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ON THE DIFFERENTIABILITY OF MAPPINGS IN BANACH SPACES

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1. Introduction. Throughout this paper E, E_1 denote the real Banach spaces, R (or N) the set of all real (or natural) numbers, $F: E \to E_1$ a mapping of E into E_1. Let E' be the dual space of E, (x,e) the value of $e \in E'$ at the point $x \in E$. Let $K_r = \{x \in E ; \|x\| \le r\}$ denote the closed ball in E of radius $r > 0$ about the origin; let S_r denote the boundary of K_r. By $(E \to E_1)$ there is meant the space of all linear bounded mappings of E into E_1 (with the topology of uniform convergence on K_1). We shall use the symbols " $\longrightarrow$ " and " $\xrightarrow{w}$ " to denote the strong and weak convergence in E (or in E'). A mapping $F: E \to E_1$ is said to be weakly (strongly) continuous if $x_n \xrightarrow{w} x$ implies $F(x_n) \xrightarrow{w} F(x)$ $(F(x_n) \longrightarrow F(x))$. The symbol $[x_0, y_0]$, where $x_0, y_0 \in E$, denotes the element of $E \times E$ and a neighbourhood of $[x_0, y_0]$ is taken in $E \times E$. By $VF(x_0, h)$ $(DF(x_0, h))$ we denote the Gâteaux (linear Gâteaux) differential of a mapping $F: E \to E_1$ at $x_0 \in E$. If $DF(x_0, h)$ is continuous in $h \in E$, $F: E \to E_1$ is said to have the Gâteaux derivative $F'(x_0)$ at x_0. We shall say that a mapping $F: E \to E_1$ has the Fréchet differential $dF(x_0, h)$ at $x_0 \in E$ if

$F(x_0+h)-F(x_0)=dF(x_0,h)+\omega(x_0,h)$, $h\in E$, where $dF(x_0,h)$ is linear in h and $\lim\limits_{\|h\|\to 0}\frac{\|\omega(x_0,h)\|}{\|h\|}=0$.

A mapping $F: E\to E_1$ is said to have the Fréchet derivative $F'(x_0)$ at $x_0\in E$ if $dF(x_0,h)$ is bounded on S_1. By the symbol " neighbourhood of x_0 " there is always meant the convex symmetric neighbourhood of $x_0\in E$. In order to omit the assumption of linearity of $dF(x_0,h)$ in h Suchomlinov ([8]) introduced the concept of a bounded differential as follows:

<u>Definition 1</u>. The mapping $F: E\to E_1$ is said to have a bounded differential $dVF(x_0,h)$ at $x_0\in E$ if the following conditions are satisfied:

1) $\lim\limits_{t\to 0}\dfrac{F(x_0+th)-F(x_0)}{t}=dVF(x_0,h)$ uniformly

with respect to $\|h\|=1$, $h\in E$,

2) $dVF(x_0,h)$ is bounded on $S_1\subset E$.

The connections between the existence of the Gâteaux and Fréchet differentials for mappings in Banach spaces were studied in [1], [2],[3],[4],[5],[6],[7],[8],[9]. L.A. Ljusternik, V.I. Sobolev ([7],chapt.8,§ 3) derived that if $VF(x,h)$ is continuous in $h\in E$ and uniformly continuous in a neighbourhood of $x_0\in E$ in the sense of $(E\to E_1)$, then F has the Fréchet derivative at x_0. The following result is due to M.M. Vajnberg ([1] th.3.3): If the Gâteaux derivative exists in some neighbourhood of x_0 and is continuous at x in the topology of $(E\longrightarrow E_1)$, then F possesses the Fréchet derivative at x_0. Another result has been established by G. Marinescu ([8]): Suppose that the Gâteaux differential $VF(x, h)$ is continuous in x in a neighbourhood $\mathcal{U}(x_0)$ of x_0 (for an arbitrary but fixed $h \in E$) and $VF(x,h)$ is continuous at h = 0 for every fixed $x \in \mathcal{U}(x_0)$. If $VF(x, h)$ is directionally continuous at x_0 uniformly with respect to $h \in E$, $\| h \| = 1$, then F has the Fréchet derivative at $x_0 \in E$. The result of N.N.Ivanov ([5]) is as follows: Let X be a finite-dimensional Banach space, $f : X \to R$ a real functional on X. If there exists the Gâteaux differential $Vf(x_0, h)$ and f satisfies the Lipschitz condition in a neighbourhood of $x_0 \in E$, then f has a bounded differential at $x_0 \in E$. J. Kolomý ([6]) has proved that if $VF(x, h)$ exists in a neighbourhood of $x_0 \in E$ (E is reflexive) and is strongly continuous jointly in $[x_0, h]$ (h is an arbitrary element of E), then $F : E \to E_1$ possesses the Fréchet derivative $F'(x_0)$ at $x_0 \in E$.

The purpose of this paper is to show some other conditions for the existence of bounded and Fréchet differentials. I wish to thank J. Kolomý for the suggestion of this problem.

2. Theorem 1. Suppose that a mapping $F : E \to E_1$ has the Gâteaux differential $VF(x, h)$ in some neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in E$. Let the following conditions be fulfilled:

1) $$\lim_{t \to 0} \| VF(x_0 + th, h) - VF(x_0, h) \|_{E_1} = 0$$

uniformly with respect to $\| h \| = r$, $h \in E$, where $r > 0$

is some fixed real number.

2) $VF(x_0, h)$ is bounded on $S_\varkappa$.

Then F possesses a bounded differential $dVF(x_0, h)$ at $x_0 \in E$.

Proof. Let h be an arbitrary element of E . Since

$$F(x_0+th)-F(x_0)=VF(x_0,th)+\omega(x_0,th) \ ,$$

$$(1) \qquad \lim_{t\to 0} \left\| \frac{\omega(x_0, th)}{t} \right\| = 0 \qquad (h \text{ is a fixed element}).$$

Assume that this limit is not uniform on $S_\varkappa$, where $\varkappa > 0$ is such real number that $x_0 + K_\varkappa \subset \mathcal{U}(x_0)$. Then there exists $\varepsilon > 0$ with the following property:

For every $n \in N$ there exist $h_n \in S_\varkappa$ and t_n such that $0 < |t_n| < \frac{1}{n}$ and

$$(2) \qquad \left\| \frac{\omega(x_0, t_n h_n)}{t_n} \right\| \geq \varepsilon \ .$$

Let $h \in S_\varkappa$ be an arbitrary element of $S_\varkappa$, then for any $\varepsilon > 0$ there exists $n_0 \in N$ such that for every $n \geq n_0$, $n \in N$ there is

$$(3) \qquad \left\| \frac{\omega(x_0, t_n h)}{t_n} \right\| \leq \frac{\varepsilon}{2} \ .$$

Since

$$F(x_0+t_n h)-F(x_0)=VF(x_0,t_n h)+\omega(x_0,t_n h) \ ,$$

$$F(x_0+t_n h_n)-F(x_0)=VF(x_0,t_n h_n)+\omega(x_0,t_n h_n) \ ,$$

we have

This fact implies

$$|\left(\frac{\omega(x_0, t_n h_n)}{t_n}, e_n\right)| \leq \| \frac{\omega(x_0, t_n h)}{t_n} \| \cdot \| e_n \| +$$

$$+ \| VF(x_0 + \tau_n t_n h_n, h_n) - VF(x_0, h_n) \| \cdot \| e_n \| +$$

$$+ \| VF(x_0, h) - VF(x_0 + \tau_n' t_n h, h) \| \cdot \| e_n \| .$$

In view of Hahn-Banach theorem there exist $e_n \in E_1'$ such that $\| e_n \|_{E_1'} = 1$ and

$$|\left(\frac{\omega(x_0, t_n h_n)}{t_n}, e_n\right)| = \| \frac{\omega(x_0, t_n h_n)}{t_n} \| .$$

Therefore

$$\| \frac{\omega(x_0, t_n h_n)}{t_n} \| \leq \| \frac{\omega(x_0, t_n h)}{t_n} \| + \tag{4}$$

$$+ \| VF(x_0 + \tau_n t_n h_n, h_n) - VF(x_0, h_n) \| +$$

$$+ \| VF(x_0, h) - VF(x_0 + \tau_n' t_n h, h) \| .$$

For $\varepsilon > 0$ there exists $n_0 \in N$ such that for $n \geq n_0$, $n \in N$

$$\| VF(x_0 + \tau_n t_n h_n, h_n) - VF(x_0, h_n) \| + \tag{5}$$

$$+ \| VF(x_0 + \tau_n' t_n h, h) - VF(x_0, h) \| \leq \frac{\varepsilon}{4} .$$

But (4) together with (5) and (3) contradicts the relation (2). Hence the limit

$$\lim_{t \to 0} \frac{F(x_0 + t h) - F(x_0)}{t} = d\, VF(x_0, h)$$

is uniform on $S_{\varkappa}$. The boundedness of $d\,VF(x_0, h)$ follows immediately from the second condition of theorem 1. This completes the proof.

Lemma 1. ([9], § 26.7) Let $F: E \to E_1$ be a continuous mapping in some neighbourhood of x_0. If there exists $DF(x_0, h)$, then $DF(x_0, h)$ is continuous in $h \in E$.

Corollary 1. Let $F: E \to E_1$ be a mapping of E into E_1 continuous in some neighbourhood of x_0. Suppose that there exists $VF(x, h)$ in a neighbourhood of x_0 and is such that $\lim\limits_{t\to 0} \| VF(x_0 + th, h) - VF(x_0, h) \| = 0$ holds uniformly with respect to $h \in E$, $\|h\| = 1$. Let there exist $DF(x_0, h)$. Then the mapping F has the Fréchet derivative $F'(x_0)$ at $x_0 \in E$.

Definition 2. We shall say that a mapping $F: E \to E_1$ is directionally continuous in a convex symmetric neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in E$ if F is continuous along any line-segment in $\mathcal{U}(x_0)$.

Theorem 2. Let E be a reflexive Banach space. Suppose that $F: E \to E_1$ is strongly continuous in $(K_{\varkappa} + x_0)$ where $\varkappa > 0$ is some real number. Assume that there exists $VF(x, h)$ in $(K_{\varkappa} + x_0)$ and is directionally continuous in $(K_{\varkappa} + x_0)$ along the line-segment connecting x_0, x $(x \in x_0 + K_{\varkappa})$. Let $VF(x_0, h)$ be strongly continuous in $h \in E$. Let us define a nonlinear functional by

$$\varphi_t(h) = \left\| \int_0^1 (VF(x_0 + \tau t h, h) - VF(x_0, h))\, d\tau \right\|, \quad h \in K_{\varkappa}$$

and suppose that $\varphi_t(h)$ has the following property: There exists ε, $1 > \varepsilon > 0$, such that if $|t| \leq \varepsilon$, $|t_1| \leq \varepsilon, |t_1| \leq |t|$, then

(6) $\varphi_{t_1}(h) \leq \varphi_t(h)$ for every $h \in K_r$.

Then F possesses the bounded differential $dVF(x_0, h)$ at $x_0 \in E$.

<u>Proof</u>. Let h be an arbitrary (buf fixed) element of K_r. Since $\varepsilon \in (0, 1)$, $x_0 + th \in x_0 + K_r$ and for $t \neq 0$, $|t| \leq \varepsilon$, according to theorem 2.7 [1] we have

$$\frac{F(x_0 + th) - F(x_0)}{t} = \int_0^1 VF(x_0 + \tau th, h)\,d\tau .$$

Suppose $h_n \in K_r$, $h_n \xrightarrow{w} h$, $h \in K_r$. Since F is strongly continuous on $(K_r + x_0)$,

$$\frac{F(x_0 + th_n) - F(x_0)}{t} \to \frac{F(x_0 + th) - F(x_0)}{t}$$

for any fixed t, $t \neq 0$, $|t| \leq \varepsilon$ whenever $n \to \infty$. Therefore $h_n \xrightarrow{w} h$, $h_n, h \in K_r$, $|t| \leq \varepsilon$ imply $\int_0^1 VF(x_0 + \tau th_n, h_n)\,d\tau \to \int_0^1 VF(x_0 + \tau th, h)\,d\tau$ whenever $n \to \infty$. Since $VF(x_0, h)$ is strongly continuous in h, $\int_0^1 VF(x_0, h_n)\,d\tau \to \int_0^1 VF(x_0, h)\,d\tau$.

Hence

$$\lim_{n\to\infty} \left\| \int_0^1 (VF(x_0 + \tau th_n, h_n) - VF(x_0, h_n))\,d\tau \right\| =$$

$$\left\| \int_0^1 (VF(x_0 + \tau th, h) - VF(x_0, h))\,d\tau \right\| .$$

Thus $\varphi_t(h)$ is strongly continuous in h on K_r for an arbitrary (but fixed) t, $|t| \leq \varepsilon$. Suppose that $\{t_n\}$ is a sequence of real numbers such that $|t_n| \leq \varepsilon$,

$\lim_{n\to\infty} t_n = 0$, $|t_{n+1}| \leqq |t_n|$. According to (6) $\varphi_{t_n}(h)$ is a monotonic sequence of strongly continuous functionals in K_r.

Employing our assumptions we see that $\varphi_{t_n}(h)$ weakly converges to the zero-functional on K_r. By the slight generalization of the Dini theorem in Banach spaces ([1] § 22.4) $\lim_{n\to\infty} \varphi_{t_n}(h) = 0$ uniformly on K_r. Now let us assume that $\lim_{t\to 0} \varphi_t(h) = 0$ is not uniform on S_r. Then there exists $\varepsilon_0 > 0$ such that for every $n \in N$ there exist $h_n \in S_r$ and t_n with the property that $0 < |t_n| < \frac{1}{n}$ and $\varphi_{t_n}(h_n) \geqq \varepsilon_0$. Passing to subsequences $\{t_{n_k}\}$, $\{h_{n_k}\}$ such that $\lim_{k\to\infty} t_{n_k} = 0$, $|t_{n_{k+1}}| \leqq |t_{n_k}|$ we obtain $\varphi_{t_{n_k}}(h_{n_k}) \geqq \varepsilon_0$. But this contradicts the fact that $\lim_{k\to\infty} \varphi_{t_{n_k}}(h) = 0$ is uniform on K_r. But the strong continuity of $VF(x_0, h)$ in h implies the boundedness of $VF(x_0, h)$ on S_r. This completes the proof.

<u>Corollary 2</u>. Let E be a reflexive Banach space, $F: E \to E_1$ a strongly continuous mapping in $(x_0 + K_r)$, $r > 0$ such that $VF(x, h)$ is directionally continuous in x on $x_0 + K_r$ along the line-segments connecting x_0, x, $x \in x_0 + K_r$ (h is any fixed element of E). Suppose that there exists ε, $0 < \varepsilon < 1$ such that for $|t| \leqq \varepsilon$, $|t_1| \leqq \varepsilon$, $|t_1| \leqq |t|$

$$\left\| \left(\int_0^1 (VF(x_0 + t_1 \tau h, h) - VF(x_0, h)) \, d\tau \right\| \leqq$$

$$\leq \left\| \int_0^1 (VF(x_0 + t\tau h, h) - VF(x_0, h)) d\tau \right\| .$$

If $DF(x_0, h)$ is strongly continuous in $h \in E$, then F has the Fréchet derivative $F'(x_0)$ at x_0.

<u>Definition 3</u>. A mapping $F : E \to E_1$ is said to be completely compact on a bounded set $\omega \subset E$ if F is compact and uniformly continuous on ω.

<u>Lemma 2</u>. ([1], § 1.4) A mapping $F : E \to E_1$ is completely compact if and only if the following condition is fulfilled: If $\{x_n\}, \{x'_n\}$ are the arbitrary sequences of ω such that $\lim_{n\to\infty} \| x_n - x'_n \| = 0$, then there exist the subsequences $\{x_{n_k}\}, \{x'_{n_k}\}$ with the property

$$\lim_{k\to\infty} F(x_{n_k}) = \lim_{k\to\infty} F(x'_{n_k}) = y_0 \in E_1 .$$

<u>Theorem 3</u>.*) Let F be a mapping of E into E_1. Suppose that there exists the Gâteaux differential $VF(x, h)$ for every $x \in (x_0 + K_r)$ $(r > 0)$. If $VF(x, h)$ is completely compact in $(x_0 + K_r) \times K_r \subset E \times E$, then F has the Fréchet derivative $F'(x_0)$ at $x_0 \in E$.

<u>Proof</u>. Let h_n, h be the arbitrary elements of E such that $h_n \in S_r$, $h \in K_r$. We have

$$F(x_0 + t_n h_n) - F(x_0) = VF(x_0, t_n h_n) + \omega(x_0, t_n h_n) ,$$

$$F(x_0 + t_n h) - F(x_0) = VF(x_0, t_n h) + \omega(x_0, t_n h) .$$

Suppose that the limit

$$\lim_{t\to 0} \left\| \frac{\omega(x_0, th)}{t} \right\| = 0$$

is not uniform on $S_r \subset E$. Then there exists $\varepsilon > 0$

*) A ~~[illegible]~~ consequence of Th. 3.3(1)

with the following property: There exist $h_n \in S_k$ and t_n such that $0 < |t_n| < \frac{1}{n}$ and

(7) $$\| \frac{\omega(x_0, t_n h_n)}{t_n} \| \geq \varepsilon .$$

Let $e_n \in E_1'$ be any arbitrary elements of E_1'. By the mean-value theorem

$$\begin{aligned}(\frac{\omega(x_0, t_n h_n)}{t_n}, e_n) = (\frac{\omega(x_0, t_n h)}{t_n}, e_n) &+ \\ + (VF(x_0 + \tau_n t_n h_n, h_n), e_n) &- \\ - (VF(x_0 + \tau_n' t_n h, h), e_n) &+ \\ + ((VF(x_0, h) - VF(x_0, h_n)), e_n) &.\end{aligned}$$

According to Hahn-Banach theorem there exist $e_n \in E_1'$ such that $\| e_n \|_{E_1'} = 1$ and

$$| (\frac{\omega(x_0, t_n h_n)}{t_n}, e_n) | = \| \frac{\omega(x_0, t_n h_n)}{t_n} \| .$$

Hence

$$\begin{aligned}\| \frac{\omega(x_0, t_n h_n)}{t_n} \| \leq \| \frac{\omega(x_0, t_n h)}{t_n} \| + \| VF(x_0 + \tau_n t_n h_n, h_n) &- \\ - VF(x_0, h_n) \| + \| VF(x_0 + \tau_n' t_n h, h) &- \\ - VF(x_0, h) \| &.\end{aligned}$$

Since $VF(x, h)$ is completely compact on $(x_0 + K_k) \times K_k$, passing to the subsequences $\{[x_0 + \tau_{n_k} t_{n_k} h_{n_k}, h_{n_k}]\}, \{[x_0, h_{n_k}]\}$, we have that

$$\lim_{k \to \infty} VF(x_0 + \tau_{n_k} t_{n_k} h_{n_k}, h_{n_k}) = \lim_{k \to \infty} VF(x_0, h_{n_k}) .$$

Again we can extract a subsequence $\{n_{k_\ell}\}$ such that for the sequences $\{[x_0+\tau'_{n_{k_\ell}} t_{n_{k_\ell}} h, h]\}$, $\{[x_0, h]\}$ there is

$$\lim_{\ell\to\infty} \| VF(x_0+\tau'_{n_{k_\ell}} t_{n_{k_\ell}} h, h) - VF(x_0, h)\| = 0 .$$

Since F has the Gâteaux differential at x_0,

$$\lim_{\ell\to\infty} \left\| \frac{\omega(x_0, t_{n_{k_\ell}}, h)}{t_{n_{k_\ell}}} \right\| = 0 .$$

These facts give the contradiction with (7).

Thus F has the bounded differential $dVF(x_0, h)$ at $x_0 \in E$. By the Vajnberg theorem ([1], th. 3.1) $dVF(x_0, h)$ must be linear in $h \in E$. Therefore $dVF(x_0, h) = dF(x_0, h) = F'(x_0)h$, where $F'(x_0)$ denotes the Fréchet derivative of F at $x_0 \in E$.

<u>Lemma 3</u>. Let E be a reflexive Banach space, $F: E \to E_1$ a mapping of E into E_1 such that there exists $VF(x, h)$ in some neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in E$. Suppose that $VF(x, h)$ is directionally continuous in $\mathcal{U}(x_0)$ for every (but fixed) $h \in E$. Let $VF(x_0, h)$ be strongly continuous in $h \in E$. Let $t_n \to 0$, $h_n \xrightarrow{w} 0$, $h_n \in K_r$, $r > 0$ imply

$$(8) \qquad \lim_{n\to\infty} \left\| \int_0^1 VF(x_0+t_n h+t t_n h_n, h_n)\,dt \right\|_{E_1} = 0 .$$

Then F possesses the bounded differential $dVF(x_0, h)$ at $x_0 \in E$.

<u>Proof</u>. Let $h_n \in S_r$, $h \in K_r$, $t_n \to 0$. We have

$$(9) \quad F(x_0+t_n h_n) - F(x_0) = VF(x_0, t_n h_n) + \omega(x_0, t_n h_n),$$

$$F(x_0+t_n h)-F(x_0)=VF(x_0,t_n h)+\omega(x_0,t_n h)\ .$$

Assume that

$$\lim_{t\to 0}\left\|\frac{\omega(x_0,th)}{t}\right\|=0$$

is not uniform on S_r. Then there exist $\varepsilon>0$, $0<|t_n|<$ $<\frac{1}{n}$, $h_n\in S_r$ such that

$$\left\|\frac{\omega(x_0,t_n h_n)}{t_n}\right\|\geq\varepsilon\ . \tag{10}$$

Since E is a reflexive Banach space, passing to a subsequence $\{h_{n_k}\}$, we may assume that $h_{n_k}\xrightarrow{w}h$. From (9) we obtain

$$\left(\frac{\omega(x_0,t_{n_k}h_{n_k})}{t_{n_k}},e_{n_k}\right)=\left(\frac{\omega(x_0,t_{n_k}h)}{t_{n_k}},e_{n_k}\right)+$$

$$+\left(\frac{F(x_0+t_{n_k}h_{n_k})-F(x_0+t_{n_k}h)}{t_{n_k}},e_{n_k}\right)+$$

$$+\left(\left(VF(x_0,h)-VF(x_0,h_{n_k})\right),e_{n_k}\right),$$

where $e_{n_k}\in E_1'$ are any elements of E_1'. Let us choose $e_{n_k}\in E_1'$ such that

$$\left|\left(\frac{\omega(x_0,t_{n_k}h_{n_k})}{t_{n_k}},e_{n_k}\right)\right|=\left\|\frac{\omega(x_0,t_{n_k}h_{n_k})}{t_{n_k}}\right\|,$$

$\|e_{n_k}\|_{E_1'}=1$.

Hence

$$\left\|\frac{\omega(x_0,t_{n_k}h_{n_k})}{t_{n_k}}\right\|\leq\left\|\frac{\omega(x_0,t_{n_k}h)}{t_{n_k}}\right\|+$$

$$+\left\|\int_0^1 VF(x_0+t_{n_k}h+tt_{n_k}(h_{n_k}-h),h_{n_k}-h)\,dt\right\|+$$

$$+ \| VF(x_0, h_{n_k}) - VF(x_0, h) \| .$$

Since $VF(x_0, h)$ is strongly continuous in h and in view of (8), we have the contradiction with (10).

<u>Theorem 4</u>. Let E be a reflexive Banach space, $F: E \to E_1$ a mapping of E into E_1 having the following properties:

1) there exists the Gâteaux differential $VF(x_0 + x, h)$ for $x \in K_{\kappa_0}$, is directionally continuous in $x \in K_{\kappa_0}$ (for any fixed $h \in E$) and $\| VF(x_0 + x, h) \| \leq K$ for every $x \in K_{\kappa_0}$, $h \in K_{\kappa_0}$, where K_{κ_0} is some closed ball in E, $VF(x_0, h)$ is strongly continuous in $h \in E$

2) $\| VF(x_0 + x_n, h_n \| \to 0$ whenever $x_n \to x_0$, $h_n \xrightarrow{w} 0$.

Then F has the bounded differential $dVF(x_0, h)$ at $x_0 \in E$.

<u>Proof</u>. Suppose that the conditions of our theorem are satisfied. Let $t \in \langle 0, 1 \rangle$. Then we have

$$\| VF(x_0 + t_n h + t\, t_n (h_n - h), h_n - h) \| \to 0$$

whenever $|t_n| < 1$, $t_n \to 0$, $h_n \in S_{\frac{\kappa_0}{4}}$, $h_n \xrightarrow{w} h$.
Thus $\varphi_n(t) = VF(x_0 + t_n h + t\, t_n (h_n - h), h_n - h)$ are continuous abstract functions on $\langle 0, 1 \rangle$, $\| \varphi_n(t) \| \leq K$,

$\lim\limits_{n \to \infty} \varphi_n(t) = 0$ in $\langle 0, 1 \rangle$. By the Lebesgue theorem ([10],chapt.III,§ 6.16)

$$\lim_{n \to \infty} \| \int_0^1 VF(x_0 + t_n h + t t_n (h_n - h), h_n - h) \| = 0 .$$

Thus the conditions of the lemma 3 are fulfilled and our theorem is proved.

Corollary 3. Under the conditions of the theorem 4, let there exist $DF(x_0, h)$. Then F possesses the Fréchet derivative $F'(x_0)$ at $x_0 \in E$.

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