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NETS AND GROUPOIDS

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In this article we shall investigate general "(half)-nets" which correspond canonically to general (half) groupoids in an analogous manner as the usual 3-webs correspond to quasi-groups or loops, respectively. This concept of a general "half-net" seems to be new. Our interest is then concentrated onto the discovering of the expected more general parallelism between closure conditions in "halfnets" and algebraic laws in halfgroupoids as in the usual cases of 3-webs contra loops (for this usual case see [1],[3],[4],[5],[6]). We shall also take notice of general "double nets" and their corresponding double groupoids and give an elementary construction of all double groupoids over a given groupoid using left multiplications. Finally, special halfgroupoids called the binars are studied and some results on homomorphism theory of normal binars are obtained, generalizing certain results of Kiokemeister [2].

The author wishes to express his thanks to Professor J. Aczél for his suggestions leading to the realization of this paper.

1. Definition of a halfnet. Let S be a nonempty set and $\mathfrak{X}$ a partition in $S \times S$ 1) such that there exists

1) A partition $\mathcal{P}$ in a set $S \neq \emptyset$ is a nonempty set of nonempty subsets in S which are mutually disjoint. If moreover $\mathcal{P}$ covers S then $\mathcal{P}$ is called a partition on S.

an injective mapping $\varsigma : \mathcal{Z} \to S$. Then $\mathcal{N} = (S, \mathcal{Z}, \varsigma)$ will be called a halfnet with injection [2]. We shall need some denotations: Let $\mathcal{D}om\, \mathcal{Z}$ be equal to $\bigcup_{Z \in \mathcal{Z}} Z$, $\mathcal{X}$ be a partition on $\mathcal{D}om\, \mathcal{Z}$ consisting of nonempty sets of the form $\xi(b) = \{(x, y) \mid y = b\}$ and $\mathcal{Y}$ be a partition on $\mathcal{D}om\, \mathcal{Z}$ consisting of nonempty sets of the form $\eta(a) = \{(x, y) \mid x = a\}$. Call elements of $\mathcal{D}om\, \mathcal{Z}$ the points and elements of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ respectively the $\mathcal{X}$-lines, $\mathcal{Y}$-lines, $\mathcal{Z}$-lines, respectively.

If $\mathcal{D}om\, \mathcal{Z} = S \times S$ we get a net (with injection).

Note that in any halfnet $\mathcal{N} = (S, \mathcal{Z}, \varsigma)$ there holds $card\,(X \cap Y) \leq 1$ for each $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and there is always precisely one $\mathcal{X}$-, $\mathcal{Y}$-, $\mathcal{Z}$-line, respectively, passing through any point of $\mathcal{N}$.

In the following, we restrict ourselves to the case $card\, S \geq 2$. An $\mathcal{X}$-line $\xi(b)$ is said to be full if $(x, b) \in \xi(b)$ for all $x \in S$. Similarly a full $\mathcal{Y}$-line will be defined. We say that a halfnet $\mathcal{N} = (S, \mathcal{Z}, \varsigma)$ admits an $\mathcal{X}$-axis, if there is a $b \in S$ such that (i) $\xi(b)$ is a full $\mathcal{X}$-line and (ii) for each $Z \in \mathcal{Z}$, it holds $(\varsigma Z, b) \in Z$. The admitting of a $\mathcal{Y}$-axis will be defined analogously. We say that $\mathcal{N}$ admits a frame if

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2) Thus a halfnet is the above couple $(S, \mathcal{Z})$ where only the existence of a preceding injection ς is postulated. If one such injection ς is fixed, then the halfnet becomes already some algebraization. In the sequel the term "halfnet" shall mean already the halfnet with injection ς even so the fixing of ς is on many places not essential.

it admits an $\mathcal{X}$ -axis $\xi(b)$ and a $\mathcal{Y}$ -axis $\eta(a)$ such that $a = b$; (a, a) is then called the *origin* of the frame.

Proposition 1. In a halfnet $\mathcal{N} = (S, \mathcal{Z}, \xi)$ admitting an $\mathcal{X}$ -axis, each $\mathcal{Z}$ -line intersects the $\mathcal{X}$ -axis in exactly one point.

Proof. If there are different points $(x_1, b), (x_2, b)$ lying on any $Z \in \mathcal{Z}$, then $Z = \xi^{-1}x_1 = \xi^{-1}x_2$, which contradicts to the assumption that ξ is single-valued. Q.E.D.

Be given two halfnets $\mathcal{N}_i = (S_i, \mathcal{Z}_i, \xi_i)$; $i = 1, 2$. A surjective mapping $\sigma : \mathcal{D}om\, \mathcal{Z}_1 \longrightarrow \mathcal{D}om\, \mathcal{Z}_2$ will be called an *epimorphism* between $\mathcal{N}_1$ and $\mathcal{N}_2$, if $X \in \mathcal{X}_1 \Rightarrow \sigma X \in \mathcal{X}_2$, $Y \in \mathcal{Y}_1 \Rightarrow \sigma Y \in \mathcal{Y}$, $Z \in \mathcal{Z}_1 \Rightarrow \sigma Z \in \mathcal{Z}_2$. Given an epimorphism σ between $\mathcal{N}_1$ and $\mathcal{N}_2$ define derived mappings $\sigma_x : \{x \in S_1 \mid \exists y \in S_1 \text{ s.t. } (x,y) \in \mathcal{D}om\, \mathcal{Z}_1\} \to \{x \in S_2 \mid \exists y \in S_2 \text{ s.t. } (x,y) \in \mathcal{D}om\, \mathcal{Z}_2\}$,

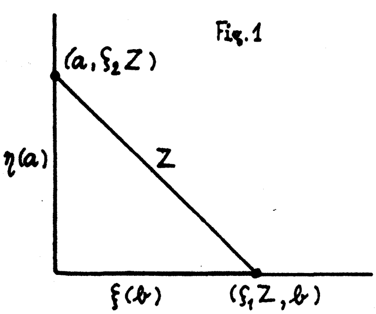
$$\sigma_y : \{y \in S_1 \mid \exists x \in S_1 \text{ s.t. } (x,y) \in \mathcal{D}om\, \mathcal{Z}_1\} \to \{y \in S_2 \mid \exists x \in S_2 \text{ s.t. } (x,y) \in \mathcal{D}om\, \mathcal{Z}_2\},$$

$$\sigma_z : \xi_1^{-1}\mathcal{Z}_1 \to \xi_2^{-1}\mathcal{Z}_2 \text{ by } \sigma\eta_1(x) = \eta_2(\sigma_x x),\ \sigma\xi_1(y) = \xi_2(\sigma_y y),\ \sigma Z = \xi_2^{-1}\sigma_z\, \xi_1 Z .$$

An *isomorphism* is, as usually, a bijective epimorphism.

Proposition 2. A halfnet $\mathcal{N} = (S, \mathcal{Z}, \xi)$ is isomorphic to a halfnet admitting an $\mathcal{X}$ -axis iff there is a full $\mathcal{X}$ - line in $\mathcal{N}$ which intersects each $\mathcal{Z}$ -line always in exactly one point. A halfnet $\mathcal{N} = (S, \mathcal{Z}, \xi)$ is isomorphic to a halfnet admitting a frame iff there is in $\mathcal{N}$ a full $\mathcal{X}$ - line and a full $\mathcal{Y}$ -line each of which intersects any $\mathcal{Z}$ - line always in exactly one point.

Proof. The necessity follows from Proposition 1. We shall prove the sufficiency: Suppose there exists $\xi(\mathscr{b})$ such that $(x,\mathscr{b}) \in \xi(\mathscr{b})$ for each $x \in S$ and that $\mathrm{card}\,(Z \cap \xi(\mathscr{b})) = 1$ for each $Z \in \mathscr{Z}$. Define a mapping $\varsigma^*: \mathscr{Z} \to S$ in such a way that $\varsigma^* Z$ is equal to the first coordinate of the common point of Z and $\xi(\mathscr{b})$. Form a new halfnet $\mathscr{N}^* = (S, \mathscr{Z}, \varsigma^*)$. Then there is an isomorphism σ between $\mathscr{N}$ and $\mathscr{N}^*$ for $\sigma_x = id_S$, $\sigma_y = id_S$ and σ_z determinated by $\sigma_z = \varsigma^*(\varsigma^{-1} c)$. – Suppose now that there exist $\xi(\mathscr{b})$ and $\eta(a)$ such that $(x,\mathscr{b}) \in \xi(\mathscr{b})$ and $(a,x) \in \eta(a)$ for all $x \in S$ and that $\mathrm{card}\,(Z \cap \xi(\mathscr{b})) = \mathrm{card}\,(Z \cap \eta(a)) = 1$ for all $Z \in \mathscr{Z}$. Define mappings $\varsigma_1: \mathscr{Z} \to S$, $\varsigma_2: \mathscr{Z} \to S$ by $\varsigma_1 Z$ to be equal to the first coordinate of the point of intersection of Z, $\xi(\mathscr{b})$ and $\varsigma_2 Z$ to be equal to the second coordinate of the point of intersection of Z, $\eta(a)$ (Fig.1).



Define mappings $\sigma_1: S \to S$ and $\sigma_2: S \to S$ so that $\sigma_1(\varsigma_1 Z) = \sigma_2(\varsigma_2 Z) = \sigma Z$ for each $Z \in \mathscr{Z}$. Let $\mathscr{N}^* = (S, \mathscr{Z}^*, \varsigma^*)$ be a halfnet where $\mathscr{Z}^*$ consists of all Z^* such that $(x,y) \in Z \in \mathscr{Z} \Leftrightarrow (\varsigma_1 x, \varsigma_2 y) \in Z^* \in \mathscr{Z}^*$ and $\varsigma^* Z^* = \varsigma Z$. There is an isomorphism σ between $\mathscr{N}$ and $\mathscr{N}^*$ such that $\sigma_x = \varsigma_1$, $\sigma_y = \varsigma_2$ and $\sigma_z = id_S$. Q.E.D.

2. Closure conditions and their algebraic counterparts

Let $\mathcal{N} = (S, \mathcal{X}, \mathcal{Y})$ be a halfnet possessing a frame. By an $\mathcal{N}$ -rectangle we shall mean any quadruple of points A, B, C, D in $\mathcal{N}$ such that A, B lie on the same $\mathcal{X}$ -line, C, D lie on the same $\mathcal{X}$ -line, B, C lie on the same $\mathcal{Y}$ -line and D, A lie on the same $\mathcal{Y}$ -line. (Fig.2)

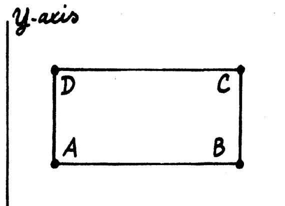


Fig 2

We shall investigate the following closure conditions in a halfnet $\mathcal{N} = (S, \mathcal{Z}, \mathcal{Y})$ possessing a frame.

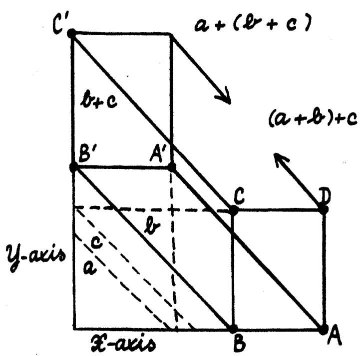


Fig. 3

Reidemeister: If (A, B, C, D), (A′, B′, C′, D′) are $\mathcal{N}$ -rectangles with A, B on the $\mathcal{X}$ -axis, B′, C′ on the $\mathcal{Y}$ -axis, A, A′ on the same $\mathcal{Z}$ -line, B, B′ on the same $\mathcal{Z}$ -line and C, C′ on the same $\mathcal{Z}$ -line, then D, D′ lie also on the same $\mathcal{Z}$ -line (Fig.3),

A particular form of Reidemeister with A, D, B′, C′ on the same $\mathcal{Y}$ -line, A′, B′, C′ on the same $\mathcal{X}$ -line and A, A′, C, C′ on the same $\mathcal{Z}$ -line, respectively, will be called $\mathcal{X}$ -Bol, $\mathcal{Y}$ -Bol, $\mathcal{Z}$ -Bol, respectively.

Hexagonality: If A is a point on the $\mathcal{X}$ -axis then there are $\mathcal{N}$ -rectangles (A,B,C,D), (A′,B′,C′,D′) , with

C' on the $\mathcal{X}$ -axis and with B', D' being equal to the origin, so that A, A' are on the same $\mathcal{Z}$ -line, B, B', D, D' are on the same $\mathcal{Z}$ -line and C, C' are on the same $\mathcal{Z}$ -line (Fig.4).

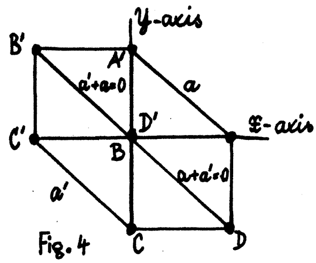


Fig. 4

<u>Thomsen:</u> Let A, B, C, A', B', C' be points such that A, B lie on the $\mathcal{X}$ -axis, A', C' on the same $\mathcal{X}$ -line, B', C' on the same $\mathcal{X}$ -line, A', B' on the $\mathcal{Y}$ -axis, A, C' on the same $\mathcal{Y}$ -line, B, C on the same $\mathcal{Y}$ -line, A, A' on the same $\mathcal{Z}$ -line and B, B' on the same $\mathcal{Z}$ -line.

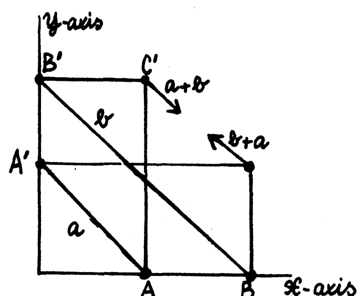


Fig. 5

Then C, C' lie on the same $\mathcal{Z}$ -line (Fig.5).

<u>ℓ - Bol</u>: Let (A,B,C,D), (A', B', C', D') be $\mathcal{N}$ -rectangles with A, B on the $\mathcal{X}$ -axis, A, D, B', C' on the $\mathcal{Y}$ -axis, A, A' on the same $\mathcal{Z}$ -line, B, B' on the same $\mathcal{Z}$ -line and C, C' on the same $\mathcal{Z}$ -line. Then D, D' are on the same $\mathcal{Z}$ -line (Fig.6).

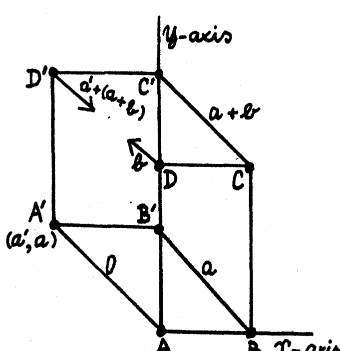


Fig. 6

<u>r -Bol</u>: Let (A,B,C,D), (A', B', C', D') be $\mathcal{N}$ -rectangles with A, B on the

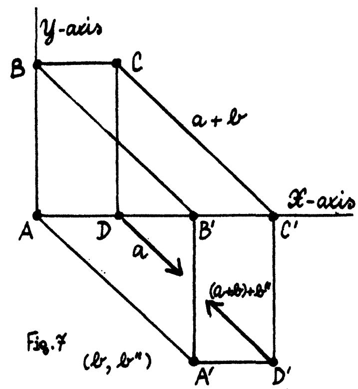


Fig.7 (b, b'')

$\mathcal{Y}$-axis, A,D,B$'$,C$'$ on the $\mathcal{X}$-axis, A,A$'$ on the same $\mathcal{Z}$-line, B,B$'$ on the same $\mathcal{Z}$-line and C,C$'$ on the same $\mathcal{Z}$-line. Then D,D$'$ lie on the same $\mathcal{Z}$-line (Fig.7).

A <u>halfgroupoid</u> is defined as a couple $\mathcal{G} = (S, \beta)$ where S is a nonempty set and β a mapping of some set $\mathcal{D}om\,\beta \subseteq S \times S$ into S. If $\mathcal{D}om\,\beta = S \times S$ we get a <u>groupoid</u>. - In the following we shall restrict ourselves to the case $card\, S \geq 2$, $\mathcal{D}om\,\beta \neq \emptyset$.

If $\mathcal{G} = (S, \beta)$ is a halfgroupoid then the halfnet $(S, \mathcal{Z}, \varsigma)$ defined by $\beta(x, y) = z \Longleftrightarrow (x, y) \in \varsigma z$ for all possible $x, y, z \in S$, is called <u>associated</u> to $\mathcal{G}$ and it is denoted by $\mathcal{A}s\, \mathcal{G}$. If $\mathcal{N} = (S, \mathcal{X}, \varsigma)$ is a halfnet then the halfgroupoid (S, β) defined by $\beta(x, y) = z \Longleftrightarrow (x, y) \in \varsigma z$ for all possible $x, y, z \in S$ is called <u>associated</u> to $\mathcal{N}$ and it is denoted by $\mathcal{A}s\, \mathcal{N}$. Clearly, $\mathcal{A}s\,(\mathcal{A}s\,\mathcal{G}) = \mathcal{G}$, $\mathcal{A}s\,(\mathcal{A}s\,\mathcal{N}) = \mathcal{N}$ for all halfgroupoids $\mathcal{G}$ and all halfnets $\mathcal{N}$.

Recall that in a halfgroupoid $\mathcal{G} = (S, \beta)$ any <u>left neutral</u> element a is characterized by the validity of $\beta(a, y) = y$ for all $y \in S$ (so that $(a, y) \in \mathcal{D}om\,\beta$ for all $y \in S$ is included).
Similarly one defines a <u>right neutral</u> element in $\mathcal{G}$. An element a is called <u>neutral</u> if it is simultaneously left

and right neutral. The <u>cancellation laws</u> will mean: if $\beta(x,b)=\beta(y,b)$ then $x=y$ and if $\beta(a,x)=$ $=\beta(a,y)$ then x = y .

<u>Proposition 3</u>. Let $\mathcal{N}=(S,\mathcal{Z},\xi)$ be a halfnet. It follows:

a) $\mathcal{N}$ admits an $\mathcal{X}$-axis iff $\mathcal{As}\,\mathcal{N}$ possesses a right neutral element.

b) $\mathcal{N}$ admits a frame, iff $\mathcal{As}\,\mathcal{N}$ possesses a neutral element.

c) $\operatorname{card}(\xi(a)\cap Z)\leq 1$, $\operatorname{card}(\eta(a)\cap Z)\leq 1$ for all $a\in S$, $Z\in\mathcal{Z}$ is true iff in $\mathcal{As}\,\mathcal{N}$ the both cancellation laws hold.

d) $\operatorname{card}(\xi(a)\cap Z)=\operatorname{card}(\eta(a)\cap Z)=1$ for all $a\in S$, $Z\in\mathcal{Z}$ iff $\mathcal{As}\,\mathcal{N}$ is a quasigroup.

<u>Proof</u>. a) Let $\mathcal{N}$ admit an $\mathcal{X}$-axis $\xi(b)$. By Proposition 2 and by the definition of the $\mathcal{X}$-axis and $\mathcal{As}\,\mathcal{N}$ it follows that $\beta(x,b)=x$ for every $x\in S$. Conversely, if there exists a $b\in S$ such that $\beta(x,b)=x$ for all $x\in S$ then by the definition of $\mathcal{As}\,\mathcal{N}$ it follows that $\operatorname{card}(Z\cap\xi(b))=1$ for each $Z\in\mathcal{Z}$ and $(\xi Z,b)$ so that $\xi(b)$ must be an $\mathcal{X}$-axis.

b) Let $\mathcal{N}$ admit a frame with the $\mathcal{X}$-axis $\xi(a)$ and the $\mathcal{Y}$-axis $\eta(a)$. Then, according to part a), it follows $\beta(x,a)=\beta(a,x)=a$ for all $x\in S$, and conversely.

c)-d) Clearly, $\operatorname{card}(\xi(a)\cap Z)\leq 1$ or = 1, means that the equation $\beta(x,a)=\xi Z$ has not at most one solution $x\in S$ or exactly one solution $x\in S$ respectively. From this it follows the required conclusion. Q.E.D.

Let $\mathcal{G} = (S, \beta)$ be a halfgroupoid. We shall still need some algebraical laws which we wish to formulate as follows:

Associativity: $\beta(\beta(a,b),c) = \beta(a,\beta(b,c))$ for all $a, b, c \in S$ for which (a,b), $(\beta(a,b),c)$; (b,c), $(a,\beta(b,c)) \in \mathcal{D}om\ \beta$.

The particular case of associativity for $a = b$, $b = c$, $\beta(a,b) = \beta(b,c)$ respectively is termed a left, right and middle alternativity respectively.

Now let $\mathcal{G}$ have a neutral element e.

Existence of inverse elements: To any $a \in S$ there is at least one element $a' \in S$ such that $\beta(a', a) = \beta(a, a') = e$.

The left inverse property: For all triples $(a', a, b) \in S \times S \times S$ such that $(a', a) \in \mathcal{D}om\ \beta$, $\beta(a', a) = e$ it holds $\beta(a', \beta(a, b)) = b$.

The right inverse property: For all triples $(a, b, b'') \in S \times S \times S$ such that $(b, b'') \in \mathcal{D}om\ \beta$, $\beta(b, b'') = e$ it holds $a = \beta(\beta(a, b), b'')$.

Proposition 4. Let $\mathcal{N} = (S, \mathcal{X}, \mathfrak{f})$ be a halfnet admitting a frame. Then the following couples of conditions for $\mathcal{N}$ and $\mathcal{A}s\ \mathcal{N}$ are equivalent:
(i) Reidemeister and associativity, (ii) $\mathcal{X}$-, $\mathcal{Y}$-, $\mathcal{Z}$-Bol, respectively, and left, right, middle alternativity respectively, (iii) hexagonality and existence of inverse elements, (iv) ℓ-, κ-Bol, respectively, and the left inverse property, the right inverse property, respectively, (v) Thomsen and commutativity.

Proof. Denote the composition β of $\mathcal{A}s\ \mathcal{N}$ also by $+$ and the neutral element of $\mathcal{A}s\ \mathcal{N}$ by 0. Part (i) is described on Fig.3. Part (ii) is only a particular case of (i). The schema for (iii) is given on Fig.4. If an element a' exists with $a' + a = 0$ then (a', a) must lie on $\varsigma^{-1}(0)$ so that (a', a) can be situated into A$'$. The situation for (iv) is demonstrated in Fig.6-7. Finally, Fig.5 illustrates the situation for (v).

<u>An isotopy</u> between halfgroupoids $\mathcal{H}_i = (S_i, \beta_i)$; $i = 1, 2$, is defined as a triple $(\sigma_1, \sigma_2, \sigma_3)$ of bijective mappings $\sigma_j : S_1 \to S_2$; $j = 1, 2, 3$, such that $\{(\sigma_1 x, \sigma_2 y) \mid (x, y) \in \mathcal{D}om\, \beta_1\} = \mathcal{D}om\, \beta_2$ and that $\beta_2(\sigma_1 x, \sigma_2 y) = \sigma_3 \beta(x, y)$ for all $(x, y) \in \mathcal{D}om\, \beta_1$.

<u>Proposition 5</u>. Be given two halfnets $\mathcal{N}_i = (S_i, \mathcal{Z}_i, \varsigma_i)$; $i = 1, 2$. They are isomorphic iff the halfgroupoids $\mathcal{A}s\, \mathcal{N}_1$, $\mathcal{A}s\, \mathcal{N}_2$ are isotopic.

<u>Proof</u>. Let there exist an isomorphism σ between $\mathcal{N}_1$ and $\mathcal{N}_2$. Then $(\sigma_x, \sigma_y, \sigma_z)$ is (up to restrictions) the required isotopy between $\mathcal{A}s\, \mathcal{N}_1$ and $\mathcal{A}s\, \mathcal{N}_2$. - If there exists an isotopy $(\sigma_1, \sigma_2, \sigma_3)$ between $\mathcal{A}s\, \mathcal{N}_1$ and $\mathcal{A}s\, \mathcal{N}_2$, then there is ~~an~~ isomorphism σ between $\mathcal{N}_1$ and $\mathcal{N}_2$ such that $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, $\sigma_z = \sigma_3$ (up to restrictions). Q.E.D.

3. <u>Double groupoids and double nets</u>. A <u>double groupoid</u> $\mathcal{G}$ is defined as a couple of groupoids $\mathcal{G}_+ = (S, \alpha)$, $\mathcal{G}_\cdot = (S, \beta)$ 3) where $\mathcal{G}_+$ has a neutral element 0 such

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3) We shall use the denotations $(S, \alpha) \equiv (S, +)$, $\alpha(x, y) = x + y$ and $(S, \beta) \equiv (S, \cdot)$, $\beta(x, y) = x \cdot y$.

that $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$.

Proposition 6. Let a groupoid $\mathcal{G}_+ = (S, +)$ with a neutral element 0 be given. Then every double groupoid over $\mathcal{G}_+$ can be constructed as follows: Let φ be some mapping of S into the set $\mathfrak{S}$ of all mappings from S into S reproducing the neutral element 0 and such that $\sigma(0)$ sends each $x \in S$ onto 0. Define $\cdot$ by $x \cdot y = (\varphi(x))y$ for all $x, y = S$.

Proof. Clearly, the described construction gives rise to a double groupoid. Secondly, in each double groupoid over a given $\mathcal{G}_+$, with second composition $\cdot$, each mapping L_x defined by $L_x y = x \cdot y$ for all $y \in S$ is a mapping of S into S satisfying the above conditions.Q.E.D.

Corollaries: $(S, \cdot)$ contains a neutral element 1 (the unity) iff $\sigma(1) = id_S$ and, for each $x \in S$, $\sigma(x)$ takes 1 onto x. The left distributivity $x(y + z) = x \cdot y + x \cdot z$ holds iff each $\sigma(x)$ is an endomorphism of $\mathcal{G}_+$. $\mathcal{G}^\circ_\cdot = (S \setminus \{0\}, \cdot)$ is a quasigroup iff each $\sigma(x)$ is a permutation of S and $\sigma(S \setminus \{0\})$ acts simply transitively upon $S \setminus \{0\}$. $\mathcal{G}$ is without zero divisors if $card\, \sigma^{-1}x(0) = 1$ for all $x \in S$.

The proof follows at once applying the definition of $\cdot$.

A double net $\mathcal{N}$ is defined as a couple of nets $\mathcal{N}_+(S, \mathfrak{X}_+, \varsigma_+)$, $\mathcal{N}_\cdot = (S, \mathfrak{X}_\cdot, \varsigma_\cdot)$ such that $\mathcal{N}_+$ admits a frame with $\mathcal{X}_+$- and $\mathcal{Y}_+$-axis lying in the same $\mathfrak{X}_\cdot$-line. Moreover we shall suppose that (i) the union of the $\mathcal{X}_+$-and the $\mathcal{Y}_+$-axis is a $\mathfrak{X}_\cdot$-line θ which will be called singular, (ii) the reduced net $\mathcal{N}^\circ_\cdot = (S^\circ, \mathfrak{X}^\circ_\cdot, \varsigma^\circ_\cdot)$ where $S^\circ = S \setminus \{0\}$, $\mathfrak{X}^\circ_\cdot = \mathfrak{X}_\cdot \setminus \{\theta\}$, $\varsigma^\circ_\cdot = \varsigma_\cdot|_{\mathfrak{X}^\circ_\cdot}$ admits a frame.

Clearly, if $\mathcal{N} = (\mathcal{N}_+, \mathcal{N}_\cdot)$ is a double net then $(\mathcal{A}s\,\mathcal{N}_+, \mathcal{A}s\,\mathcal{N}_\cdot)$ is a double groupoid without zero divisors and with unity. Conversely, if $\mathcal{G} = (\mathcal{G}_+, \mathcal{G}_\cdot)$ is a double groupoid without zero divisors and with unity then $(\mathcal{A}s\,\mathcal{G}_+, \mathcal{A}s\,\mathcal{G}_\cdot)$ is a double net.

A double groupoid $\mathcal{G} = (\mathcal{G}_+, \mathcal{G}_\cdot)$ such that both $\mathcal{G}_+, \mathcal{G}_\cdot^\circ$ are loops is a well-known <u>double loop</u>.
Let $\mathcal{G} = (\mathcal{G}_+, \mathcal{G}_\cdot)$ be a double groupoid without zero divisors and with unity. The geometric counterparts in $(\mathcal{A}s\,\mathcal{G}_+, \mathcal{A}s\,\mathcal{G}_\cdot)$ which correspond to the left of the right distributivity are much complicated. Although the coordinatization of 4-webs given in [3], pp.61-63 is geometrically more convenient, the preceding concept may possibly present a further view upon the question of the coordinatization.

<u>4. Binars</u>. <u>A binar</u> $\mathcal{B}$ is defined as a quadruple (S_1, S_2, S_3, β) where S_1, S_2, S_3 are nonempty sets and β is a mapping of $S_1 \times S_2$ onto S_3. Clearly a binar (S_1, S_2, S_3, β) can be regarded as a halfgroupoid $(S_1 \cup S_2 \cup S_3, \beta)$ with $\mathcal{D}om\,\beta = S_1 \times S_2$.

Some suggestions for the study of binars are given in [3], p.24 or in [4], p.448, respectively.

The concept of isotopy between two groupoids can be subordinated (after some arrangements) to the concept of an epimorphism between two binars:

Let $\mathcal{B} = (S_1, S_2, S_3, \beta)$, $\mathcal{B}' = (S_1', S_2', S_3', \beta')$ be two binars. By a <u>mapping</u> σ of $\mathcal{B}$ onto $\mathcal{B}'$ we shall mean a triple $(\sigma_1, \sigma_2, \sigma_3)$ such that σ_i is a mapping of S_i onto S_i' ; $i = 1, 2, 3$. Such a mapping σ is called epimorphism between $\mathcal{B}$ and $\mathcal{B}'$ if $\beta'(\sigma_1 x, \sigma_2 x) = \sigma_3 \beta(x, y)$ for each $(x, y) \in S_1 \times S_2$. By a partition $\mathcal{P}$ on $\mathcal{B}$ we mean a triple $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ where $\mathcal{P}_i$ is a partition on S_i; $i = 1, 2, 3$. $\mathcal{P}$ is said to be generating if, for each $(X, Y) \in \mathcal{P}_1 \times \mathcal{P}_2$, $\beta(X, Y)$ is contained in some member of $\mathcal{P}_3$.

Some properties of generating partitions: [4]To each generating partition $\mathcal{P}$ on $\mathcal{B}$ there corresponds a canonical epimorphism between $\mathcal{B}$ and $\mathcal{B}/\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \beta/\mathcal{P})$ where $\beta/\mathcal{P}(X, Y) = Z$ for $\beta(X, Y) \subseteq Z \in \mathcal{P}_3$. To each epimorphism σ between $\mathcal{B}$ and $\mathcal{B}'$ there corresponds an induced generating partition $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ such that $\mathcal{P}_i = \{\sigma_i^{-1} r \mid r \in S_i'\}$; $i = 1, 2, 3$, and consequently, $\mathcal{B}'$ is isomorphic with $\mathcal{B}/\mathcal{P}$. It is also well-known that for any system $\mathfrak{S} = (\mathcal{P}^{\iota})_{\iota \in \mathcal{J}}$ of generating partitions on $\mathcal{B}$, the partitions $\sup \mathfrak{S} = (\sup \mathfrak{S}_1, \sup \mathfrak{S}_2, \sup \mathfrak{S}_3)$ and $\inf \mathfrak{S} = (\inf \mathfrak{S}_1, \inf \mathfrak{S}_2, \inf \mathfrak{S}_3)$ with $\mathfrak{S}_i = (\mathcal{P}_i^{\iota})_{\iota \in \mathcal{J}}$ for $i = 1, 2, 3$ are also generating.

A binar $\mathcal{B} = (S_1, S_2, S_3, \beta)$ is said to be left reversible, if, for any $(b, c) \in S \times S$ there is a unique $x \in S_1$, such that $\beta(x, b) = c$. Similarly a right reversibility is defined.

A generating partition $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ on a binar $\mathcal{B}$ is called left normal if $\mathcal{B}/\mathcal{P}$ is left reversible. Similarly a right normality will be defined.

Proposition 7. Let $\mathcal{B} = (S_1, S_2, S_3, \beta)$ be a left

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4) See for example: Čas.pěst.mat.91(1966),246-253.

reversible binar and $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$, $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3)$ left normal partitions on $\mathcal{B}$. Then $inf(\mathcal{P}, \mathcal{Q})$ and $sup(\mathcal{P}, \mathcal{Q})$ are left normal too.

Proof. First, consider the equation $\beta/inf(\mathcal{P},\mathcal{Q})(x,b) = c$ for given $(b, c) \in inf(\mathcal{P}_2, \mathcal{Q}_2) \times inf(\mathcal{P}_3, \mathcal{Q}_3)$. We have $b = p_2 \cap q_2$, $c = p_3 \cap q_3$ for convenient $(p_2, q_2) \in \mathcal{P}_2 \times \mathcal{Q}_2$, $(p_3, q_3) \in \mathcal{P}_3 \times \mathcal{Q}_3$. There exist uniquely determined solutions $p_1 \in \mathcal{P}_1$, $q_1 \in \mathcal{Q}_1$ of $\beta/\mathcal{P}(p_1, p_2) = p_3$ and $\beta/\mathcal{Q}(q_1, q_2) = q_3$ respectively. By the left reversibility of $\mathcal{B}$ it follows $a = p_1 \cap q_1 \neq \emptyset$. Thus $a \in inf(\mathcal{P}_1, \mathcal{Q}_1)$ is the required unique solution of the given equation. - Secondly, consider the equation $\beta/sup(\mathcal{P},\mathcal{Q})(x,b) = c$ for given $(b, c) \in sup(\mathcal{P}_2, \mathcal{Q}_2) \times sup(\mathcal{P}_3, \mathcal{Q}_3)$. Let b_1, b_2 be two members of $\mathcal{P}_2$ lying in b and c_1, c_2 two members of $\mathcal{P}_3$ lying in c. Thus there must exist "chainings" [5)] $b_1 \mathrel{\text{I}} v_1' \mathrel{\text{I}} v_1 \mathrel{\text{I}} v_2' \mathrel{\text{I}} \dots \mathrel{\text{I}} v_r' \mathrel{\text{I}} b_2$ ($v_i \in \mathcal{P}_2, v_i' \in \mathcal{Q}_2$ for $i = 1,\dots,r$) and $c_1 \mathrel{\text{I}} w_1' \mathrel{\text{I}} w_1 \mathrel{\text{I}} w_2' \mathrel{\text{I}} \dots \mathrel{\text{I}} w_r' \mathrel{\text{I}} c_2$ ($w_i \in \mathcal{P}_3, w_i' \in \mathcal{Q}_3$ for $i = 1,\dots,r$) of the same length $2r + 1$. Find uniquely determined solutions $u_i \in \mathcal{P}_1$, $u_i' \in \mathcal{Q}_1$ of $\beta/\mathcal{P}(u_i, v_i) = w_i$, $\beta/\mathcal{Q}(u_i', v_i') = w_i'$ for $i = 1,\dots, r$ and a unique solution $a_j \in \mathcal{P}_1$ of $\beta/\mathcal{P}(a_j, b_j) = c_j$; $j = 1, 2$. Then by left reversibility of $\mathcal{B}$ there exists a chaining $a_1 \mathrel{\text{I}} u_1' \mathrel{\text{I}} u_1 \mathrel{\text{I}} u_2' \mathrel{\text{I}} \dots \mathrel{\text{I}} u_r' \mathrel{\text{I}} a_2$. Consequently a_1, a_2 must lie in the same member a of $sup(\mathcal{P}_1, \mathcal{Q}_1)$ and the given equation has a unique solution $a \in sup(\mathcal{P}_1, \mathcal{Q}_1)$. Q.E.D.

Lemma. Let $\mathcal{B} = (S_1, S_2, S_3, \beta)$ be a left reversible binar. Let $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ be a left normal

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5) If A,B are sets then $A \mathrel{\text{I}} B$ means $A \cap B \neq \emptyset$.

partition on $\mathcal{B}$. If $P \in \mathcal{P}_1$ and $q \in S_2$ then $R \in \mathcal{P}_3$, $R \mathrel{I} \beta(P, q) \Longrightarrow R \subseteq \beta(P, q)$.

<u>Proof</u>. Let $p \in P$, $u \in R$. Let $x \in S_1$ be a unique solution of the equation $\beta(x, q) = u$.
Then from $\beta(p, q)$, $u \in R$ it follows, by left normality of $\mathcal{P}$, that $u \in \beta(P, q)$.

<u>Proposition 8</u>. Let $\mathcal{B} = (S_1, S_2, S_3, \beta)$ be a left and right reversible binar. Let $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ be a left normal partition on $\mathcal{B}$ and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3)$ be a right normal partition on $\mathcal{B}$. Then $P \in \mathcal{P}_1$, $Q \in \mathcal{Q}_1 \Rightarrow \beta(P, Q) \in \in \sup(\mathcal{P}_3, \mathcal{Q}_3)$ and $\mathcal{P}_0, \mathcal{Q}_0$ is an associable pair 6).

<u>Proof</u>. Let $\beta(p, q) \in Z \in \sup(\mathcal{P}_0, \mathcal{Q}_0)$ for some $(p, q) \in P \times Q$. First, we shall show that each $z \in Z$ lies in $\beta(P, Q)$. Since z and $\beta(p, q)$ belong to Z, there exists a finite sequence $z = p_0, p_1, \dots, p_{k-1}, p_k = \beta(p, q)$ such that any two consecutive elements lie in the same member of $\mathcal{P}_0$ or $\mathcal{Q}_0$ respectively. As $p_k = \beta(p, q) \in \beta(P, Q)$ then, by gradual using of the preceding lemma, all $p_{k-1}, \dots, p_1, p_0$ must belong also to $\beta(P, Q)$.
Secondly, take any $(p', q') \in P \times Q$. As $\mathcal{P}, \mathcal{Q}$ are generating, $\beta(p', q')$, $\beta(p, q')$ lie in the same member of $\mathcal{P}_0$ and $\beta(p, q')$ $\beta(p, q)$ also lie in the same

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6) A pair of partitions $\mathcal{A}, \mathcal{B}$ on a set $S \neq \emptyset$ is said to be <u>associable</u> if each member C of $\sup(\mathcal{A}, \mathcal{B})$ satisfies the following conditions: if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are contained in C then $A \mathrel{I} B$. See f.e. Journ.de math. pure et appl. 18(1939),p.72.

member of $\mathcal{Q}_0$ so that $\beta(p, q)$ $\beta(p', q')$ belong to the same member of $sup(\mathcal{P}_0, \mathcal{Q}_0)$ and $\beta(P, Q) \subseteq Z$.
In the whole, we have $\beta(P, Q) = Z$, as it was required.
From the preceding it follows, by the way, the associability of $\mathcal{P}_0$ and $\mathcal{Q}_0$. Q.E.D.

__Corollary__. Let $\mathcal{L}$ be the lattice of all left and right normal partitions on B . Then the lattice of the third components of these partitions is modular.

The proof follows by the well-known fact (from the theory of equivalence relations) that associability implies modularity. Q.E.D.

Propositions 7 - 8 generalize the results of [2] where normal partitions on quasigroups are studied.

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