounded, evidently.

(b) Set $\omega(x,h) = F(x+h) - Fx - G(x,h)$ for $x, h \in C(G)$. There exists a function ϑ on $G \times G$ such that

$$K(s,t,x(t)+h(t)) - K(s,t,x(t)) = K'_u(s,t,x(t)+\vartheta(s,t)h(t))h(t)$$

and $0 < \vartheta(s,t) < 1$ for every $s \in G$ and almost every $t \in G$. The left side of this identity and hence also the right one is a measurable function of t in G for every $s \in G$ and a continuous function of s on G for almost every $t \in G$. Now, similarly as above, we can prove that $\int_G K'_u(\cdot,t,x(t)+\vartheta(\cdot,t)h(t))\,dt$ is a continuous function on G.

Setting $\alpha(s,t,u,v) = K'_u(s,t,u+v) - K'_u(s,t,u)$

we have

$$\frac{1}{\|h\|_C}\cdot\|\omega(x,h)\|_C=\frac{1}{\|h\|_C}\cdot\|\int_G\alpha(s,t,x(t),\vartheta(s,t)h(t))h(t)dt\|_C\le$$

$$\le\|\int_G\alpha(s,t,x(t),\vartheta(s,t)h(t))\,dt\|_C\ .$$

Since $\int_G\alpha(\cdot,t,x(t),\vartheta(\cdot,t)h(t))dt$ is a continuous function on G, there exists $s_0\in G$ such that

$$\|\int_G\alpha(s,t,x(t),\vartheta(s,t)h(t))dt\|_C=|\int_G\alpha(s_0,t,x(t),\vartheta(s_0,t)h(t))dt|.$$

From the continuity $K'_u(s,t,\cdot)$ it follows that

$\alpha(s_0,t,x(t),\vartheta(s_0,t)h(t))\to 0$ as $\vartheta(s_0,t)h(t)\to 0$

for almost every $t\in G$. Furthermore,

$$|\alpha(s_0,t,x(t),\vartheta(s_0,t)h(t))|\le 2H_1(t)\cdot\max_{u\in\langle-\|x\|,+\|x\|\rangle}H_2(u)+$$

$$+L_1(t)\cdot\max_{u\in\langle-\|x\|,+\|x\|\rangle}L_2(u)$$

for almost every $t\in G$. If $\|h\|_C\to 0$, then $\vartheta(s_0,t)h(t)\to 0$ for all $t\in G$ and according to lemma 1

$$|\int_G\alpha(s_0,t,x(t),\vartheta(s_0,t)h(t))dt|\to 0\ .$$

Hence

$$\frac{1}{\|h\|_C}\cdot\|\omega(x,h)\|_C\le|\int_G\alpha(s_0,t,x(t),\vartheta(s_0,t)h(t))dt|\to 0$$

whenever $\|h\|_C\to 0$. This proves that $G(x,\cdot)$ is the Fréchet derivative of F at the point $x\in C(G)$. Denoting $G(x,\cdot)$ by $F'(x)$ we see the formula (3.1) holds for any $s\in G$.

(c) Suppose $R>0$; then for every $x\in C(G)$, $\|x\|_C\le R$

$$\|F'(x)\|_{[C\to C]}=\sup_{\|h\|_C\le 1}\|F'(x)h\|_C=$$

$$=\sup_{\|h\|_C\le 1}\max_{s\in G}|\int_G K'_u(s,t,x(t))h(t)dt|\le$$

$$\leq \sup_{\|h\|_C \leq 1} \max_{s \in G} \left[\int_G L_1(t) \cdot \max_{u \in \langle -\|x\|, +\|x\| \rangle} L_2(u)\, dt \cdot \|h\|_C \right] \leq$$

$$\leq \int_G L_1(t)\, dt \cdot \max_{u \in \langle -R, +R \rangle} L_2(u) \ .$$

We have proved that F' is bounded on every bounded set in $C(G)$; hence, F is Lipschitzian on every bounded set. This completes the proof.

Theorem 6. Let K be a U_C-function on $G \times G \times E_1$, F the Urysohn operator generated by the function K. Let there exist $K'_u(s, t, u)$ for every $s \in G$, $u \in E_1$ and almost every $t \in G$, having the property that K'_u is a U_C-function on $G \times G \times E_1$. Suppose there exist functions H, L which are defined on $G \times E_1$ and bounded on every bounded set, such that

$$|K(s, t, u)| \leq H(t, u), \quad |K'_u(s, t, u)| \leq L(t, u)$$

for every $s \in G$, $u \in E_1$ and almost every $t \in G$. Then F maps $C(G)$ into $C(G)$, is Lipschitzian on every bounded set and possesses the bounded (on every bounded set) Fréchet derivative F' on all the space $C(G)$. If x, $h \in C(G)$, then (3.1) holds for every $s \in G$.

The proof of this theorem is similar to the one of theorem 5; in the estimates of the type (3.2), one has to write " $\max_{[t,u] \in G \times \langle -\|x\|, +\|x\| \rangle} H(t, u)$ " instead of " $H_1(t) \cdot$

$\cdot \max_{u \in \langle -\|x\|, +\|x\| \rangle} H_2(u)$ "

Theorem 7. Let K be a U_C-function on $G \times G \times J$, $J = \langle -R, +R \rangle$, $R > 0$, F the Urysohn operator generated by the function K. Let there exist $K'_u(s, t, u)$

for every $s \in G, u \in J$ and almost every $t \in G$, having the property that K'_u is a U_C-function on $G \times G \times J$. Assume there are functions $H, L \in L(G)$ such that

$$|K(s,t,u)| \leq H(t), \quad |K'_u(s,t,u)| \leq L(t)$$

for every $s \in G, u \in J$ and almost every $t \in G$. Then F is the Lipschitzian operation from $D_R = \{x \in C(G): \|x\|_C \leq R\}$ into $C(G)$ having the bounded Fréchet derivative F' on $D^o_R = \{x \in C(G): \|x\|_C < R\}$. If $x \in D^o_R$, $h \in C(G)$, then the formula (3.1) is valid for all $s \in G$.

The proof of our theorem is almost similar to that of theorem 5.

Similar theorems can be derived for the second derivative of the Urysohn operator, too. The principle of their proofs remain the same and so I present these theorems withouht their proofs.

<u>Theorem 8</u>. Let K be a U_C-function on $G \times G \times E_1$, F the Urysohn operator generated by the function K. Suppose $K''_{u^2}(s,t,u)$ exists for every $s \in G, u \in E_1$ and almost every $t \in G$; let both K'_u and K''_{u^2} be U_C-functions on $G \times G \times E_1$.

Assume there are functions $H_1, L_1, M_1 \in L(G)$ and functions H_2, L_2, M_2 defined on E_1 and bounded on every bounded interval such that

$$|K(s,t,u)| \leq H_1(t) \cdot H_2(u), \quad |K'_u(s,t,u)| \leq L_1(t) \cdot L_2(u),$$

$$|K''_{u^2}(s,t,u)| \leq M_1(t) \cdot M_2(u)$$

for every $s \in G, u \in E_1$ and almost every $t \in G$. Then F is the operator from $C(G)$ into itself, Lipschitzian on every bounded set and having on $C(G)$ the Fréchet derivative F' which is Lipschitzian on every bounded set, too.

Moreover, F possesses the bounded (on every bounded set) FF -derivative F'' on all $C(G)$ and

(3.3) $$F'(x)h(s) = \int_G K'_u(s,t,x(t))h(t)\,dt,$$

$$F''(x)(h,k)(s) = \int_G K''_{u^2}(s,t,x(t))h(t)\,dt$$

for every $s \in G$, $x, h, k \in C(G)$.

<u>Theorem 9</u>. Let K be a U_c -function on $G \times G \times E_1$, F the Urysohn operator generated by the function K. Let there exist $K''_{u^2}(s,t,u)$ for every $s \in G$, $u \in E_1$ and almost every $t \in G$; suppose K'_u, K''_{u^2} are U_c -functions on $G \times G \times E_1$. If there exist functions H, L, M, defined on $G \times E_1$ and bounded on every bounded set, such that

$$|K(s,t,u)| \leq H(t,u), \quad |K'_u(s,t,u)| \leq L(t,u),$$

$$|K''_{u^2}(s,t,u)| \leq M(t,u)$$

for every $s \in G$, $u \in E_1$ and almost every $t \in G$, then F is the mapping of $C(G)$ into itself Lipschitzian on every bounded set. The operator F possesses the Fréchet derivative F' on $C(G)$ Lipschitzian on every bounded set, and the bounded (on every bounded set) FF -derivative F'' on $C(G)$. The formula (3.3) is valid for all $s \in G$ and $x, h, k \in C(G)$.

<u>Theorem 10</u>. Let K be a U_c -function on $G \times G \times J$, $J = \langle -R, +R \rangle$, $R > 0$, F the Urysohn operator generated by the function K. Let there exist $K''_{u^2}(s,t,u)$ for every $s \in G$, $u \in J$ and almost every $t \in G$; suppose that both K'_u and K''_{u^2} are U_c -functions on $G \times G \times J$. If there exist functions H, L, $M \in L(G)$ such

that

$|K(s,t,u)| \le H(t)$, $|K'_u(s,t,u)| \le L(t)$, $|K''_{u^2}(s,t,u)| \le M(t)$

for every $s \in G$, $u \in J$ and almost every $t \in G$, then F is the Lipschitzian operation from $D_R = \{x \in C(G) : \|x\|_C \le R\}$ into $C(G)$ having the Lipschitzian Fréchet derivative F' and the bounded FF-derivative F'' on the open set $D_R^o = \{x \in C(G) : \|x\|_C < R\}$. Moreover, the formula (3.3) is valid for all $s \in G$ whenever $x \in D_R^o$ and $h, k \in C(G)$.

4. The differentiability of the Nemyckij operator in the spaces $L_p(G)$, $p \ge 2$

Unless otherwise stated, G denotes a measurable bounded subset of E_n. By θ there is meant the zero-element of $L_2(G)$.

Theorem 11. Let g be a function of two variables on $G \times E_1$, $D_R = \{x \in L_2(G) : \|x\|_{L_2} < R\}$, h an operator defined by $hx(t) = g(t, x(t))$ for almost every $t \in G$ and every $x \in D_R$ $(0 < R \le \infty)$. If h maps D_R into $L_2(G)$ and

$$(4.1) \qquad \lim_{\|x\|_{L_2} \to 0} \frac{\|hx\|_{L_2}}{\|x\|_{L_2}} = 0 ,$$

then h is the zero-operator on $D_R - \{\theta\}$.

Proof (The proof depends on the arguments of M.M. Vajnberg [2,pp.91-92]). Suppose there is a function $x_0 \in D_R - \{\theta\}$ such that $hx_0(t) \ne 0$ for all t of some set with a positive measure. Then there exist $a > 0$ and $F \subset G$ with $mes\, F > 0$ such that

(4.2) $$|hx_0(t)| \geq a$$

for almost every $t \in F$. According to the well-known theorem (see [3],p.92) there exist a number $b > 0$ and a set $E \subset F$ with $mes\, E > 0$ such that

(4.3) $$|x_0(t)| \leq b$$

for all $t \in E$. Since $mes\, E > 0$, one may find a point $t_0 \in E$ such that if U is an arbitrary neighbourhood of t_0, there exists a subset of E with a positive measure lying in U (in the opposite case we obtain a contradiction the Borel-Lebesgue covering theorem).

Now, let $D(t_0, r)$ denote the ball in E_n with the centre in t_0 and the radius r and put $A_k = E \cap D(t_0, k^{-1})$ for $k = 1, 2, \ldots$. Define the sequence $\{x_k\}$ by

$$x_k(t) = 0 \quad \text{if } t \in G - A_k, \qquad x_k(t) = x_0(t) \quad \text{if } t \in A_k .$$

The absolute continuity of an integral implies that

$$\| x_k \|_{L_2} \longrightarrow 0 \quad \text{if} \quad k \to 0 .$$

From (4.2) and (4.3) it follows that

$$\frac{\| hx_k \|_{L_2}}{\| x_k \|_{L_2}} = \frac{(\int |hx_k(t)|^2 dt)^{\frac{1}{2}}}{(\int_{A_k} |x_0(t)|^2 dt)^{\frac{1}{2}}} \geq \frac{(\int_{A_k} |q(t, x_k(t))|^2 dt)^{\frac{1}{2}}}{(\int_{A_k} |x_0(t)|^2 dt)^{\frac{1}{2}}} =$$

$$= \frac{(\int_{A_k} |q(t, x_0(t))|^2 dt)^{\frac{1}{2}}}{(\int_{A_k} |x_0(t)|^2 dt)^{\frac{1}{2}}} \geq \frac{a \sqrt{mes\, A_k}}{b \sqrt{mes\, A_k}} = \frac{a}{b} .$$

But this contradicts (4.1). Hence $hx = \theta$ for all $x \in D_R - \{\theta\}$; this concludes the proof.

Using this theorem it is easy to prove the following

<u>Theorem 12</u>. Let q be an N-function on $G \times E_1$,

$x_0 \in L_2(G)$. Suppose that an operator of Nemyckij h is generated by the function g and maps a neighbourhood of the point x_0 in $L_2(G)$ into the space $L_2(G)$. Assume $b \in L_2(G)$. Then the operator h possesses at x_0 the Fréchet differential

$$(4.4) \qquad dh(x_0, x) = b \cdot x \quad (x \in L_2(G))$$

($b \cdot x$ denotes the point-product of the functions b and x), if and only if there is a function a on G such that

$$(4.5) \qquad g(t, u) = a(t) + u \cdot b(t)$$

for almost every $t \in G$ and every $u \in E_1$.

<u>Proof</u> 1) Suppose g satisfies the assumptions of our theorem and (4.5) holds. Set

$$H(x_0, x) = x \cdot b, \; \omega(x_0, x) = h(x_0 + x) - hx_0 - H(x_0, x)$$

for $x \in L_2(G)$; we have $\omega(x_0, x) = \theta$ as $\overset{\circ}{x} \in L_2(G)$. Hence $H(x_0, \cdot)$ is the Fréchet differential of h at the point x_0.

2) Let $b \in L_2(G)$ and suppose that h possesses the Fréchet differential at the point x_0 with the form (4.4). Put

$$\omega_{x_0}(x) = h(x_0 + x) - hx_0 - dh(x_0, x)$$

for $x \in L_2(G)$. The operator ω_{x_0} satisfies all the assumptions of theorem 11 (with $R = \infty$). Hence $\omega_{x_0}(x) = \theta$ for all $x \in L_2(G)$ and

$$g(t, x_0(t) + x(t)) = g(t, x_0(t)) + dh(x_0, x)(t) =$$

$$= g(t, x_0(t)) + b(t)\, x(t) =$$

$= [g(t, x_0(t)) - b(t)x_0(t)] + [x_0(t) + x(t)] \cdot b(t)$

for almost every $t \in G$ whenever $x \in L_2(G)$. But this is equivalent to (4.5).

The following theorem is a local form of theorem 20.2 [1,§ 20]; its proof is quite similar to that one.

Theorem 13. Let g be an N-function on $G \times E_1$, $x_0 \in L_2(G)$. Suppose $g'_u(t, u)$ exists and is bounded for almost every $t \in G$ and every $u \in U$ where U is some neighbourhood of the set $\{x_0(t) : t \in G\}$ in E_1. Assume $g'_u(t, \cdot)$ is continuous in the points $x_0(t)$ for almost every $t \in G$. If h is the operator of Nemyckij generated by the function g, then h maps a neighbourhood of x_0 in $L_2(G)$ into the space $L_2(G)$. Moreover, the operator h possesses at x_0 the Gâteaux derivative $h'x_0$ and $h'x_0(y) = g'_u(\cdot, x_0) \cdot y \quad (y \in L_2(G))$.

Now, we shall state some theorems concerning the first Fréchet derivative of the Nemyckij operator in the spaces $L_p(G)$ $(p > 2)$. There is a theorem 20.1 in [1] giving a sufficient condition for the existence of the Gâteaux differential of that operator; the following theorem shows this condition is also sufficient for the existence of a bounded continuous Fréchet derivative.

Theorem 14. Let g be an N-function on $G \times E_1$, suppose $g'_u(t, u)$ exists for almost every $t \in G$ and every $u \in E_1$. Denote by h the Nemyckij operator generated by the function g, by h_1 the operator defined by

(4.6) $$h_1 x = g'_u(\cdot, x) \quad (x \in L_p(G)).$$

Assume $p > 2$ and put $q = \frac{p}{p-1}$, $r = \frac{p}{p-2}$; suppose

h_1 is a continuous mapping from $L_p(G)$ into $L_q(G)$. Then h is the operation from $L_p(G)$ into $L_q(G)$, Lipschitzian on every bounded set and having on $L_p(G)$ the continuous Fréchet derivative $h': L_p \to [L_p \to L_q]$; the operation h' is bounded on every bounded set and $h'x(y) = (h_1 x)\cdot y, \quad x, y \in L_p(G)$.

The proof of this theorem can be omitted. Its principle will be evident from the proof of theorem 16. For completeness, we present a lemma which is used in a verification of theorem 14:

<u>Lemma 3</u>. Let g be an N-function on $G \times E_1$. Suppose $g'_u(t, u)$ exists for almost every $t \in G$ and every $u \in E_1$ and define the operators h, h_1 and H for $x, y \in L_p(G)$ by

$$h\,x(t) = g(t, x(t)), \quad h_1 x(t) = g'_u(t, x(t)),$$

$$H(x, y)(t) = g'_u(t, x(t))\cdot y(t) \qquad (t \in G).$$

Assume $p > 2$ and set $q = \frac{p}{p-1}$, $r = \frac{p}{p-2}$. If h_1 is a continuous mapping from $L_p(G)$ into $L_r(G)$, then:

(a) h is a continuous mapping of $L_p(G)$ into $L_q(G)$, Lipschitzian on each closed ball $D_R = \{x \in L_p(G) : \|x\|_{L_p} \leq R\}$.

(b) $H(x, \cdot)$ ($x \in L_p(G)$ is fixed) is a continuous linear operation from $L_p(G)$ into $L_q(G)$; $H(\cdot, y)$ is a continuous bounded (on every bounded set) operation from $L_p(G)$ into $L_q(G)$.

Similarly as theorem 14, we can prove the following local

Theorem 15. Let g be an N-function on $G \times E_1$, suppose $g'_u(t,u)$ exists for almost every $t \in G$ and every $u \in E_1$. Assume $p > 2$, $x_0 \in L_p(G)$, U is a neighbourhood of x_0 in $L_p(G)$. Let h be the operator of Nemyckij generated on U by the function g, h_1 the operator defined on U by (4.6). Put $q = \frac{p}{p-1}$, $r = \frac{p}{p-2}$; assume h is a mapping from U into $L_q(G)$ and h_1 is a mapping from U into $L_r(G)$. Then the following assertions are valid:

(a) If the mapping h_1 is continuous at x_0, then the operator h has the Fréchet derivative $h'x_0 \in [L_p(G) \to L_q(G)]$ at the point x_0 and $h'x_0(y) = (h_1 x_0)\cdot y$ for every $y \in L_p(G)$.

(b) If h_1 is continuous on the set U and if V is an open convex neighbourhood of x_0 in $L_p(G)$, $V \subset U$, then the operator h possesses the continuous bounded (on every bounded set) Fréchet derivative $h' : V \to [L_p(G) \to L_q(G)]$ and $h'x(y) = (h_1 x)\cdot y$, $x \in V$, $y \in L_p(G)$.

Now, we shall deal with the second derivatives of the Nemyckij operators in the spaces $L_p(G)$ $(p \geq 2)$.

Lemma 4. Let g be an N-function on $G \times E_1$, suppose $g''_{u^2}(t,u)$ exists for almost every $t \in G$ and every $u \in E_1$. Let h, h_1, h_2, H and K be the operators defined by

$$h x(t) = g(t, x(t)), \qquad h_1 x(t) = g'_u(t, x(t)),$$

$$h_2 x(t) = g''_{u^2}(t, x(t)), \qquad H(x,y)(t) = h_1 x(t) \cdot y(t),$$

$$K(x,y,z)(t)=h_2x(t)\cdot y(t)\cdot z(t), \quad x,y,z\in L_p(G).$$

Assume $p>3$ and set $q=\frac{p}{p-1}$, $r=\frac{p}{p-2}$, $s=\frac{p}{p-3}$.

Let h_2 be a continuous mapping of $L_p(G)$ into $L_s(G)$; then:

(a) h is a continuous operation from $L_p(G)$ into $L_q(G)$, Lipschitzian on each closed ball $D_R = \{x\in L_p(G):\ \|x\|_{L_p}\le R\}$.

(b) h_1 is a continuous operation from $L_p(G)$ into $L_r(G)$, Lipschitzian on each closed ball D_R.

(c) $H(x,\cdot)$ $(x\in L_p(G))$ is a continuous linear operation from $L_p(G)$ into $L_q(G)$, $H(\cdot,y)$ $(y\in L_p(G))$ is a continuous bounded (on every bounded set) operation from $L_p(G)$ into $L_q(G)$.

(d) $K(x,\cdot,\cdot)$ $(x\in L_p(G))$ is a continuous bilinear operation from $L_p(G)\times L_p(G)$ into $L_q(G)$, $K(\cdot,y,z)$ $(y,z\in L_p(G))$ is a continuous bounded (on every bounded set) operation from $L_p(G)$ into $L_q(G)$.

Proof. ad (b) Let x, y be the arbitrary elements of $L_p(G)$. The function g'_u is an N-function on $G\times E_1$. A continuity of $g'_u(t,\cdot)$ for almost every $t\in G$ follows from the existence $g''_{u^2}(t,u)$ for almost every $t\in G$ and every $u\in E_1$; that $g'_u(\cdot,u)$ is measurable for all $u\in E_1$, it follows from the measurability of $g(\cdot,u)$ for all $u\in E_1$. Hence, both the function hx and h_1x are measurable on G. A continuity of h_2 implies (see [1],th.19.1) the function

$g''_{u^2}(\cdot, t)$ is bounded on $\langle x(t), y(t)\rangle$ for almost every $t \in G$ and hence [11]

$$h_1 x - h_1 y = (x-y)\cdot \int_0^1 h_2(y+\tau(x-y))\,d\tau .$$

Using the Hölder inequality (with exponents $p-2, \frac{p-2}{p-3}$)

we have

$$\int_G |h_1 x(t) - h_1 y(t)|^p dt = \int_G |(x(t)-y(t))\cdot \int_0^1 h_2(y+\tau(x-y))(t)\,d\tau|^p dt \le$$

$$\le \int_G [\,|x(t)-y(t)|^p \cdot (\int_0^1 |g''_{u^2}(t, y(t)+\tau(x(t)-y(t)))|\,d\tau)^p]\,dt \le$$

$$\le (\int_G |x(t)-y(t)|^p dt)^{\frac{1}{p-2}} \cdot$$

$$\cdot [\int_G (\int_0^1 |g''_{u^2}(t, y(t)+\tau(x(t)-y(t)))|^{\frac{p}{p-2}} d\tau)^{\frac{p-2}{p-3}} dt]^{\frac{p-3}{p-2}} \le$$

$$\le \|x-y\|_{L_p}^{\frac{p}{p-2}} \cdot [\int_G \int_0^1 |g''_{u^2}(t, y(t)+\tau(x(t)-$$

$$-y(t)))|^{\frac{p}{p-3}} d\tau\, dt]^{\frac{p-3}{p-2}} =$$

$$= \|x-y\|_{L_p}^{p} \cdot [\int_0^1 (\int_G |g''_{u^2}(t, y(t)+\tau(x(t)-y(t)))|^{s} dt)\,d\tau]^{\frac{1}{s}\cdot p} .$$

Since $h_2 : L_p(G) \to L_s(G)$ is continuous, then there exist $a, b \ge 0$ such that [1,th.19.1]

$$\| h_2(y+\tau(x-y))\|_{L_s} \le a + b\cdot \| y+\tau(x-y)\|_{L_p} .$$

Suppose $x, y \in D_R$ $(R > 0)$; then $\| y+\tau(x-y)\|_{L_p} \le R$ $(0 \le \tau \le 1)$.

Hence, there is a function c which depends on R, only, such that

$$[\int_G |g''_{u^2}(t, y(t)+\tau(x(t)-y(t)))|^s dt]^{\frac{1}{s}} \le c(R) .$$

Therefore,

$$\| h_1 x - h_1 y \|_{L_n} \leq \| x - y \|_{L_p} \cdot c(R)$$

and so h_1 is Lipschitzian on D_R and continuous on $L_p(G)$.

The assertions (a) and (c) follow immediately from the assertion (b) and from lemma 3.

ad (d) Let $x, y, z \in L_p(G)$; according to the elementary Hölder inequality we obtain

$$|K(x,y,z)(t)|^q = |g''_{u^2}(t, x(t))|^q \cdot |y(t)|^q \cdot |z(t)|^q \leq$$

$$\leq \frac{1}{p-1} \cdot |y(t)|^p + \frac{1}{p-1} \cdot |z(t)|^p + \frac{p-3}{p-1} \cdot |g''_{u^2}(t, x(t))|^s .$$

Hence

$$\int_G |K(x,y,z)(t)|^q dt \leq \frac{1}{p-1} \cdot$$

$$\cdot [\| y \|^p_{L_p} + \| z \|^p_{L_p} + (p-3) \| h_2 x \|^s_{L_s}] < \infty ;$$

the function $K(x, y, z)$ is measurable on G, evidently.

Hence $K(x, y, z) \in L_q(G)$.

Let x be a fixed element of $L_p(G)$; then $K(x,\cdot,\cdot)$ is a bilinear mapping of $L_p(G) \times L_p(G)$ into $L_q(G)$. According to the Hölder inequality (with exponents $\frac{p-1}{p-3}$, $p-1$, $p-1$)

$$\| K(x,y,z) \|_{L_q} = [\int_G |g''_{u^2}(t, x(t))|^q \cdot |y(t)|^q \cdot |z(t)|^q dt]^{\frac{1}{q}} \leq$$

$$\leq [\int_G | g''_{u^2}(t, x(t))|^s dt]^{\frac{1}{s}} \cdot [\int_G |y(t)|^p dt]^{\frac{1}{p}} \cdot [\int_G |z(t)|^p dt]^{\frac{1}{p}} \leq \| h_2 x \|_{L_s}$$

for every $y, z \in L_p(G)$ with $\| y \|_{L_p} \leq 1$, $\| z \|_{L_p} \leq 1$.

This relation proves a boundedness and hence also a continuity of the mapping $K(x,\cdot,\cdot)$ on $L_p(G)\times L_p(G)$.

Let $y, z \in L_p(G)$ be fixed. According to the Hölder inequality

$$\|K(x,y,z)\|_{L_q} \le \|h_2 x\|_{L_s} \cdot \|y\|_{L_p} \cdot \|z\|_{L_p}$$

for every $x \in L_p(G)$. Since h_2 is a continuous mapping from $L_p(G)$ into $L_s(G)$, there is a constant $c(R)$ for every $R>0$ such that $\|h_2 x\|_{L_s} \le c(R)$ whenever $\|x\|_{L_p} \le R$ (see corollary 19.1 in [1]). This proves a boundedness of $K(\cdot,y,z): L_p(G) \to L_q(G)$. If $x_1, x_2 \in L_p(G)$, then

$$\|K(x_1,y,z)-K(x_2,y,z)\|_{L_q} \le \|h_2 x_1 - h_2 x_2\|_{L_s} \cdot \|y\|_{L_p} \cdot \|z\|_{L_p}.$$

So a contimuity of the mapping h_2 implies a continuity of $K(\cdot,y,z)$, which completes the proof.

Theorem 16. Let g be an N-function on $G\times E_1$. Suppose $g''_{u^2}(t,u)$ exists for almost every $t\in G$ and every $u \in E_1$; let h be the Nemyckij operator generated by the function g, h_1 and h_2 the operators defined by

$$(4.7)\qquad h_1 x(t) = g'_u(t,x(t)),\ h_2 x(t) = g''_{u^2}(t,x(t)).$$

Assume $p>3$ and set $q=\frac{p}{p-1}$, $s=\frac{p}{p-3}$. If h_2 is a continuous mapping from $L_p(G)$ into $L_s(G)$, then the following assertions are valid: h is the operation from $L_p(G)$ into $L_q(G)$, Lipschitzian on every bounded set, having the Fréchet derivative $h': L_p(G) \to [L_p(G) \to L_q(G)]$, Lipschitzian on every bounded set, too.

Furthermore, h possesses the continuous bounded (on every bounded set) FF-derivative $h'': L_p(G) \longrightarrow$

$\to [L_p(G)\times L_p(G)\to L_q(G)]$ and $(x, y, z\in L_p(G))$

$$h'x(y)=(h_1x)\cdot y,\quad h''x(y,z)=(h_2x)\cdot y\cdot z\ .$$

Proof. The assertions concerning the mappings h, h' follow from lemma 4 and theorem 14.

Put $K(x,y,z)=h_2x\cdot y\cdot z$, $\omega_2(x,z)=h'(x+z)-h'x-K(x,\cdot,z)$. According to lemma 4, ω_2 is a mapping from $L_p(G)\times L_p(G)$ into $[L_p(G)\to L_q(G)]$. Let $x, z\in L_p(G)$. There is $\vartheta(t)\in(0,1)$ for almost every $Z\in G$ such that

$$g'_u(t,x(t)+z(t))-g'_u(t,x(t))=g''_{u^2}(t,x(t)+\vartheta(t)z(t))\cdot z(t)\,.$$

Hence, according to the Hölder inequality with exponents $\frac{p-1}{p-3}$, $p-1$, $p-1$:

$$\|\omega_2(x,z)\|_{[L_p\to L_q]}=\sup_{\|y\|_{L_p}\le 1}\|\omega_2(x,z)(y)\|_{L_q}=$$

$$=\sup_{\|y\|_{L_p}\le 1}\Big[\int_G|g'_u(t,x(t)+z(t))y(t)-g'_u(t,x(t))y(t)-$$

$$-g''_{u^2}(t,x(t))y(t)z(t)|^q dt\Big]^{\frac{1}{q}}\le$$

$$\le\sup_{\|y\|_{L_p}\le 1}\Big[\int_G|g''_{u^2}(t,x(t)+\vartheta(t)z(t))-g''_{u^2}(t,x(t))|^{\frac{p}{p-3}}dt\Big]^{\frac{p-3}{p}}\cdot$$

$$\cdot\Big[\int_G|y(t)|^r dt\Big]^{\frac{1}{r}}\cdot\Big[\int_G|z(t)|^r dt\Big]^{\frac{1}{r}}=$$

$$=\|h_2(x+\vartheta z)-h_2x\|_{L_s}\cdot\|z\|_{L_p}\ .$$

Let $\varepsilon>0$ be an arbitrary number, $x\in L_p(G)$. There is $\delta>0$ such that $\|h_2(x+\vartheta z)-h_2x\|_{L_s}<\varepsilon$ whenever $\|x\|_{L_p}<\delta$.

Hence

$$\frac{\|\omega_2(x,z)\|_{[L_p\to L_q]}}{\|z\|_{L_p}}\le\|h_2(x+\vartheta z)-h_2x\|_{L_s}<\varepsilon$$

for all $z \in L_p(G)$ with $\|z\|_{L_p} < \sigma$. This means that $d\,d\,h(x,y,z) = K(x,y,z)$. According to lemma 4, $K(x,\cdot,\cdot) \in [L_p \times L_p \to L_q]$, which proves the existence of the FF-derivative of h at every point $x \in L_p(G)$.

Denote this derivative by h''. Let x_1, x_2 be any elements of $L_p(G)$, then

$$\|h''x_1 - h''x_2\|_{[L_p \times L_p \to L_q]} =$$

$$= \sup_{\|y\|_{L_p} \le 1, \|z\|_{L_p} \le 1} \|h''x_1(y,z) - h''x_2(y,z)\|_{L_q} =$$

$$= \sup_{\|y\|_{L_p} \le 1, \|z\|_{L_p} \le 1} \Big[\int_G |g''_{u^2}(t, x_1(t))y(t)z(t) -$$

$$- g''_{u^2}(t, x_2(t))y(t)z(t)|^q dt\Big]^{\frac{1}{q}} \le$$

$$\le \sup_{\|y\|_{L_p} \le 1, \|z\|_{L_p} \le 1} \Big[\int_G |g''_{u^2}(t,x_1(t)) - g''_{u^2}(t,x_2(t))|^{\frac{p}{p-3}} dt\Big]^{\frac{p-3}{p}} \cdot$$

$$\cdot \Big[\int_G |y(t)|^p dt\Big]^{\frac{1}{p}} \cdot \Big[\int_G |z(t)|^p dt\Big]^{\frac{1}{p}} =$$

$$= \|h_2 x_1 - h_2 x_2\|_{L_s},$$

which proves a continuity of h'' on $L_p(G)$ (considered as a mapping of L_p into $[L_p \times L_p \to L_q]$. As h_2 is a continuous operation from $L_p(G)$ into $L_s(G)$, there is a number $c(R)$ $(R > 0)$ such that $\|h_2 x\|_{L_s} \le c(R)$ whenever $\|x\|_{L_p} \le R$ (see corollary 19.1, [1]). Hence

$$\|h''x\|_{[L_p \times L_p \to L_q]} = \sup_{\|y\|_{L_p} \le 1, \|z\|_{L_p} \le 1} \|h''x(y,z)\|_{L_q} \le$$

$$\le \sup_{\|y\|_{L_p} \le 1, \|z\|_{L_p} \le 1} \|h_2 x\|_{L_s} \cdot \|y\|_{L_p} \cdot \|z\|_{L_p} \le c(R)$$

and it proves a boundedness of h'' on each bounded set.

__Theorem 17.__ Let g be an N -function on $G \times E_1$, $x_0 \in L_p(G)$ $(p > 3)$, U a neighbourhood of x_0 in $L_p(G)$. Suppose that $g''_{u^2}(t, u)$ exists for almost every $t \in G$ and every $u \in E_1$. Let h be the Nemyckij operator generated on U by the function g, h_1 and h_2 the operators defined on U by (4.7), Set $q = \frac{p}{p-1}$, $r = \frac{p}{p-2}$, $s = \frac{p}{p-3}$; suppose h_1 is a continuous mapping from U into $L_r(G)$ and h_2 is a mapping from U into $L_s(G)$. If V is an open convex neighbourhood of x_0 into $L_p(G)$ such that $V \subset U$, then the following assertions are valid:

(a) If h_2 is continuous at x_0, then h is a mapping from V into $L_q(G)$, Lipschitzian on every bounded set and having on V the continuous bounded (on every bounded subset of V) Fréchet derivative $h' : V \to$ $\to [L_p(G) \to L_q(G)]$. Moreover, h possesses at the point x_0 the FF -derivative $h'' x_0 \in$ $\in [L_p \times L_p \to L_q]$ and $h' x(y) = (h_1 x) \cdot y$, $(h_2 x_0) y \cdot z =$ $= h'' x_0(y, z)$, $x \in V$, $y, z \in L_p(G)$.

(b) If h_2 is continuous on U, then h' is Lipschitzian on every bounded subset of V and h has the continuous bounded (on every bounded subset of V) FF -derivative h'' on V. Furthermore, $h'' x(y, z) = (h_2 x) \cdot$ $\cdot y \cdot z$ if $x \in V$, $y, z \in L_p(G)$.

The proof is similar to that of theorem 16.

The disadvantage of theorems 16,17 consists in the assumption $p > 3$. The following theorem is valid for $p \geq 2$, but its assertion is weakened.

Theorem 18. Let g be an N-function on $G \times E_1$, $x_0 \in L_p(G)$, U a neighbourhood of x_0 in $L_p(G)$. Suppose $g''_{u^2}(t, u)$ exists for every $u \in E_1$ and almost every $t \in G$. Let h be the Nemyckij operator generated on U by the function g, h_1 the operator defined on U by (4.6) and k the operation defined on $U \times L_p(G)$ by

$$k(x,y) = g''_{u^2}(\cdot, x)\cdot y \quad (x \in U, y \in L_p(G)).$$

Assume $p > 2$ and set $q = \frac{p}{p-1}$, $r = \frac{p}{p-2}$. Suppose

h_1 is a continuous mapping from U into $L_r(G)$ and $k(\cdot, y)$ a mapping of U into $L_r(G)$ which is continuous at x_0 for every $y \in L_p(G)$. Let V be an open convex neighbourhood of x_0 lying in U. Then h is an operation from V into $L_q(G)$, Lipschitzian on each bounded subset of V and having on V the continuous bounded (on every bounded subset of V) Fréchet derivative $h': V \to [L_p(G) \to L_q(G)]$. Moreover, h possesses the bilinear FG-differential $h''x_0 : L_p(G) \times L_p(G) \to L_q(G)$ at the point x_0 and $h'x(y) = (h_1 x)\cdot y$, $h''x_0(y,z) =$

$= k(x_0, y)\cdot z = k(x_0, z)\cdot y$ as $x \in V$, $y, z \in L_p(G)$.

Proof. The assertions concerning the operators h, h' follow from the proof of theorem 14. Set $K(x, y, z) =$ $= k(x,y)\cdot z$, $\omega_2(x,z)(y) = h'(x+z)(y) - h'x(y) - K(x,y,z)$; then $k(x,y)\cdot z = k(x,z)\cdot y$, evidently. The inequality

$$|K(x,y)z)(t)|^2 = |k(x,y)(t)|^2 \cdot |z(t)|^2 \leq$$

$$\leq \frac{1}{p-1}\cdot|z(t)|^p+\frac{p-2}{p-1}\cdot|h(x,y)(t)|^p$$

holds for almost every $t\in G$ whenever $x\in U$, $y,z\in$ $\in L_p(G)$.

Hence

$$\int_G|K(x,y,z)(t)|^q dt\leq\frac{1}{p-1}\cdot\|z\|_{L_p}^p+\frac{p-2}{p-1}\cdot\|h(x,y)\|_{L_p}^p<\infty,$$

which means that $K(x,y,z)\in L_q(G)$.

Now, let z be an arbitrary point of $L_p(G)$, τ a sufficiently small number such that $x_0+\tau z\in V$. There is $\vartheta(t)\in(0,1)$ for almost every $t\in G$ such that

$$g'_u(t,x_0(t)+\tau z(t))-g'_u(t,x_0(t))=g''_{u^2}(t,x_0(t)+\tau\vartheta(t)z(t))\cdot\tau z(t).$$

Hence

$$\|\omega_2(x_0,\tau z)\|_{[L_p\to L_q]}=\sup_{\|y\|_{L_p}\leq 1}\|\omega_2(x_0,\tau z)(y)\|_{L_q}=$$

$$=\sup_{\|y\|_{L_p}\leq 1}\Big[\int_G|g'_u(t,x_0(t)+\tau z(t))y(t)-g'_u(t,x_0(t))y(t)-$$
$$-g''_{u^2}(t,x_0(t))y(t)z(t)\tau|^q dt\Big]^{\frac{1}{q}}=$$

$$=\sup_{\|y\|_{L_p}\leq 1}\Big[\int_G|g''_{u^2}(t,x_0(t)+\tau\vartheta(t))z(t)\tau-$$
$$-g''_{u^2}(t,x_0(t))z(t)\tau|^q\cdot|y(t)|^q dt\Big]^{\frac{1}{q}}\leq$$

$$\leq\sup_{\|y\|_{L_p}\leq 1}|\tau|\cdot\Big[\int_G|g''_{u^2}(t,x_0(t)+\tau\vartheta(t)z(t))z(t)-$$
$$-g''_{u^2}(t,x_0(t))z(t)|^{\frac{p}{p-2}}dt\Big]^{\frac{p-2}{p}}\cdot\Big[\int_G|y(t)|^p dt\Big]^{\frac{1}{p}}=$$

$$|\tau|\cdot\|h(x_0+\tau\vartheta z,z)-h(x_0,z)\|_{L_p}.$$

Given $\varepsilon>0$, there is $\sigma>0$ such that $\|h(x_0+$ $+\tau\vartheta z,z)-h(x_0,z)\|_{L_p}<\varepsilon$ as $\|\tau\vartheta z\|_{L_p}<\sigma$. Hence, for all $\tau \neq 0$ with $|\tau| < \frac{\sigma}{\|z\|_{L_r}}$ ($z \in L_r(G)$ is fixed) the inequalities

$$\frac{1}{|\tau|}\cdot \|\omega_2(x_0,\tau z)\|_{[L_p \to L_2]} \leq \|k(x_0+\tau\vartheta z, z) - k(x_0,z)\|_{L_\varkappa} < \varepsilon$$

are valid, which proves that $K(x_0, \cdot, \cdot)$ is the FG-differential of the mapping h at x_0. The mapping $K(x_0, \cdot, \cdot)$ is bilinear and hence we can write

$$K(x_0,y,z) = k(x_0,y)\cdot z = h''x_0(y,z), \quad y,z \in L_r(G).$$

This completes the proof.

The following theorem shows that in the most of non-linear cases being important for applications the second differential of the Nemyckij operator in $L_2(G)$ does not exist.

Theorem 19. Let G be a bounded measurable set in E_1, g an N-function on $G \times E_1$. Suppose that there exists $g''_{u^2}(t,u)$ for almost every $t \in G$ and every $u \in E_1$ and that g''_{u^2} is an N-function on $G \times E_1$. Assume there is a constant $M > 0$ such that $|g'_u(t,u)| \leq M$ for almost every $t \in G$ and every $u \in E_1$. Denote by h the operator of Nemyckij generated by the function g (according to theorem 20.2, [1, § 20], h maps $L_2(G)$ into itself). Let x_0 be an element of $L_2(G)$. If there is an interval $J \in G$ and a constant $m > 0$ such that $|g''_{u^2}(t, x_0(t))| \geq m$ for almost every $t \in J$, then the mapping h has not any second differential at x_0.

Proof. According to theorem 20.2 in [1] the mapping h possesses on $L_2(G)$ the Gâteaux derivative h'.

Put

$$L_{y,z}(\tau) = \frac{1}{\tau}\cdot[h'(x_0+\tau z)(y) - h'x_0(y)]$$

for $y, z \in L_2(G)$, $\tau \neq 0$. There is $\vartheta(t) \in (0,1)$ for almost every $t \in G$ such that

$$g'_u(t, x_0(t)+\tau z(t)) - g'_u(t, x_0(t)) = g''_{u^2}(t, x_0(t)+\tau\vartheta(t)z(t))\cdot\tau z(t).$$

From that and from continuity $g''_{u^2}(t,\cdot)$ for almost every $t \in G$ it follows that

$$\lim_{\tau\to 0} L_{y,z}(\tau)(x) = \lim_{\tau\to 0} g''_{u^2}(t, x_0(t)+\tau\vartheta(t)z(t))\cdot$$
$$\cdot y(t)z(t) = g''_{u^2}(t, x_0(t))y(t)z(t)$$

for almost every $t \in G$. Hence, if any second differential of h at x_0 existed, it would be equal to $g''_{u^2}(\cdot, x_0)\cdot y\cdot z$ (almost everywhere on G). Let $t_0 \in J$ and put $y(t) = z(t) = |t-t_0|^{-\frac{1}{4}}$ for $t \in G$. Then $y, z \in L_2(G)$ and

$$\int_G |g''_{u^2}(t, x_0(t))y(t)z(t)|^2 dt \geq \int_J |g''_{u^2}(t, x_0(t))|^2\cdot$$
$$\cdot|t-t_0|^{-1}dt \geq m\cdot\int_J |t-t_0|^{-1}dt;$$

But the last integral diverges and hence $g''_{u^2}(\cdot, x_0)\cdot y\cdot z \notin L_2(G)$. This proves our theorem.

Remark. If $g''_{u^2}(t,u) = 0$ for all u of some neighbourhood of a set $\{x_0(t): t\in G\}$ and almost every $t \in G$, then according to theorem 12 the mapping h has the FF-derivative at x_0 equalling to the zero-operation on $L_2(G)$.

Remark. When this paper was submitted to press, we acquainted by means of [14, § 5] with the recent results of P.P. Zabrejko concerning the differentiability of mappings in functional spaces. But our results are quite different than Zabrejko's assertions.

R e f e r e n c e s

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